A Thermally Driven Disc Dynamo

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A new simple idealised model for the geodynamo is proposed. In this model, the energy required for the self-exciting dynamo action is provided by thermal convection in a Welander loop; this is coupled with two Bullard disc dynamos. It is found that two discs with opposite 'twists' are required in order to achieve reversals of the magnetic field, and the model can then be described by a dynamical system of 5th order. This system is found to be identical to the 5th order system of Kennett (1976) resulting from a truncation of the full magnetohydrodynamic equations. The bifurcation structure of the system is studied, and numerical solutions are presented.

1 INTRODUCTION

A simple mechanical model for self-exciting dynamo action was proposed by Bullard (1955). In this model, a conducting disc (driven by a constant external torque G, say) rotates in the presence of a magnetic field at angular velocity $2\pi\Omega(t)$; the current $I(t)$ induced by Faraday's law flows in a wire connecting the rim of the disc (through a sliding contact) and the axle. This system is traditionally described by a second order dynamical system

$$\begin{aligned} L\dot{I} &= -RI + M\Omega I\,, \\ C\dot{\Omega} &= G - k\Omega - MI^2\,, \end{aligned} \tag{1}$$

where L and R are the self-inductance and resistance of the total current circuit respectively, M is the mutual inductance between the wire loop and the rim of the disc, C is the moment of inertia of the disc about the axle, and k is a frictional resistance parameter. When G is positive and large enough, steady dynamo states with non-zero current can be found. Note that

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this model has a preferred parity (or 'handedness') in the sense that dynamo action cannot be found for negative values of G.

Idealised systems with a few degrees of freedom such as the one above are often studied in the hope that they may illuminate the essential physics for the real problem. However, the Bullard disc model is so simplistic that the magnetic field does not reverse; whereas magnetic field reversals are commonly found in astronomical bodies such as the Sun and the Earth. The first idealised model exhibiting reversals was proposed by Rikitake (1958); this consisted of two coupled discs resulting in a third order system. Other third order models (Malkus 1972; Moffatt 1979) derived from a single Bullard disc can also exhibit reversals; they are either identical with, or closely related to, the Lorenz system. Again, all such models have preferred parities so that there is no dynamo action if the direction of applied torque is in the opposite sense relative to the twist in the wire.

There is however an argument suggesting that a good model for geodynamo should have no preferred parity. Helicity (and so the α-effect) results from an interaction between buoyancy and Coriolis effects and may therefore be expected to be proportional to the pseudo-scalar $\mathbf{g}\cdot\boldsymbol{\Omega}$, (where $\mathbf{g}$ is the gravitational acceleration, and $\boldsymbol{\Omega}$ is the angular velocity of the Earth); it is therefore antisymmetric about the equatorial plane (Steenbeck, Krause & Rädler 1966). We therefore build an analogue model (Figure 1) with two coupled Bullard discs mounted on the same horizontal axle but with wires twisted in *opposite* directions in order to provide antisymmetry about the vertical centre-plane; and the discs are thermally driven by a fluid loop heated from below. This model has no global preferred parity, as for the geodynamo.

It turns out that the 5th order system governing this model is identical with that found by Kennett (1976) in a truncation of the full magnetohydrodynamic equations for a convecting layer.

2 THE MODEL

2.1 The mechanical device

The model consists of two ingredients (Figure 1); the first is a coupled Bullard disc system in which two discs are mounted on the same axle and allowed to rotate at the same angular velocity. The wires are twisted in opposite senses so that when the discs rotate in one sense, the right-hand disc acts as a dynamo and the left-hand disc acts as an anti-dynamo; if the discs rotate in the opposite sense, then the right-hand disc becomes an anti-dynamo and the left-hand disc becomes a dynamo. The second model ingredient is a circular fluid loop in a vertical plane, heated from below and cooled from above (Welander 1967); thermal convection must occur when the thermal forcing exceeds some critical value. Suppose now we combine these two ingredients

