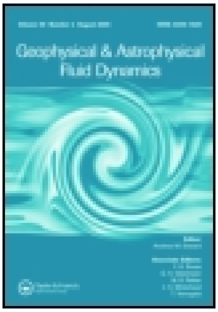

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Report on workshop on small-diffusivity dynamos and dynamical systems

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REPORT ON WORKSHOP ON
SMALL-DIFFUSIVITY DYNAMOS AND DYNAMICAL SYSTEMS
HELD AT
OBSERVATOIRE DE NICE 25–30 JUNE 1989

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In recent years there has been rapidly growing interest in deterministic dynamo action: to what extent can non-random conducting flows sustain magnetic fields, and efficiently so, at the high magnetic Reynolds numbers which are common in the natural environment? For a dynamo to be relevant to astrophysical applications, it can be crucial that it be "fast" in the sense of Vainshtein and Zeldovich (1972): the exponential growth rate of the magnetic field should remain positive and bounded away from zero as the magnetic Reynolds number tends to infinity. Research in this area involves a mixture of advanced numerical simulations and analytical work which touches on delicate issues in the theory of dynamical systems. A workshop was held in the Observatoire de Nice, where recent advances, controversies and the possibilities for future collaboration were discussed in an informal atmosphere, reflected in this report. Note that to make the report more readable, references are not given for papers presented at the workshop itself.

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One central question, repeatedly discussed at the workshop, is the relation between the rate of growth of the dynamo in the limit of zero diffusivity and the Lyapunov exponents controlling the separation of infinitesimally close fluid particles. Arnol'd *et al.* (1981) were the first to point out that there may be such a relation. Indeed, when the magnetic diffusivity is set equal to zero, magnetic fields behave exactly as infinitesimally close fluid particles. In the presence of diffusion, and for a certain class of "orientable" steady flows, in which the unstable direction does not change too rapidly, it now appears possible to relate the growth rate of the magnetic field to what may be called the "free energy at inverse temperature one" of the local expansion rate of a suitable dynamical system to be defined below. The free energy $F(p)$ at inverse temperature p is defined as the limit for large times, t, of $1/t$ times the logarithm of the average of the pth power of the expansion rate of the dynamical system. The usual Lyapunov exponent is $(dF(p)/dp)|_{p=0}$. For non-vanishing p, $F(p)$ is controlled by large deviations in the fluctuations of the instantaneous Lyapunov exponent. The relevant dynamical system for the dynamo is a stochastic white noise perturbation of the dynamical system of the steady fluid motion, which provides the necessary diffusive effect. The above result presented at the workshop by Collet (Ecole Polytechnique, Palaiseau) will hopefully soon be put on a completely rigorous footing; related work may be found in Bayly (1986) and Vishik (1989). It is likely that the result will also hold for the case of zero diffusivity. The underlying dynamical system then becomes deterministic but fluctuations in the Lyapunov exponents will generally not go away (except in special cases such as the 3-D flow based on the "cat map" considered by Arnol'd *et al.* (1981)). For zero diffusivity, results relating the growth rate of the magnetic field to $F(1)$ were presented by Finn (University of Maryland) (with strong numerical support) and also by Paladin and Vulpiani (Università La Sapienza, Rome), who stressed the relation with multifractality in the magnetic field. For non-orientable steady flows (non-uniformly hyperbolic flows or flows with a mixture of chaotic and integrable domains) the situation is still confused, as it is for flows with stagnation points, such as certain ABC flows (Arnol'd, 1965; Dombre *et al.*, 1986). These contribute possible exceptions to results of Oseledets (1984) and Vishik (1989), ruling out a fast dynamo when the flow has vanishing Lyapunov exponents. Part of the confusion stems from the massive cancellation between fluid elements that are all stretched, but in different directions: can one still have exponential growth of the magnetic field when the direction fluctuates from point to point? Although this more realistic problem does not fit within the standard thermodynamic formalism, it is possible that it may be treated using the zeta function methods discussed by Aurell (Observatoire de Nice).

The workshop helped to clarify how diffusion actually enters into fast dynamos. The fast dynamos of Soward (1987) and Ponomarenko (1973) are diffusive in nature, because the geometry allows the magnetic field to adjust its scale down to diffusive values. However, with the exception of these "diffusive examples", diffusion is not the operating mechanism of the "generic" fast dynamo; the work of Vishik (1989) suggests that exponential stretching is the key element. Soward (University of Newcastle) described work with Childress (New York University) on steady laminar flows which are more complicated than Soward's (1987) fast dynamo, but are still simpler than flows with exponential stretching. One of their examples involves dense Lagrangian trajectories modulo the periodicity of the flow. It displays anisotropic but intense dynamo activity generated by the stretching out of fields in thin "channels" which thread through the flow.

