THEORY OF NON-AXISYMMETRIC BURGERS VORTEX WITH ARBITRARY REYNOLDS NUMBER

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Abstract. We develop an asymptotic theory of the steady state of a rectilinear vortex in linear straning flow. In the special case of axisymmetric strain the solution is the familiar Burgers vortex. In the more general, non-axisymmetric situation the asymptotic theories were developed for low Reynolds number (Robinson & Saffman 1984) and for high Reynolds number (Moffatt, Kida & Ohkitani 1994). In the present paper we develop a new expansion in the parameter λ which characterises the departure from axisymmetry. Hence we obtain an expansion valid uniformly for all Reynolds numbers and thus bridgeing the gap between the low and the high Reynolds number theories. In practice the new expansion is useful when the non-axisymetric deformation of the vortex is not too large.

1. Introduction

G. I. Taylor (1938) recognised the fact that the competition between stretching and viscous diffusion of vorticity must be the mechanism controlling the dissipation of energy in turbulence. A decade later Burgers (1948) obtained exact solutions describing steady vortex tubes and layers in locally uniform straining flow where the two effects are in balance. The discovery of the

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exact solutions stimulated the development of the models of the dissipative scales of turbulence as random collections of vortex tubes and/or sheets.

The intermittent nature of the vorticity field was observed in experiments by taking statistical measurements which indicated the existence of the small-scale localised structures (Townsend 1951). Only recently have these structures been directly observed, first in the numerical simulations (see, for example Vincent & Meneguzzi 1991) and then in the laboratory experiments where a new visualisation technique was employed (Douady, Couder & Brachet 1991).

The Burgers vortex has axial symmetry unlikely to be found in real flows, hence the need to find solutions describing non-axisymmetric stretched vortices. Let us consider incompressible fluid with viscosity ν. We look for a steady state of a vortex having stream-function $\Psi(x, y)$, vorticity $\boldsymbol{\omega} = -(\nabla^2\Psi)\hat{\mathbf{z}}$ and total circulation Γ subjected to the ambient irrotational straining flow

$$\boldsymbol{U} = (\alpha x, \beta y, \gamma z) \quad , \qquad \alpha + \beta + \gamma = 0 \quad , \qquad \alpha < 0, \quad \gamma > 0 \tag{1}$$

characterised by the parameter

$$0 \leq \lambda = \frac{\alpha - \beta}{\alpha + \beta} \tag{2}$$

which measures the departure from axisymmetry. Taking $\sqrt{\nu/\gamma}$, γ^{-1}, $\Gamma/2\pi$, $\gamma\Gamma/2\pi\nu$ to be the units of length, time, stream-function and vorticity respectively respectively we obtain the steady state equations in polar coordinates (r, θ),

$$\frac{1}{r}\frac{\partial(\Psi, \omega)}{\partial(r, \theta)} = -R_\Gamma^{-1}\left(L_0\omega + \lambda L_1\omega\right), \tag{3}$$

$$\omega = -\nabla^2\Psi, \tag{4}$$

where $R_\Gamma = \Gamma/2\pi\nu$ is the Reynolds number of the vortex and L_0, L_1 are linear operators,

$$L_0 = 1 + \tfrac{1}{2}r\partial_r + \nabla^2 \tag{5}$$

$$L_1 = \tfrac{1}{2}\cos(2\theta)\, r\partial_r - \tfrac{1}{2}\sin(2\theta)\, \partial_\theta. \tag{6}$$

Robinson and Saffman (RS84) solved this equation numerically for $0 < R_\Gamma < 100$ and found that the vortex has a quasi-elliptical shape with the minor axis inclined at an angle $\Phi(R_\Gamma, \lambda)$ to the principal axis of strain. They found that $\Phi(R_\Gamma, \lambda) \to 0$ as $R_\Gamma \to 0$ and developed a theory for $R_\Gamma \ll 1$ involving *double* expansion in powers of both λ and R_Γ. Their computations showed that as R_Γ increases $\Phi(R_\Gamma, \lambda)$ settles to a constant value $\Phi_c \approx 45°$. The value of Φ_c was theoretically derived by Moffatt, Kida & Ohkitani (MKO94) who developed an asymptotic theory for $R_\Gamma \gg 1$ later adapted to diffusing vortices in two-dimensional strain for which much more numerical data, including details of the vortex structure, are available (Jiménez, Moffatt & Vasco 1996). Here we develop a new theory based on the expansion in powers of λ.