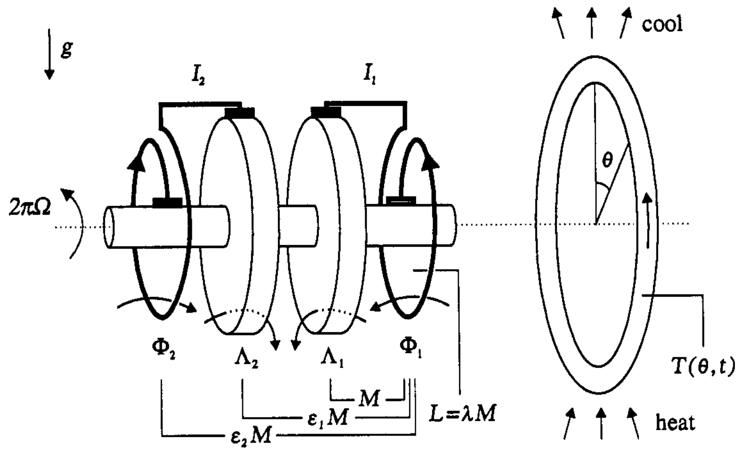

Figure 1. In this model, the coupled discs are driven by a fluid loop heated from below. The fluid loop must be imagined to be dynamically coupled to the rims of the discs, so that both fluid and discs have the same angular velocity $\Omega(t)$.

by identifying their angular velocities, which can be done by placing the fluid loop around the rim of the discs so that the thermal forcing is communicated to the discs; the composite system can now be described by a 5th order nonlinear dynamical system. A similar construction was first proposed by Moffatt (1992).

2.2 The model equations

Let $I_i\,(i=1,2)$ be the currents in the wires, positive when flowing outward from the axle; and Φ_i be the magnetic fluxes across the wire loops. The directions of fluxes are chosen (conventionally) so that the self inductances L_i of the wire loops are positive. Similarly we define Λ_i to be the fluxes across the discs, with the directions chosen so that the mutual inductance M_i between the wire loop and the disc in each circuit is positive (see Figure 1). For simplicity, we shall assume that the two circuits are identical except for the 'twists', hence $M_1 = M_2 \overset{\text{def}}{=} M$ and $L_1 = L_2 \overset{\text{def}}{=} \lambda M$ $(\lambda > 1)$. If we further let $-\varepsilon_1 M$ be the mutual inductance between the wire loop and the disc on the different circuit, and $-\varepsilon_2 M$ be the mutual inductance between the two wire loops, then the fluxes and the currents are related by the following equations:

$$\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = M\begin{pmatrix}\lambda & -\varepsilon_2\\ -\varepsilon_2 & \lambda\end{pmatrix}\begin{pmatrix}I_1\\ I_2\end{pmatrix}, \quad \begin{pmatrix}\Lambda_1\\ \Lambda_2\end{pmatrix} = M\begin{pmatrix}1 & -\varepsilon_1\\ -\varepsilon_1 & 1\end{pmatrix}\begin{pmatrix}I_1\\ I_2\end{pmatrix}. \tag{2}$$

Note that the mutual inductances $-\varepsilon_1 M$ and $-\varepsilon_2 M$ are negative because the wires are twisted in opposite directions, hence ε_1 and ε_2 are positive. The induction equations for the two circuits are (cf. Moffatt 1979)

$$\dot{\Phi}_1 = -RI_1 + \Omega\Lambda_1, \quad \dot{\Phi}_2 = -RI_2 - \Omega\Lambda_2, \tag{3}$$

and the equation of motion is

$$C\dot{\Omega} = -k\Omega + G - I_1\Lambda_1 + I_2\Lambda_2 \,, \tag{4}$$

where R, C, k and G are defined as in (1). Denoting

$$x = \Phi_1 + \Phi_2 \,, \quad y = \Phi_1 - \Phi_2 \,, \quad z = \Omega \,, \tag{5}$$

we eliminate I_i and Λ_i using (2) and obtain

$$\begin{aligned} \dot{x} &= -\frac{R}{M(\lambda - \varepsilon_2)} x + \frac{1 + \varepsilon_1}{\lambda + \varepsilon_2} yz \,, \\ \dot{y} &= -\frac{R}{M(\lambda + \varepsilon_2)} y + \frac{1 - \varepsilon_1}{\lambda - \varepsilon_2} xz \,, \\ C\dot{z} &= G - kz - \frac{xy}{M(\lambda^2 - \varepsilon_2^2)} \,. \end{aligned} \tag{6}$$

Now let us derive equations for the fluid loop. Let $T(\theta, t)$ be the temperature perturbation along the fluid loop; then it satisfies the one dimensional heat diffusion equation

$$\frac{\partial T}{\partial t} + 2\pi\Omega \frac{\partial T}{\partial \theta} = D \frac{\partial^2 T}{\partial \theta^2} + S(\theta) \,, \tag{7}$$

where D is an effective thermal diffusivity in the θ-direction and $S(\theta)$ is the differential heating, which for simplicity we may take to be

$$S(\theta) = -\sigma \sin\theta, \qquad \sigma > 0 \,. \tag{8}$$

The solution of (7) then has the form

$$T(\theta, t) = u(t)\cos\theta + v(t)\sin\theta \,, \tag{9}$$

where

$$\dot{u} = -Du - 2\pi vz \,, \qquad \dot{v} = -Dv - \sigma + 2\pi uz \,. \tag{10}$$