The dominating effect of exponential stretching is perhaps most evident in time-dependent flows. The unsteady stretch-fold-shear (SFS) fast dynamo (Bayly and Childress, 1987) is suggestive of "non-diffusive" fast dynamos in that, while diffusion is active in all parts of space, its real effect is to smooth and cancel small scales, not cause dynamo action. Indeed in this model the growth occurs on large scales and so is best measured by a projection of the magnetic field which excludes small scales. On the other hand, diffusion "cannot be neglected anywhere" à la Moffatt and Proctor (1985), simply because the "generic" fast dynamo produces a sea of small scale activity which dissipates strongly. One can think of waves on water—the production of a tsunami is accompanied by waves on all scales down to events where viscosity and surface tension are important. On the other hand, this generic diffusion is a passive component, which rides on the basic object (dynamo or tsunami) without being able to arrest it.

There has been a considerable amount of study by plasma physicists on area-preserving maps which is likely to be of considerable importance for the next steps in fast dynamo theory. At the same time the study of dynamos should illuminate some new features of transport theory in chaotic flows. Perhaps a steady fast dynamo might be established in a flow with a stochastic web, discussed by Zaslavsky (Institute of Space Research, Moscow) and by Gilbert (Observatoire de Nice), who illustrated how the streamline problem may be reduced to a discrete map. In any case the theory of transport in stochastic webs seems poised for rapid development. The reason is simply that the intractable part of the problem—the chaos in all of its glory—is geometrically under control, so that useful approximations become possible. Any saving in the geometric complexity is crucial since computation must play a key role. An important unanswered question is the role that trapping of field near to KAM tori and the stretching out of field by a Lévy flight, could have on the induction mechanism. One possibility is that flux expulsion occurs, into the chaotic regions. This could lead to a multifractal structure, but one which is entirely chaotic and perhaps subject to rather simple scaling laws. A general conclusion along these lines is that the ABC family of flows may have something to reveal by a close study of the magnetic structures which are formed. In this respect there was a call for more computations, with graphical output as an important ingredient. It is now possible to increase the numerical resolution and magnetic Reynolds number considerably beyond those of Galloway and Frisch (1986), allowing a detailed exploration of the topology of the magnetic field. On the other hand there is the danger of too much focus on one family; the flows discussed by Bajer (University of Cambridge), which show unusual forms of layered chaos, should be mentioned in this regard.

The role of flow geometry on transport phenomena is the essential mathematical problem, and one consensus seems to be that the classical Lyapunov exponent provides too limited an index of the flow. One needs an index of twisting, in effect a phase. This is much harder to implement than the Lyapunov measure, since one needs precise local orientation, not average orientation. As Klapper (New York University) and others suggested, there are probably deep topological reasons for the full cancellation of flux at the level of the largest Lyapunov exponent in R^3. One hopeful point is the fact that once stretching is present the orientation issue seems to reduce to a parity issue. The direction of the field is known locally, but the orientation is not. The issues here are suggestive of WKB analysis, which indeed has been applied by Vishik (1989) and was discussed by Olivé (Moscow State University); one can draw an analogy between stretching and the transport equations, orientation and the phase propagation. Perhaps there is a sort of local complex Lyapunov exponent, as Bayly (University of Arizona) suggested, or one might think in terms of the velocity derivative matrix being systematically summed along a Lagrangian orbit.

Finally there was an undercurrent of thinking about vorticity kinematics and the role of these studies on turbulence theory. Dynamo theorists are dealing with a number of very specific physical models, where the flow fields are studied in detail. Turbulence theory is moving in this direction, propelled by the increasing resolution of numerical experiments on the Navier–Stokes equations; those presented at the workshop by Meneguzzi (CERFACS, Toulouse) provide strong evidence that in high Reynolds number flow with no boundaries the vorticity organises itself into ropelike structures. However, the two problems are very different in detail: in a fast dynamo magnetic energy increases because of stretching by a given large scale fluid motion, while in fluid turbulence an inertial-range eddy breaks up while conserving energy. On the other hand, ideas first introduced in the statistical theories and experimental studies of turbulence, such as intermittency and multifractality, which were discussed by Frisch (Observatoire de Nice), appear now to be relevant for further advances in the deterministic fast dynamo problem.