2. λ-expansion

We look for solutions of (3-4) in the form of the series

$$\omega = \omega_0 + \lambda\omega_1 + \lambda^2\omega_2 + \dots \quad , \tag{7}$$

$$\Psi = \Psi_0 + \lambda\Psi_1 + \lambda^2\Psi_2 + \dots \quad . \tag{8}$$

The lowest order gives an equation for the vortex in *axisymmetric* strain:

$$\frac{1}{r}\frac{\partial(\Psi_0,\omega_0)}{\partial(r,\theta)} = -R_\Gamma^{-1}L_0\omega_0 \tag{9}$$

whose solution is the familiar Burgers vortex,

$$\omega_0 = \tfrac{1}{2}\mathrm{e}^{-\frac{1}{4}r^2}, \tag{10}$$

$$\Psi_0 = \int_0^r x^{-1}\left(\mathrm{e}^{-\frac{1}{4}x^2} - 1\right) dx. \tag{11}$$

The calculations of the $O(\lambda)$ terms become much simpler in the variables

$$w = \tfrac{1}{4}r^2 \quad , \qquad \varphi = 2\theta. \tag{12}$$

From (3-4) we obtain

$$R_\Gamma\left[\frac{d\Psi_0}{dw}\frac{\partial\omega_1}{\partial\varphi} - \frac{d\omega_0}{dw}\frac{\partial\Psi_1}{\partial\varphi}\right] + L_0\omega_1 + L_1\omega_0 = 0, \tag{13}$$

$$\nabla^2\Psi_1 = \omega_1, \tag{14}$$

where now

$$\omega_0 = \tfrac{1}{2}\mathrm{e}^{-w} \quad , \frac{d\Psi_0}{dw} = \tfrac{1}{2}w^{-1}\left(\mathrm{e}^{-w} - 1\right), \tag{15}$$

and the operators ∇^2, L_0 and L_1 now take the form:

$$\nabla^2 = \partial_w w\partial_w + w^{-1}\partial_\varphi^2 \tag{16}$$

$$L_0 = w\partial_w^2 + (w+1)\partial_w + w^{-1}\partial_\varphi^2 + 1 \tag{17}$$