The density perturbation is $-\alpha T$ where α is the coefficient of thermal expansion, and the gravitational torque acting on the fluid is then

$$G = \int_0^{2\pi} \alpha g T a \cos\theta (V/2\pi)\, d\theta = \frac{1}{2}\alpha g a V u(t) \overset{\text{def}}{=} \gamma u(t) \,, \tag{11}$$

where V is the volume of the fluid.

For the composite system, we now identify G in (6c) and (11), and reinterpret C as the total moment of inertia and k as the total frictional resistance coefficient. By rescaling the variables suitably and shifting the origin, we obtain the model equations:

$$\begin{aligned} \dot{x} &= \alpha(-\eta x + \omega y z)\,, \\ \dot{y} &= -\eta y + \omega x z\,, \\ \dot{z} &= \kappa(u - z - xy)\,, \\ \dot{u} &= -u + \xi z - v z\,, \\ \dot{v} &= -v + u z\,, \end{aligned} \tag{12}$$

where

$$\xi = \frac{2\pi\sigma R}{MD^2}\,, \qquad \kappa = \frac{k}{CD}\,, \qquad \eta = \frac{R}{MD}\frac{1}{\lambda + \varepsilon_2} \tag{13}$$

are measures of thermal forcing, frictional resistance and magnetic diffusion respectively, and

$$\alpha = \frac{\lambda + \varepsilon_2}{\lambda - \varepsilon_2}\,, \qquad \omega = \frac{1}{2\pi}\frac{\sqrt{1 - \varepsilon_1^2}}{\lambda + \varepsilon_2} \tag{14}$$

are geometrical parameters. Note that $\alpha > 1$ since $\lambda > \varepsilon_2 > 0$. We recall also the meaning of the dependent variables: x is the flux difference, y is the total flux, z is the angular velocity, and u, v give the temperature in the fluid loop, via (9).

2.3 Basic properties

The system has two symmetries: if (x, y, z, u, v) is a solution, then so are $(-x, -y, z, u, v)$ and $(-x, y, -z, -u, v)$. There are at most 7 steady states which can be divided into 3 classes: (i) the trivial state, pure heat conduction (one solution, no dynamo):

$$x = y = z = u = v = 0\,; \tag{14}$$

(ii) pure thermal convection (2 solutions, no dynamo):

$$x = y = 0\,, \quad u = z = \pm\sqrt{\xi - 1} \overset{\text{def}}{=} \pm z_0\,, \quad v = \xi - 1\,; \tag{15}$$

(iii) dynamo solutions (4 solutions, steady dynamo):

$$\begin{aligned} z = \pm\frac{\eta}{\omega} \overset{\text{def}}{=} \pm z_1\,, \quad u = \frac{\pm\xi z_1}{1 + z_1^2}\,, \quad v = \frac{\xi z_1^2}{1 + z_1^2}\,, \\ x = y = \pm\sqrt{\frac{\eta}{\omega}\left(\frac{\xi}{1 + z_1^2} - 1\right)} \overset{\text{def}}{=} \pm x_1\,. \end{aligned} \tag{16}$$

Consider the case when the thermal forcing parameter ξ is increased slowly from zero. Figure 2 shows a typical scenario (with $\alpha = 3/2$, $\omega = 1$, $\eta = 4$ and $\kappa = 1$) for bifurcations from the trivial state. The first bifurcation occurs

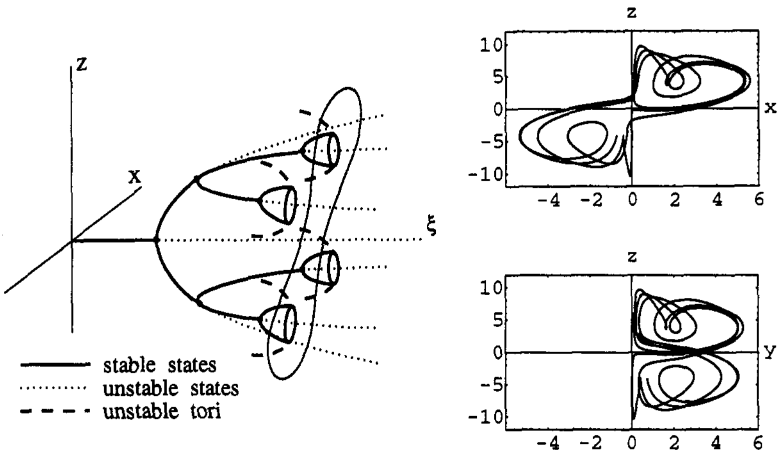

Figure 2. (left) Bifurcations through increasing thermal forcing ξ and (right) phase portraits of the strange attractor.

