

Steady Confined Stokes Flows with Chaotic Streamlines

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A family of incompressible steady flows, called STF flows, is considered. These are Stokes flows whose velocity is a quadratic function of cartesian coordinates satisfying $\mathbf{u}^{STF} \cdot \mathbf{n} = 0$ on a sphere $r = 1$. When the flow is an $O(\epsilon)$ perturbation of a flow \mathbf{u}_0^{STF} with closed streamlines the particle paths are, for a long time, constrained near some two-dimensional surfaces (adiabatic drift) but they jump randomly from one such surface to another when they approach a stagnation point. Because of these random jumps, which we call 'super-adiabatic drift', streamlines are apparently space-filling throughout entire domain. The Lyapunov exponent λ is computed and it is found to be positive on the space-filling streamlines. There is some evidence that λ remains bounded away from zero when $\epsilon \rightarrow 0$. The separation of two distinct but initially close fluid particles is also computed. It indicates efficient mixing of a passive scalar in the STF flows.

1. INTRODUCTION

It has been known for some years that even simple fluid flows may lead to exceedingly complicated particle paths. Three-dimensional steady flows can have chaotic streamlines which densely fill some regions of space – a property relevant, for example, to the kinematic dynamo problem. In two-dimensional flows even weak time dependence may result in the chaotic motion of fluid particles, which can be useful for stirring and mixing.

Several time-dependent, two-dimensional flows have been investigated so far: blinking point vortices (Aref 1984), Stokes flow between the alternately rotating, eccentric circular cylinders (Aref & Balachandar 1986; Chaiken *et al.* 1986, 1987; Ottino *et al.* 1988); and also simple models of such physical flows as: weak waves in Taylor vortices (Broomhead & Ryrie 1988) or tidal flows in shallow seas (Pasmanter 1988).

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The structure of streamlines has been studied in some space-periodic steady flows.¶ The best known are the so-called ABC flows (Henon 1966; Dombre *et al.* 1986) and Q-flows (Beloshapkin *et al.* 1989). In this paper we introduce a class of three-dimensional incompressible steady flows in a sphere which have a particularly rich Lagrangian structure. These flows have a very simple Eulerian representation: the cartesian components of the velocity are quadratic functions of the coordinates. Similar flows were considered before (Moffatt & Proctor 1985) in connection with Stretch-Twist-Fold fast dynamo processes and for this reason we adopt the name STF flows for their modified version presented below.

2. STF FLOWS

We consider a two-parameter family of quadratic flows:

$$\mathbf{u}^{STF} = (\alpha z - 8xy, 11x^2 + 3y^2 + z^2 + \beta xz - 3, -\alpha x + 2yz - \beta xy) \quad , \quad (1)$$

where α, β are positive real parameters. Clearly \mathbf{u}^{STF} satisfies: $\nabla \cdot \mathbf{u}^{STF} = 0$; $\mathbf{u}^{STF} \cdot \mathbf{n}|_S = 0$ where S is the surface of a unit sphere.*

The flow (1) is a sum of the rigid rotation \mathbf{R} around the y -axis with angular velocity α and the remaining part \mathbf{u}_0^{STF} . Streamlines of \mathbf{u}_0^{STF} are mostly closed loops with the exception of some which are heteroclinic lines linking the hyperbolic stagnation points at $(x, y, z) = (0, \pm 1, 0)$. Hence, for $\alpha \ll 1$, \mathbf{u}^{STF} is a weakly perturbed integrable flow with closed streamlines.

The STF flow, being quadratic in space coordinates, satisfies the Stokes equation $\nabla^2 \mathbf{u} = \nabla p$. Stokes flows are uniquely determined provided the velocity is prescribed on the boundary. Hence they can be realised in a viscous fluid by suitable motions of the boundaries.

