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Flow of a viscous trickle on a slowly varying incline

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Abstract

Steady flow of a thin trickle of viscous fluid down an inclined surface is considered, via a thin-film approximation Results obtained for uniform flow down the topside or underside of an inclined plane are used in a simple way to approximate flow down a non-plane surface

Keywords Viscous flow, Flow, Thin film approximation

1. Introduction

Many viscous flow problems of practical importance involve free surfaces whose effects contribute significantly to the dynamics One prototype problem that has received much attention is that of the 'draining' of viscous films down inclined surfaces In the simplest cases the film will run down the surface as a uniform 'trickle', if the film is too wide, however, then generally it will become unstable and break into 'fingers', each of which will run down the surface as a trickle We consider the steady behaviour of such a 'trickle' of viscous liquid (which we take to be supplied at a prescribed volume flux) We adopt a very simple approach to the problem, using a thin-film approximation This leads to some simple results for a uniform trickle on an inclined plane, and these results are later used in a rather crude description of flow down curved surfaces

We consider only steady flows, with fixed contact lines, so that the difficulties associated with moving contact lines are avoided (cf Davis [1], and the many references therein) Unsteady flows have been considered (within a thin-film theory) by, for example, Huppert [2], Schwartz [3], Lister [4] and Moriarty et al [5]

2. A uniform rivulet

Consider first the flow of a uniform rivulet down an inclined solid plate (see Fig 1) Aspects of this problem have been considered by Towell and Rothfeld [6] and by Allen and Biggin [7], so some of our results parallel some of theirs, though we obtain results in a form more useful later

Suppose Newtonian fluid, of constant density ρ and viscosity μ, is undergoing steady rectilinear flow in the form of a filament, down a plate inclined at an angle α to the horizontal Referred to Cartesian coordinates $Oxyz$ as indicated in Fig 1, the velocity will be of the form $\mathbf{u}=u(y, z)\mathbf{i}$, and the Navier–Stokes equations reduce to

$$0=-p_x+\rho g \sin\alpha+\mu(u_{yy}+u_{zz}),$$

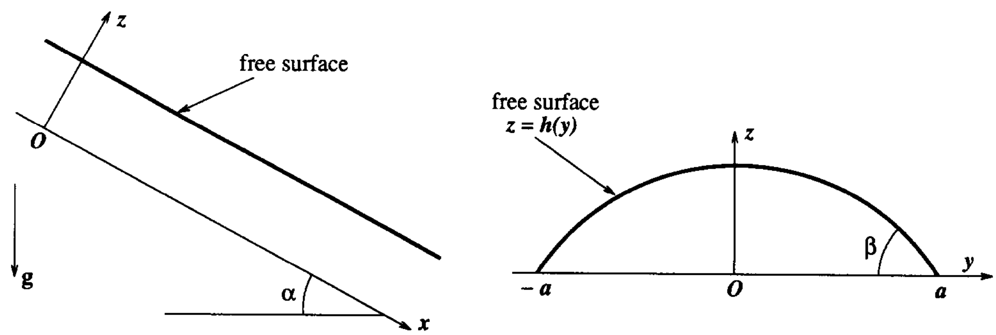

Fig 1 A trickle of viscous liquid, of width $2a$ and maximum depth h_m, flowing down a flat plate inclined at an angle α to the horizontal

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$$0=-p_y,\quad 0=-p_z-\rho g\cos\alpha \tag{1}$$

(subscripts x, y and z denoting partial differentiation) In a 'thin-film' theory, these equations approximate to

$$0=-p_x+\rho g\sin\alpha+\mu u_{zz},$$

$$0=-p_y,\quad 0=-p_z-\rho g\cos\alpha \tag{2}$$

to be integrated subject to the boundary conditions

$$u=0 \text{ on } z=0$$

$$p-p_a=-\gamma h'' \text{ and } u_z=0 \text{ on } z=h \tag{3}$$

$$h=0 \text{ and } h'=\mp\tan\beta \text{ at } y=\pm a$$

Here $z=h(y)$ is the free-surface profile, a prime denotes d/dy, p is the pressure in the liquid, p_a is atmospheric pressure, a is the semi-width of the trickle, g is the gravitational acceleration, γ is the coefficient of surface tension, and β is the contact angle at the three-phase contact line We have neglected the viscosity of the air above the liquid, and we have taken the surface curvature to be h'' Also we have taken γ and β to be prescribed constants, nominally with β small A constant γ means that any surface-tension-driven effects associated with surfactants or differential heating are ignored A constant β means that any contact-angle hysteresis is ignored, this is probably reasonable for these rivulets