at $\xi = 1$, when the trivial state becomes unstable; stable thermal convection solutions are created in a pitchfork bifurcation and become preferred states. The second pitchfork bifurcation occurs at $\xi = 1 + (\eta/\omega)^2$, at which point stable dynamo solutions are created. Note that Hopf bifurcations from thermal convection branches are also possible (when $\kappa > 2$), but we shall not consider them in this short account.

Typically, as ξ is increased further, the fixed points corresponding to dynamo solutions become unstable in a supercritical Hopf bifurcation; 4 stable periodic solutions (limit cycles) are created, one from each fixed point. They are non-reversing periodic dynamos because y does not change sign, where $y \propto \Phi_1 - \Phi_2$ is a measure of the total flux of the system. The amplitude of each periodic dynamo solution grows as ξ is increased, up to a point when a subcritical bifurcation to invariant torus takes place, the periodic solution becomes unstable and the system now has a strange attractor wandering about 2 unstable fixed points (steady dynamos). This strange attractor (Lorenz type) is believed to be associated with a homoclinic bifurcation at the origin, and always exhibits reversals in x, but not in y. A closer look at the roles taken by x and y in the model equations (12) reveals that it is necessary to make $\alpha < 1$ to allow reversals in y; this implies $\varepsilon_2 < 0$, which is not possible for the configuration of Figure 1.

2.4 A modified model for magnetic field reversals

The above dynamo model does not permit field reversals. Before we give a modified version that does exhibit reversals, let us first examine why the above model does not. Suppose the discs are rotating in an anti-clockwise direction, and the right-hand circuit, which acts as a dynamo, maintains an

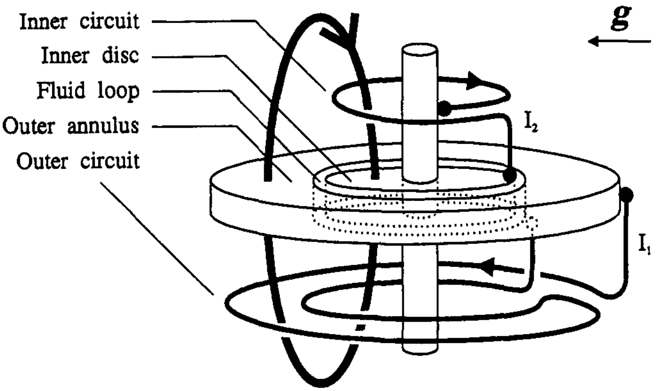

Figure 3. A modified model that exhibits reversals (gravity now acts horizontally).

positive flux through its disc. Some of the flux diffuses away, but some, however tiny it is, must pass through the left-hand disc. Now if the direction of rotation is reversed, the flux through the right-hand disc decays since the right-hand circuit now acts as an anti-dynamo; the flux through the left-hand disc increases but, because it 'remembers' its direction, it can never change sign. In order to achieve reversals, it is necessary therefore to modify the model so as to change the sign of the mutual inductance between two wire loops. A possible solution is illustrated in Figure 3: the fluid loop is now embedded in a conducting disc in such a way that it also acts as an insulator which separates the inner disc and the outer annulus. Wires with sliding contacts are added to complete two separated circuits. It is not difficult to show that the model equations remain unchanged, and the mutual inductance between the two wire loops $-\varepsilon_2 M$ is now positive (i.e. $\varepsilon_2 < 0$). Computations with $\varepsilon_2 < 0$ confirm that y does now show random reversals.

3 CONCLUSIONS

We have shown how a Welander thermal loop may be dynamically coupled with a pair of Bullard disc dynamos with opposite twists to yield a self-exciting dynamo system in which the power supplied is entirely of thermal origin. The bifurcations of the resulting 5th order nonlinear dynamical system have been studied, and the conditions for dynamo action are determined. Flux reversals do not occur if the mutual inductance between the twisted wire loops is negative; but if the geometrical arrangement is such that this mutual inductance is positive (as for example in the configuration of Figure 3), then flux reversals do occur. An analogous constraint on the distribution of helical eddies responsible for the α-effect may be necessary to explain the reversals of the Earth's magnetic field.

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