$$L_1 = \cos\varphi w\partial_w - \sin\varphi\partial_\varphi. \tag{18}$$

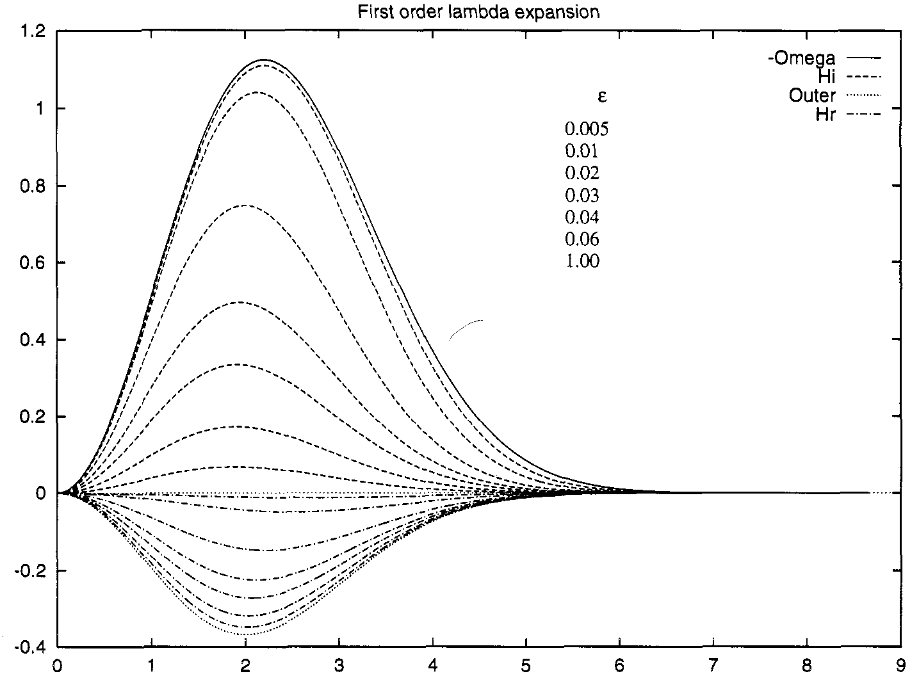

Figure 1. The functions $H_i(r)$ (positive values, dashed lines) and $H_r(r)$ (negative values, dot-dashed lines) for different values of the parameter $\epsilon = \nu/\Gamma$ listed in the figure. The peak values of H_i increase as $\epsilon \to 0$ when H_i tends to the function $-\Omega(r)$ (solid line). The peak value of $|H_r|$ increases as $\epsilon \to \infty$ ($R_\Gamma \to 0$) when H_r tend to its limiting value $-we^{-w}$.

Substituting (15) into (13) we obtain

$$\tfrac{1}{2}R_\Gamma w^{-1}\left(e^{-w}-1\right)\frac{\partial\omega_1}{\partial\varphi}+\tfrac{1}{2}R_\Gamma e^{-w}\frac{\partial\Psi_1}{\partial\varphi}+L_0\omega_1=\tfrac{1}{2}we^{-w}\cos\varphi. \qquad (19)$$

This equation reveals the nature of the λ-expansion. Lower orders become *source* terms in the equations for higher orders. This is quite different from the procedure developed in MKO94 for $R_\Gamma \gg 1$. There the first order equation appeared as the integrability condition for the *second* order.

The right-hand-side of (19) consist of a single Fourier mode, so we can look for a solution in the form

$$\Psi_1 = \mathrm{Re}\left[\tfrac{1}{2}S(r)e^{i\varphi}\right] \quad , \qquad \omega_1 = \mathrm{Re}\left[\tfrac{1}{2}H(r)e^{i\varphi}\right]. \qquad (20)$$