For brevity in writing the solution we indicate the three cases $0<\alpha<\pi/2$, $\alpha=\pi/2$ and $\pi/2<\alpha<\pi$ by (i), (ii) and (iii), respectively Then the solution is as follows The velocity is

$$u(y,z)=\frac{\rho g\sin\alpha}{2\mu}(2hz-z^2) \tag{4}$$

the free-surface velocity u_s ($=u(y,h)$) is

$$u_s(y)=\frac{\rho g\sin\alpha}{2\mu}h^2 \tag{5}$$

and the pressure is

$$p(z)=p_a-\rho gz\cos\alpha+\tan\beta\sqrt{(\gamma\rho g|\cos\alpha|)}$$

$$\times\begin{cases}\coth B & \text{(i)}\\ B^{-1} & \text{(ii)}\\ \cot B & \text{(iii)}\end{cases} \tag{6}$$

where

$$B=a(\rho g|\cos\alpha|/\gamma)^{1/2}\quad(\text{for } \alpha\neq\pi/2) \tag{7}$$

B (>0) is a Bond number for the flow For $\alpha=\pi/2$ we have $B=0$, nonetheless case (ii) may be included formally in Eq (6), the interpretation being that factors of $\cos\alpha$ are cancelled before the limit $\alpha\to\pi/2$ is taken

The free-surface profile $z=h(y)$ is given by

$$\left(\frac{\rho g|\cos\alpha|}{\gamma}\right)^{1/2}\frac{h}{\tan\beta}=\begin{cases}\dfrac{\cosh B-\cosh B\xi}{\sinh B} & \text{(i)}\\ \frac{1}{2}B(1-\xi^2) & \text{(ii)}\\ \dfrac{\cos B\xi-\cos B}{\sin B} & \text{(iii)}\end{cases} \tag{8}$$

where

$$\xi=y/a\quad(-1\leqslant\xi\leqslant1) \tag{9}$$

and the maximum depth h_m of the liquid (given by $h_m=h(0)$) satisfies

$$\left(\frac{\rho g|\cos\alpha|}{\gamma}\right)^{1/2}\frac{h_m}{\tan\beta}=\begin{cases}\tanh\frac{1}{2}B & \text{(i)}\\ \frac{1}{2}B & \text{(ii)}\\ \tan\frac{1}{2}B & \text{(iii)}\end{cases} \tag{10}$$

with interpretations for case (ii) as above Note that the scales of h_m and a in Eqs (10) and (7) differ essentially by the small factor $\tan\beta$ (and indeed in case (ii) $h_m/a=\frac{1}{2}\tan\beta$), this reflects the fact that the depth of the film is much less than its width

The solution is physically sensible only where $h(y)\geqslant0$ In case (iii), $h(y)$ in Eq (8) is positive for all y satisfying $-a\leqslant y\leqslant a$ only if B is restricted by $0\leqslant B<\pi$, the physical meaning of this upper limit is discussed below

The volume flux of liquid running down the plate is

$$Q=\int_{-a}^{a}\int_{0}^{h(y)}u\,dz\,dy \tag{11}$$

and with the nondimensionalization

$$\bar{Q}=\frac{9\mu\rho g\cos^2\alpha}{\gamma^2\tan^3\beta\sin\alpha}Q\quad(\text{for } \alpha\neq\pi/2) \tag{12}$$

we have

$$\bar{Q}=F(B) \tag{13}$$

where

$$F(B)=\begin{cases}15B\coth^3B-15\coth^2B-9B\coth B+4 & \text{(i)}\\ 12B^4/35 & \text{(ii)}\\ -15B\cot^3B+15\cot^2B-9B\cot B+4 & \text{(iii)}\end{cases} \tag{14}$$

with case (ii) interpreted as above The mean flow speed $\bar{u}$ ($=Q/\int_{-a}^{a}h(y)\,dy$) satisfies

$$\frac{18\mu}{\gamma\tan^2\beta|\tan\alpha|}\bar{u}=F(B)\times\begin{cases}(B\coth B-1)^{-1} & \text{(i)}\\ 3/B^2 & \text{(ii)}\\ (1-B\cot B)^{-1} & \text{(iii)}\end{cases} \tag{15}$$