3. THE STRUCTURE OF STREAMLINES

In fig. 1a) we show a Poincaré section of a single streamline of \mathbf{u}^{STF} with $\alpha = 0.1$ and in fig. 1b) a projective view of the same streamline. Over 8000 points of section are plotted. The streamline apparently penetrates most of the sphere but in a

¶ This problem is related to the structure of the magnetic field lines (Moffatt 1985, 1986). Despite the formal equivalence between steady incompressible flows and magnetic fields the ways of inducing them are very different. Therefore solenoidal fields regarded as physically realistic will be very different in the two cases.

* A general quadratic flow satisfying these conditions has seven independent parameters (Bajer & Moffatt 1989).

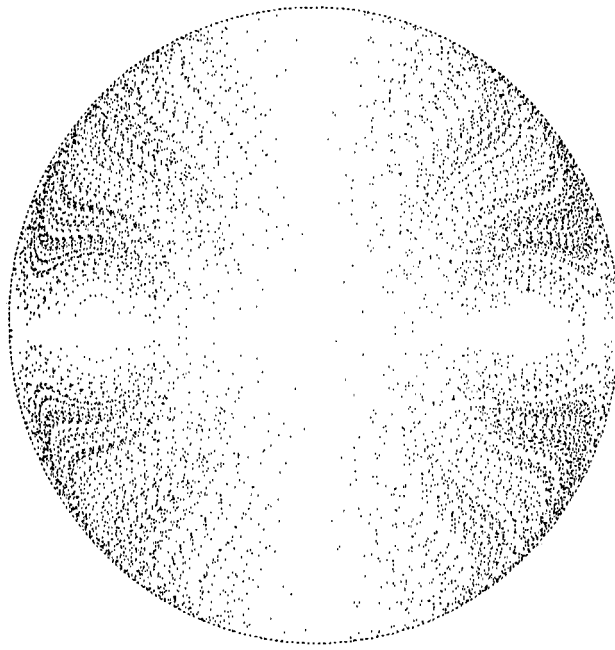
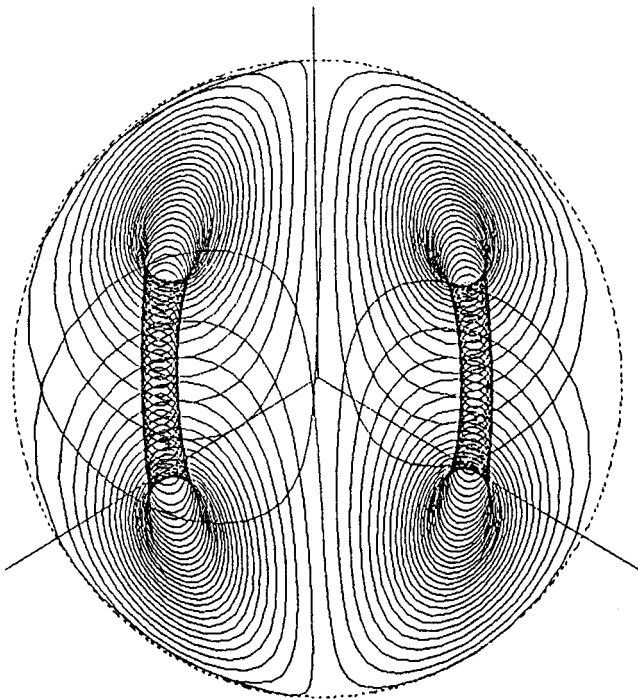
*a)**b)*

FIGURE 1. A single chaotic streamline of the STF flow: *a)* The Poincaré section; *b)* The projective view.

rather orderly manner. Watching the Poincaré section being plotted one may notice that for a long time points closely follow curves. These curves are sections of two-dimensional surfaces which meet at the stagnation points on the surface of a unit sphere and the streamline jumps from one such surface to another every time it comes near the stagnation point. The set of these curves is traced out with apparently random spacing. It forms the distinct layered pattern seen in fig. 1a).