The (complex) radial functions satisfy two coupled ordinary differential

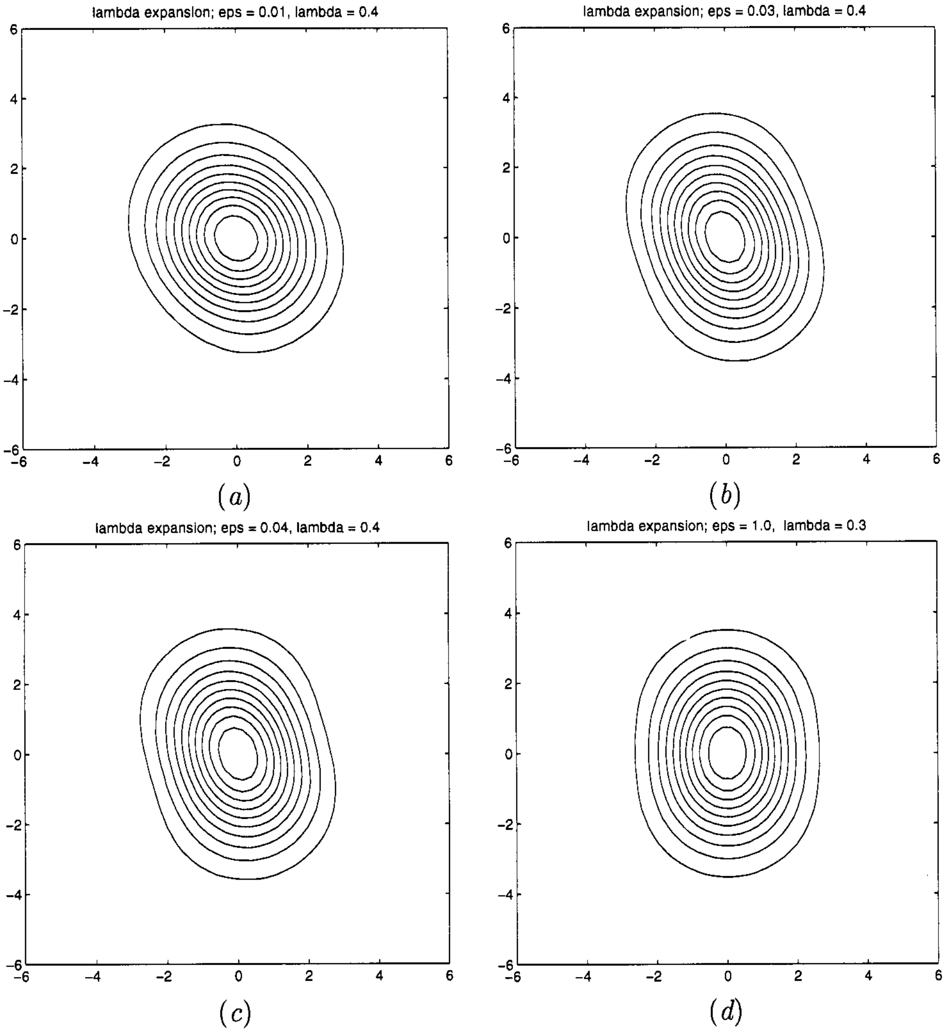

Figure 2. The level contours of $\omega_0 + \lambda\omega_1$ for $\epsilon = 0.01$, 0.03, 0.04 and 1.0. As ϵ increases the tilt of the vortex decreases from its limit value of 45° to zero while the elliptical deformation increases.

equations,

$$wH''+(w+1)H'+\left[\mathrm{i}\tfrac{1}{2}R_\Gamma w^{-1}(\mathrm{e}^{-w}-1)+1-w^{-1}\right]H+\mathrm{i}\tfrac{1}{2}R_\Gamma \mathrm{e}^{-w}S = w\mathrm{e}^{-w} \quad , \tag{21}$$

$$wS''+S'-w^{-1}S+H=0 \quad , \tag{22}$$

that can easily be solved numerically, but first we need to determine the appropriate boundary conditions.

Far away from the origin the nonlinear term on the left-hand side of (3) representing self-induced convection of vorticity is negligible and the *linearized* equation has a unique solution (MKO94),

$$\omega^E(w,\varphi) = \tfrac{1}{2}\mathrm{e}^{-w(1+\lambda\cos\varphi)}. \tag{23}$$

Expanding this external vorticity in powers of λ we can, in principle, obtain the asymptotic form for large r of all terms in the series (7-8),

$$\omega^E(w,\varphi) = \tfrac{1}{2}\mathrm{e}^{-w}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}w^n\cos^n\varphi\right]\lambda^n. \tag{24}$$

In particular, we obtain an outer boundary condition for ω_1:

$$\omega_1 \sim -\tfrac{1}{2}w\mathrm{e}^{-w}\cos\varphi \qquad \text{as} \qquad r\to\infty, \tag{25}$$

and solving (14) gives

$$\Psi_1 \sim \tfrac{1}{2}\left[\left(1+\frac{1}{w}\right)\mathrm{e}^{-w}-\frac{1}{w}\right]\cos\varphi \qquad \text{as} \qquad r\to\infty, \tag{26}$$

which gives

$$H \sim -w\mathrm{e}^{-w}, \quad S\sim -\frac{1}{w}, \qquad \text{as} \qquad r\to\infty. \tag{27}$$

The behaviour at the origin can be deduced by taking H and S in the form of a power series and balancing the dominant terms. One possible solution is

$$H\sim Cw \quad , \qquad S\sim Dw \qquad \text{as} \qquad w\to 0, \tag{28}$$

where C and D are constants. This is in agreement with the behaviour of the asymptotic solutions of MKO94, so we take (26-27) as boundary conditions for (20-21).