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STRETCH, TWIST AND FOLD

H. K. Moffat

Dynamo action, that is to say the spontaneous growth of magnetic fields associated with the motion of electrically conducting fluids, has entered a new phase. The basic question of whether fluid motion can sustain a magnetic field against its natural tendency to decay has long been settled in the affirmative. Attention has now switched to the question of the rate of growth of magnetic fields when the basic conditions for dynamo action are satisfied. This aspect of dynamo theory, involving the distinction between "fast" and "slow" dynamos introduced by Vainshtein and Zeldovich (1972), and later expanded in the book "Magnetic Fields in Astrophysics" by Zeldovich *et al.* (1983), has recently led to some surprising connections with the theory of iterated maps, connections that

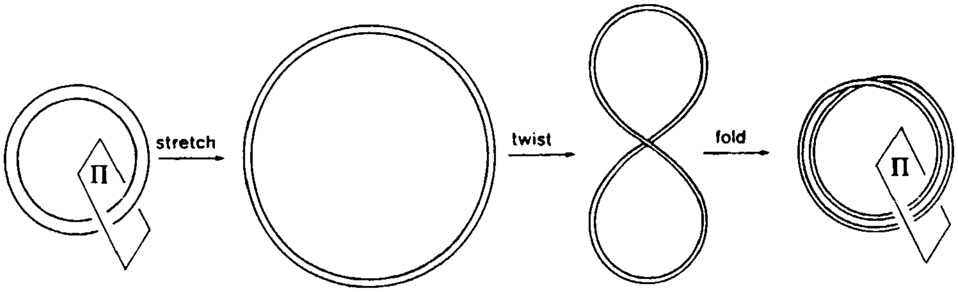

Figure 1 The stretch-twist-fold sequence, showing the plane of cross-section Π on which the associated mapping is constructed.

flow, *twisting* with convection in a rotating medium, and *folding* with geometrical constraints on such convection, the stretch-twist-fold sequence has some basis in the reality of turbulent convection in the convection zone of a rotating star.

The problem, however, is to translate the above physical argument into convincing mathematical form. The stretch-twist-fold sequence can be readily represented as a three-dimensional time-dependent velocity field, but the resulting evolution equation for the magnetic field is of such complexity that it still remains beyond available computing power. However, an alternative procedure is at hand: the same stretch-twist-fold sequence leads to a doubling of the magnetic flux in a flux tube, without change of cross-sectional area. This process can be represented by a mapping of the cross-section onto itself, with an associated mapping of the magnetic field distribution on the cross-section. A link is thereby established between dynamo theory and the theory of iterated maps, about which much is known. It was this aspect of the fast dynamo problem that received intensive study at the Nice workshop. Iterated maps that give rise to dynamo action can be described as "dynamo maps". Several examples of these have been found, notably by Bayly and Childress (1987), and more recently by Finn and Ott (1988). The problem, however, is that, although every time-periodic flow induces a map, and although every continuous map can be associated with a flow, it is not yet known whether these flows have the "orientability" property that leads to reinforcement (rather than cancellation) of the convected magnetic field. There is therefore as yet a fundamental unsolved problem in tackling dynamo theory via the theory of iterated maps.

The alternative approach is the "sledge-hammer" approach of the high-speed computer. The candidate velocity field that appears most likely to succumb to the sledge-hammer is the "ABC" flow which consists of a superposition of three circularly polarised velocity fields of the form $(0, A\sin kx, A\cos kx)$ with cyclic permutation of (x, y, z) and the parameters (A, B, C). (The flow is actually named after Arnol'd, who recognised that the particle paths associated with this flow can exhibit the phenomenon of chaos, Beltrami, who originally conceived of flows, such as this, with vorticity everywhere parallel to velocity, and Childress, who recognised the flow, through its property of maximal helicity, as a prime candidate for dynamo action.) Galloway and Frisch (1986), following Arnol'd and Korkina were explored at a recent workshop on "Small diffusivity dynamos and dynamical systems". Explicit examples of fast dynamos are known, but these involve discontinuities or singularities in the velocity field, and the question of whether there exists any smooth velocity field which is capable of fast dynamo action remains open.

Dynamo theory is important both in planetary physics and in astrophysics because magnetic fields are so ubiquitous in the cosmos: nearest to home, the Earth's magnetic field is sustained by dynamo action in the outer liquid core, and the magnetic fields of Jupiter and Saturn and possibly other planets are believed to be of fluid dynamo origin. In the absence of fluid motion, these fields would decay in a time of the order of 10^4 years, a very short time in relation to the age of the planetary system throughout which, according to geological and other evidence, the magnetic fields have persisted.

Stellar (and solar) magnetism presents a different type of problem. Here, the natural decay time for magnetic fields is nearer to 10^9 years; but we observe systematic field variations on much *shorter* time-scales—e.g. those on the solar surface associated with the 22-year sunspot cycle. Such rapid changes, which involve reversals of polarity of the global solar magnetic field, must be attributable, directly or indirectly, to fluid motion deep down in the turbulent convection zone of the sun; and insofar as the sun is typical of a wide class of stars, the *solar* dynamo process provides a first key to the understanding of *stellar* magnetism. Many astrophysicists believe also that the *galactic* magnetic field is of dynamo origin, although on such scales there is some room for controversy!