The function $\bar{Q}=F(B)$ is plotted in Fig 2 for the three cases, for the physically relevant domain $B>0$, $\bar{Q}>0$, and $B<\pi$ in case (iii) In each case, if $\bar{Q}$ is prescribed then B is determined uniquely, and hence a, $h(y)$, p and $u(y,z)$ may

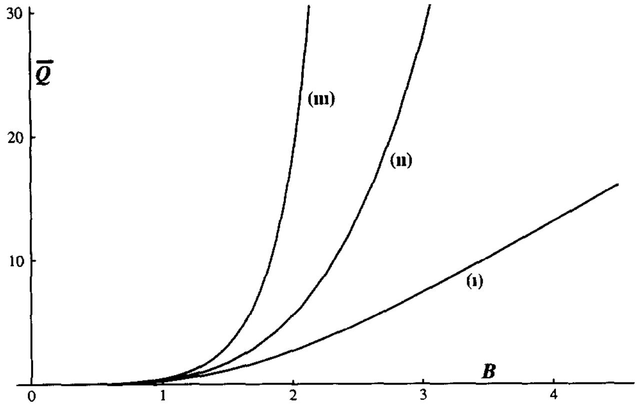

Fig 2 The nondimensional flux $\bar{Q}$ as a function of Bond number B in the three cases (i), (ii), (iii)

be found Thus the non-dimensional parameter $\bar{Q}$ essentially determines the cross-sectional shape of the trickle, the parameters $(\gamma/\rho g|\cos\alpha|)^{1/2}$ and γ/μ (together with α and β) then determine the scales of the width, depth and speed [The relationship between Q and a, involving the parameters ρ, g, γ, μ, α and β, may be reduced by dimensional analysis to a relationship between five dimensionless parameters, say $Q\rho^2 s\gamma/\mu^3$, $a^2\rho g/\gamma$, $\rho\gamma^3/g\mu^4$, α and β, in the present approximation this reduces dramatically to the relation (13) between just two parameters, $\bar{Q}$ and B]

In cases (i) and (iii) we have $\bar{Q}(B)\sim 12B^4/35$ and $h_m \sim \frac{1}{2}a\tan\beta$ as $B\to 0$ Also in case (i) we have $\bar{Q}(B)\sim 6B-11$ and $h_m\sim(\gamma/\rho g\cos\alpha)^{1/2}\tan\beta$ as $B\to\infty$ (and in practice these asymptotic forms are accurate for $B\gtrsim 4$) Result (27) of Towell and Rothfeld [6] (for a 'wide flat rivulet') corresponds here to case (i) with $B\to\infty$

For case (ii) (i e for a vertical wall) the above results are, in a more explicit form,

$$p(z)=p_a+\frac{\gamma}{a}\tan\beta,\quad h=\frac{\tan\beta}{2a}(a^2-y^2),$$

$$h_m=\frac{a}{2}\tan\beta,\quad a=\left(\frac{105\mu Q}{4\rho g\tan^3\beta}\right)^{1/4},$$

$$\bar{u}=\frac{2a^2\rho g\tan^2\beta}{35\mu} \tag{16}$$

In an exact analysis the free surface $h(y)$ in this case would be an arc of a circle, the approximation to this in Eq (16) is correct to $O(\beta^2)$ The expression for a in Eq (16) (which is independent of γ) agrees with results (24) and (25) of [6] (but note that Eq (25) in [6] should be $f(\theta)\sim 16\theta^3/35$)

In case (iii) we have $a\sim\pi(\gamma/\rho g|\cos\alpha|)^{1/2}$, $\bar{Q}(B)\sim 15\pi/(\pi-B)^3$ and $h_m/a\tan\beta\sim 2/[\pi(\pi-B)]$ as $B\to\pi$; this breakdown in the solution as $B\to\pi$ corresponds to the non-existence of a solution to the problem of a static two-dimensional drop hanging from a horizontal support when the mass of the drop is so large that the surface tension cannot hold it up (Actually our 'thin-film' approximation would be invalid well before this breakdown could occur, since $h_m\to\infty$ as $B\to\pi$; nevertheless even in a fully nonlinear analysis one would expect a solution to exist only for restricted B Stability considerations would further restrict the allowed values of B) In addition, since the minimum pressure in the liquid (occurring at the plate $z=0$ in case (iii)) must exceed the vapour pressure p_v ($<p_a$) of the liquid, B must also satisfy $0<B<B_0<\pi$, where $\cot B_0 = -(p_a-p_v)/[(\gamma\rho g|\cos\alpha|)^{1/2}\tan\beta]$ (so that $B_0>\pi/2$)