The results can be verified in two limiting cases. In the limit $R_\Gamma\to 0$ the function ω_1 must be equal to $-w\mathrm{e}^{-w}$ *everywhere*, not just at large distances. In the limit $R_\Gamma\to\infty$ the asymptotic form of ω_1 can be deduced from MKO94 who derived the function Ω, such that

$$H(r)\sim -2\mathrm{i}R_\Gamma^{-1}\Omega(r) \qquad \text{as} \qquad R_\Gamma\to\infty. \tag{29}$$

In figure 1 we plot $H_i = \frac{1}{2}R_\Gamma\mathrm{Im}(H)$ (positive) and $H_r = \frac{1}{2}R_\Gamma\mathrm{Re}(H)$ (negative), as functions of r, for different values of the parameter $\epsilon = \nu/\Gamma = (2\pi R_\Gamma)^{-1}$. The functions H_i become more curved as the value of ϵ *decreases* and they tend to $-\Omega(r)$ which means that our solution, as expected, approaches that of MKO94 as $R_\Gamma\to\infty$. The functions H_r become

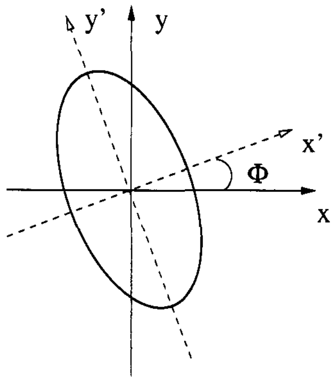

Figure 3. The frame of reference (x, y) defined by the principal axes of the ambient strain, and the frame of reference (x', y') in which the antisymmetric second moment of the vorticity distribution vanishes. The angle Φ is the tilt of the vortex.

more curved as the value of ϵ *increases* and they tend to the linearized solution $-w\mathrm{e}^{-w}$ as $R_\Gamma \to 0$.

In figure 2 we show the contour levels of vorticity in the first order solution, i.e., the contours of $\omega_0 + \lambda\omega_1$ The vortex has quasi-elliptical shape. The major axis of the ellips forms an angle Φ with the principal axis of strain (see figure 3). This tilt depends on both R_Γ (or ϵ) and λ. As $R_\Gamma \to \infty$ the angle Φ tends to 45°, as predicted by MKO94.When $R_\Gamma \to 0$ there is no self-induced rotation of the vortex, so $\Phi \to 0$. The level contours are excessively flattened along the minor axis, particularly for larger values of λ when the higher-order terms in (7) begin to play a rôle.

In order to quantify this effect the angle of inclination is defined by the rotated frame of reference,

$$x' = \cos\Phi\, x + \sin\Phi\, y \quad , \qquad y' = -\sin\Phi\, x + \cos\Phi\, y, \tag{30}$$

such that the antisymmetric second moment of the vorticity distribution vanishes in that frame (RS84),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x'y'\omega(x', y')\, dx'\, dy' = 0. \tag{31}$$

Taking $\omega = \omega_0 + \lambda\omega_1$ we obtain

$$\frac{1}{2}\int_0^{2\pi} d\theta \int_0^{\infty} dr\, r^3 \sin 2(\theta - \Phi)\left[R_\Gamma\omega_0(r) + \lambda H_r \cos 2\theta - \lambda H_i(r)\sin 2\theta\right] = 0 \tag{32}$$

Therefore, the λ-expansion yields a formula for the angle of inclination,

$$\tan 2\Phi = -\frac{\int_0^\infty dr\, r^3 H_i(r)}{\int_0^\infty dr\, r^3 H_r(r)}. \tag{33}$$