The motion along the curves can be explained in terms of an adiabatic invariant (Bajer 1989; Bajer & Moffatt 1989) and therefore we should call it adiabatic motion (or drift). The relatively large jumps mentioned above form what we term as 'super-adiabatic drift'. These two kinds of drift persist as we decrease α . For $\alpha = 0$ streamlines are closed loops, i.e. the Poincaré section of an orbit consists of only two points. For an arbitrarily small but finite α orbits spread apparently over the entire domain and there are no visible islands of regularity. This is a very different picture from, for example, the ABC flow where, in the integrable case, streamlines fill toroidal surfaces, most of them densely. Then the KAM tori surviving a small perturbation ensure confinement of the chaotic streamlines. The KAM theory does not apply to flows with closed streamlines and hence the global chaos is possible for arbitrarily small perturbations.

4. LYAPUNOV EXPONENT AND THE SEPARATION OF PARTICLES

In order to substantiate the above claim we compute the Lyapunov exponent associated with the STF flow. The Lyapunov exponent λ is the asymptotic growth-rate of the length of an infinitesimal material line element ξ advected by the flow:

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \log |\xi| \quad . \quad (2)$$

Positive λ means exponential stretching – the signature of a chaotic flow. We solve numerically the sixth order system of ODE's for ξ :

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{u}^{STF} \\ \frac{d\xi}{dt} &= (\xi \cdot \nabla) \mathbf{u}^{STF} \end{aligned} \quad (3)$$

We integrate (3) with $\alpha = 0.01$ and plot in fig. 2a) the function $\lambda(t) = \frac{1}{t} \log |\xi(t)|$. The corresponding Poincaré section of \mathbf{u}^{STF} is shown in fig. 2b). The value of the function $\lambda(t)$ was plotted (marked with a single dot in fig. 2a) every time the streamline crosses the Poincaré plane of section. Gaps in the plot of $\lambda(t)$ correspond to the long time that a particle spends near the stagnation points.

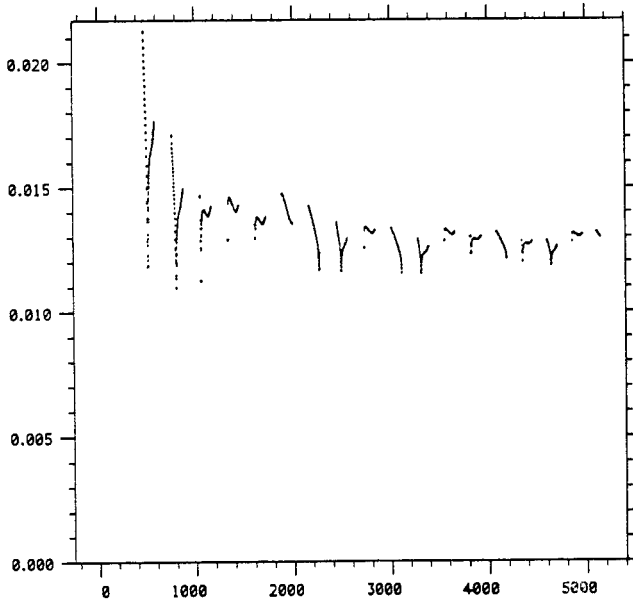
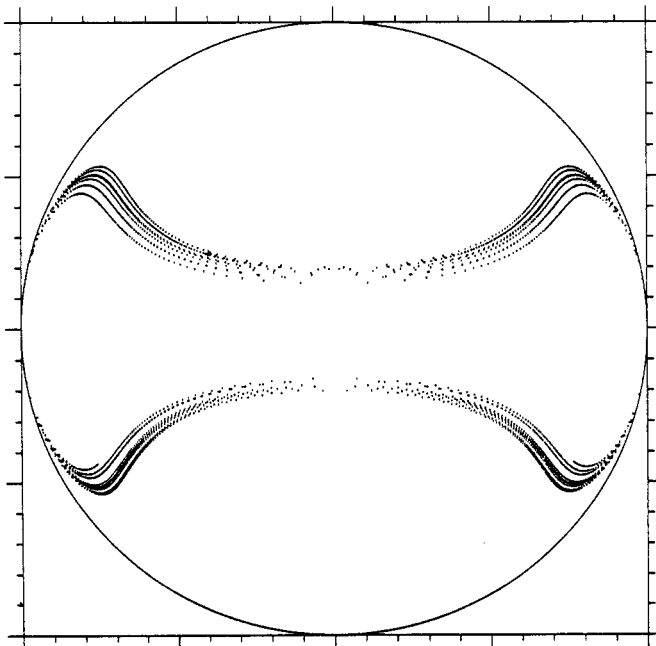
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FIGURE 2. *a)* The function $\lambda(t)$. Its asymptotic value for $t \rightarrow \infty$ is equal to the Lyapunov exponent of a streamline; *b)* Corresponding Poincaré section.