The crucial property of fluid motion that is known to be strongly conducive to dynamo action is its "helicity", which, in simple terms, means its net imbalance between right-handed and left-handed helical motions. The simplest, and in some sense purest, dynamo is the helical dynamo of Ponomarenko (1973), for which the velocity field is confined to a cylinder, the streamlines being helical within the cylinder. It has recently, and surprisingly, been shown that the Ponomarenko dynamo is "fast" (Gilbert, 1988), but only when the velocity is discontinuous across the cylinder boundary. The other known example of fast dynamo action is that of Soward (1987) who has shown that certain space-periodic velocity fields have the "fast dynamo" property, provided certain mild singularities of vorticity are introduced at the stagnation points of the field.

These examples are somewhat contrived, and one must ask why it is that the search for further, and more general, examples of fast dynamos has excited so much interest. The prototype fast dynamo in a physical, rather than a mathematical, sense is that first discussed by Vainshtein and Zeldovich (1972), and may be likened to the process of stretching, twisting and folding an elastic band in order to double the tension. Similar action applied to a magnetic flux tube in a perfectly conducting fluid (see Figure 1) will double the magnetic field strength (the Maxwell tension being exactly analogous to the elastic tension). Physically, then, it is clear that the stretch-twist-fold sequence, if repeated many times, will lead to exponential growth of the magnetic field, on a time-scale associated with the motion, and independent of any molecular diffusion process. This example is contrived also; but insofar as *stretching* is invariably associated with turbulent

(1983), have computed the dynamo growth rate as a function of magnetic Reynolds number R_m (a dimensionless measure of electrical conductivity) up to $R_m = 400$, and find two windows of dynamo action, $8 < R_m < 18$, $27 < R_m < ?$. The ? relates to the unknown behaviour when R_m increases beyond the value of 400, the present limit of reasonable accuracy of computation. An open, and exceedingly challenging question that remains here is whether the ABC flow acts as a fast dynamo or slow dynamo, or no dynamo at all, in the asymptotic limit as R_m tends to infinity.

A novel twist to this question was presented by B. J. Bayly of the University of Arizona, following ideas of A. A. Ruzmaikin (Inst. of Terrestrial Magnetism, Ionosphere and Radio Wave Propagation, Moscow). Suppose that the three ingredients of the ABC flow associated with the parameters A, B and C are switched on and off in succession and at random, separated by periods of "stasis" when diffusion operates. This type of model is reminiscent of the stasis dynamo of Backus (1958), who provided one of the first proofs of the possibility of (slow) dynamo action. It turns out that, with Bayly's prescription, fast dynamo action *can* occur, and that the wavelength of the growing magnetic field is exactly *twice* the wavelength of the underlying ABC flow ingredients. This behaviour was undetected in previous computational studies which restricted attention to a magnetic field having the *same* periodicity as the velocity field. The result, if confirmed, should stimulate renewed computational attacks on the ABC dynamo problem, allowing this additional freedom for the growing magnetic field.

Fast dynamo theory is of the greatest importance in astrophysics for the following reason. Most modern theories of solar or stellar magnetism now adopt the "mean field" formalism initiated by Steenbeck *et al.* (1966), whereby the effects of turbulence in the convection zones are parametrised in terms of a generation coefficient α and a turbulent diffusivity β. Usually α and β are expressed through simple dimensional analysis in terms of the velocity and length scales associated with the turbulent convection. The resulting dynamo growth rates are determined by α and β and by such global properties as the differential rotation and meridional circulation within the star, and do not therefore depend on the magnetic diffusivity of the medium. Such dynamos are therefore fast in character. The whole theory of stellar magnetism therefore hangs on the slender thread of dimensional analysis applied in a complex turbulent context. The mathematical difficulties inherent in fast dynamo theory are here concealed in the mean field theory that, in principle, determines the values of α and β. The paradox here is that molecular diffusivity appears to be vital to the production of an α-effect, even although the resulting value of α may be independent of the (vanishingly small) value of this diffusivity.

If the solar dynamo is indeed a fast dynamo, then it is to be expected on general grounds that the field must nearly everywhere exhibit a fine scale of order $R_m^{-1/2}$ times L, the scale of the velocity field (Moffatt and Proctor, 1985). Taking L to be of the order of 1000 kilometers (the granular length scale) and R_m of order 10^4, this implies magnetic variations on scales down to 10 kilometers, a conclusion that is not incompatible with observation. Fast dynamo theory may therefore be of much more than merely academic interest.

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