We have taken β to be constant (i e we have ignored any contact-angle hysteresis) In reality β may vary somewhat along a contact line, however at an anchored (i e fixed) three-phase line it is expected that $\beta_r\leqslant\beta\leqslant\beta_a$, where β_r and β_a are constants (the receding and advancing contact angles) If β does vary along the contact line of the rivulet then the flow will not be truly unidirectional, however the above theory may still be approximately correct if β varies only slowly Then the cross section of the filament will vary slowly with distance down the plane an increase in β (with Q fixed) will be associated with a decrease in $\bar{Q}$ (since $\bar{Q}\sim(\tan\beta)^{-3}$), with an increase in h_m (since $h_m\sim(\tan\beta)^{1/4}$), with a decrease in B (since $B\sim(\tan\beta)^{-3/4}$), and therefore with a decrease in a

3. Flow of a trickle down a surface of slowly varying slope

Nusselt, in his classic papers [8], gave, amongst many other things, the 'thin-film' description of the steady flow of viscous liquid round a circular cylinder of large diameter, with its axis horizontal He made a 'quasi-static' assumption that, at each station round the cylinder, the liquid depth and velocity have forms appropriate to flow down a *flat* plate inclined at the *local* value of the cylinder's slope He treated films of effectively infinite lateral extent (in the direction of the cylinder's generators), so that there are no contact lines, also he neglected surface tension Nusselt was primarily concerned with the thickening of a film (of water) due to condensation (of steam), if such condensation is ignored, his solution for the liquid depth h, the velocity component u down the line of greatest slope and the surface velocity $u_s(=u|_{z=h})$ may be written

$$h=\left(\frac{3\mu Q_2}{\rho g\sin\alpha}\right)^{1/3},\quad u=\frac{\rho g\sin\alpha}{2\mu}(2hz-z^2),$$

$$u_s=\frac{1}{2}\left(\frac{9Q_2{}^2\rho g\sin\alpha}{\mu}\right)^{1/3} \tag{17}$$

Here Q_2 is the prescribed volume flux (per unit width) round the cylinder, z is a local normal coordinate, and α is the local angle of slope of the cylinder's surface (with $0<\alpha<\pi$), the quantities u, h and u_s thus depend on distance s measured down the surface, this dependence arising via the slow variation of α with s. (In fact, Nusselt's results also provide the solution for flow down surfaces of more general shape, provided that the surface slope varies sufficiently slowly, and that $0<\alpha<\pi$.)

In a somewhat similar way we consider the flow of a trickle of viscous liquid down a solid cylindrical surface with horizontal generators and with slowly varying slope; more precisely, we take the width of the trickle to be much less than the radius of curvature of the solid surface. Following Nusselt we make a 'quasi-static' assumption that at each station the trickle attains an equilibrium shape and velocity appropriate to flow down a plate inclined at the local slope α (with $0<\alpha<\pi$). Thus at each station the solution is simply the uniform rivulet presented in Section 2. Since α is now a variable (depending on distance s measured down the solid surface), it is more suitable to nondimensionalize as

$$\bar{h}=\left(\frac{\rho g}{\gamma}\right)^{1/2}\cot\beta\, h,\quad \bar{y}=\left(\frac{\rho g}{\gamma}\right)^{1/2} y,$$

$$b=\left(\frac{\rho g}{\gamma}\right)^{1/2} a,\quad q=\frac{9\mu\rho g Q}{\gamma^2\tan^3\beta} \tag{18}$$

the new dimensionless variables b and q being related to the earlier ones B and $\bar{Q}$ by $B=b\sqrt{|\cos\alpha|}$ and $\bar{Q}=q\cos^2\alpha/\sin\alpha$. Then the relationship $\bar{Q}=F(B)$ in Section 2 becomes

$$\frac{q\cos^2\alpha}{\sin\alpha}=F(b\sqrt{|\cos\alpha|}) \tag{19}$$

Since $F(\ )$ is strictly monotonic in its argument we may write formally

$$b(\alpha)=\frac{1}{\sqrt{|\cos\alpha|}}F^{-1}\left(\frac{q\cos^2\alpha}{\sin\alpha}\right),\quad \text{for } \alpha\neq\frac{\pi}{2} \tag{20}$$