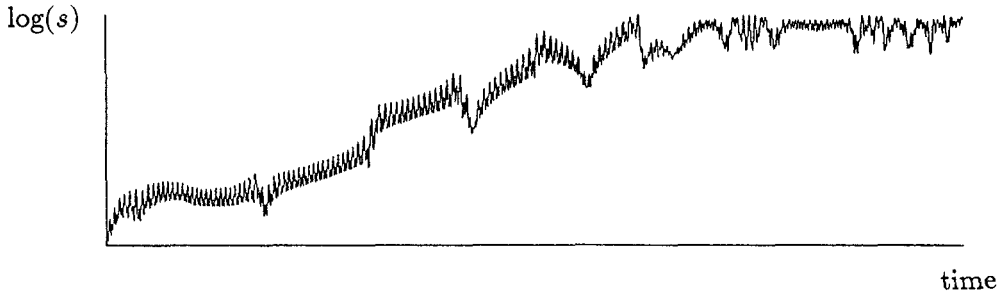


FIGURE 3. Logarithm of the separation between two close particles as a function of time.

The Lyapunov exponent has the dimension of $(\text{time})^{-1}$, the same as the small parameter α . The time-scale of the unperturbed flow \mathbf{u}_0^{STF} is not well defined. For the streamlines close to heteroclinic lines the turnover time is arbitrarily long. However, the perturbation \mathbf{R} has a unique time-scale $\frac{1}{\alpha}$. In these units the value of the Lyapunov exponent, as read from fig. 2a), is $\lambda \approx 1.2$. Further computations indicate that λ remains $O(1)$ as we increase α (up to $\alpha = 0.1$).

Finally in fig. 3 we plot the separation s between two distinct fluid particles which initially were very close (10^{-6} compared to the unit radius of the sphere). Here $\log(s)$ is plotted as a function of time for $\alpha = 0.1$. Jumps in $\log(s)$ correspond to the jumps between different adiabatic surfaces.

In a finite domain the separation, unlike the length of the infinitesimal line element, is bounded. Therefore a systematic increase in s saturates, and then s varies randomly between $s = 0$ and its maximal value $s = 2$. The time needed for the systematic increase to saturate could be regarded as the characteristic time-scale for mixing of the spatial scales of the size of the initial separation (10^{-6} in this case).

5. CONCLUSIONS

We have shown that STF flows (1) in the limit of small α exhibit Lagrangian chaos of a kind so far unknown in the context of flow kinematics. Because of the

degenerate character of the unperturbed flow \mathbf{u}_0^{STF} which has closed streamlines there are no KAM tori, and for an arbitrarily small perturbation the super-adiabatic drift spreads the streamlines over the entire domain. The Lyapunov exponent is positive, and when measured on the time-scale of the slow perturbation it appears to be bounded away from zero when $\epsilon(= \alpha) \rightarrow 0$.

This new type of chaos is likely to be typical for perturbed flows with closed streamlines. Its transport properties such as the chaotic advection of a passive scalar or the possibility of a fast dynamo deserve further investigation.

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