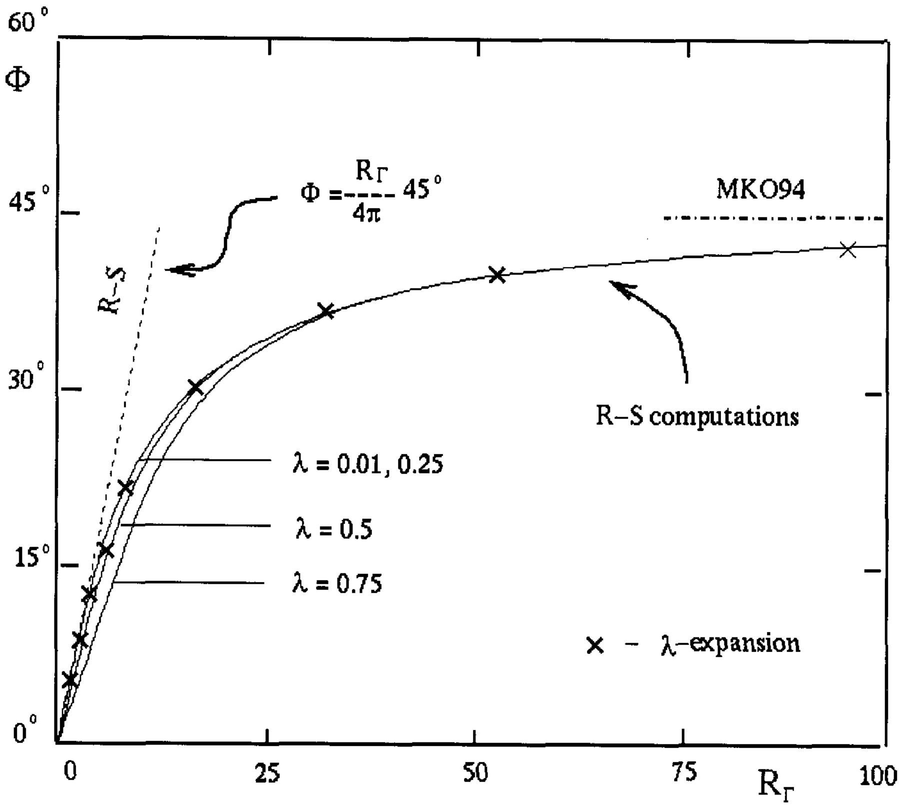

Figure 4. The tilt of the vortex computed in RS84 for four values of λ (solid lines) compared with the tilt given by (33) (crosses). The theoretical prediction of RS84 for $R_\Gamma \to 0$ (dashed line) and the asymptotic result of MKO94 for R_Γ (dot-dashed line) are also shown.

In figure 4 we show $\Phi(R_\Gamma)$ computed by Robinson and Saffman for $\lambda = 0.01$, 0.25, 0.5 and 0.75. The values calculated from (33), marked with crosses, are in good agreement, at least for $\lambda < 0.5$. The curves for $\lambda = 0.1$ and 0.25 are indistinguishable and that for $\lambda = 0.5$ differs very little. The explanation comes from (33) which shows that in the first order Φ is *independent* of λ. It means that a vortex is not easily 'tilted' by the ambient strain whose parameter λ must well exceed 0.5 to have considerable effect.

3. Conclusions

We have developed the theory describing a steady vortex in the linear irrotational ambient flow. The vortex experiences stretching along its axis and compression in the cross-sectional plane which compensates viscous diffusion of vorticity like in the classical Burgers solution which is a special axisymmetric case of this problem.

The non-linear vorticity equation is solved by expanding Ψ and ω in powers of the parameter λ measuring the departure of the ambient strain from axisymmetry. The solution is valid for any value of the Reynolds number and therefore complements previous asymptotic theories for $R_\Gamma \to 0$ and $R_\Gamma \to \infty$. For example, the tilt of the vortex can be accurately calculated for any value of R_Γ and already in the first order the results agree very well with the numerical solution of the full non-linear problem.

The first order solution is accurate only within a certain radius around the center of the vortex. Higher order corrections are needed outside this radius which depends on the value of λ and on the chosen criterion of accuracy. The first order solution is useful mainly when λ is small, say $\lambda < 0.25$, but some diagnostics, like the tilt angle, are well predicted even for larger values.

Calculating higher order terms may improve the accuracy of the result if the series is convergent, or it may make it worse if it is merely asymptotic. The convergence of the series (7) is an interesting but, due to the non-linear nature of the vorticity equation, rather difficult issue. Earlier we have developed a theory for the similar but *linear* problem of straight magnetic flux tube with a line vortex on the axis (Bajer & Moffatt 1997). The results strongly suggest that the λ-expansion is convergent in this case while the series appearing in both the low- and the high-Reynolds-nuber expansions are in fact asymptotic. The arguments used for the magnetic problem depend on its linearity, but it is plausible that the conclusion is still valid for the vortex problem.

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