with

$$b=(35q/12)^{1/4}(=b_1,\ \text{say}),\quad \text{for } \alpha=\pi/2 \tag{21}$$

(and again Eq. (21) is an instance of Eq. (20), with $F(B)=12B^4/35$ and with α set to $\pi/2$ *after* $\cos\alpha$ terms have been cancelled). Thus with the nondimensional flux q prescribed, the semi-width, b, of the trickle is determined by Eq. (20) as a function of the slope α, and hence the complete solution (valid for $0<\alpha<\pi$) is determined precisely as in Section 2, with z and y interpreted as local normal and transverse coordinates, respectively, and with α now variable.

Eq. (19) may be written

$$\lambda^2+\frac{F(B)}{q}\lambda-1=0,\quad \lambda=\sin\alpha \tag{22}$$

This leads to an equivalent parametric representation of the function $b(\alpha)$, which is somewhat easier to use in practice, namely

$$\alpha=\begin{cases}\sin^{-1}\lambda & \text{(i)}\\ \pi/2 & \text{(ii)}\\ \pi-\sin^{-1}\lambda & \text{(iii)}\end{cases} \tag{23}$$

$$b=B/(1-\lambda^2)^{1/4} \tag{24}$$

where

$$\lambda=\{[(F(B))^2+4q^2]^{1/2}-F(B)\}/2q \tag{25}$$

with the parameter B satisfying

$$\begin{cases}0<B<\infty & \text{(i)}\\ B\to 0 & \text{(ii)}\\ 0<B<\pi & \text{(iii)}\end{cases} \tag{26}$$

Then from Eq. (8) the cross-sectional profile of the trickle at each station α is given by

$$\bar{h}(\bar{y},\alpha)=\begin{cases}\dfrac{\cosh B-\cosh(\bar{y}\sqrt{\cos\alpha})}{\sqrt{\cos\alpha}\,\sinh B} & \text{(i)}\\[2ex] \dfrac{1}{2b_1}(b_1{}^2-\bar{y}^2) & \text{(ii)}\\[2ex] \dfrac{\cos(\bar{y}\sqrt{|\cos\alpha|})-\cos B}{\sqrt{|\cos\alpha|}\,\sin B} & \text{(iii)}\end{cases} \tag{27}$$

for

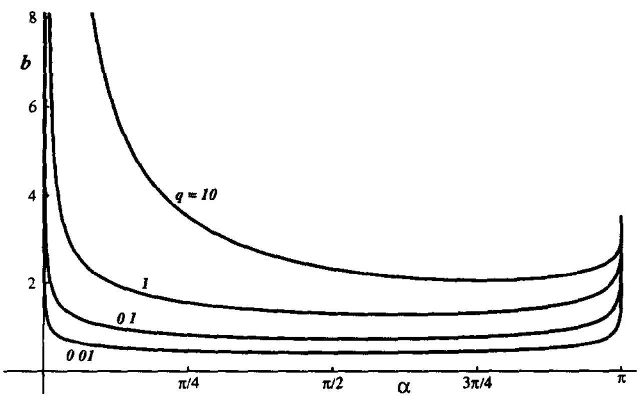

Fig. 3. The nondimensional semi-width $b(\alpha)$ of trickles running around the outside of a circular cylinder, for various values of the nondimensional flux q. Here $\alpha=0$ corresponds to the top of the cylinder and $\alpha=\pi$ corresponds to the bottom.

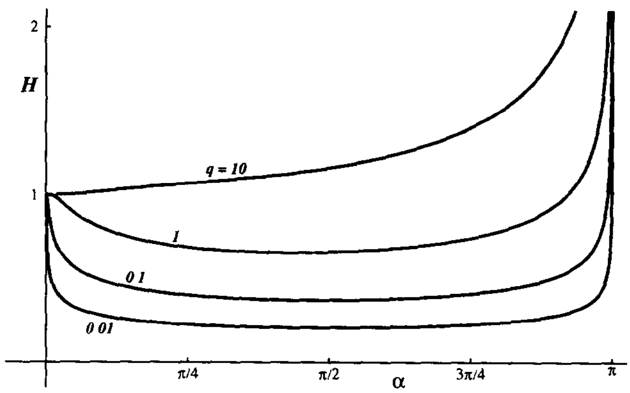

Fig. 4. The nondimensional maximum depth $H(\alpha)$ of trickles, as in Fig. 3.

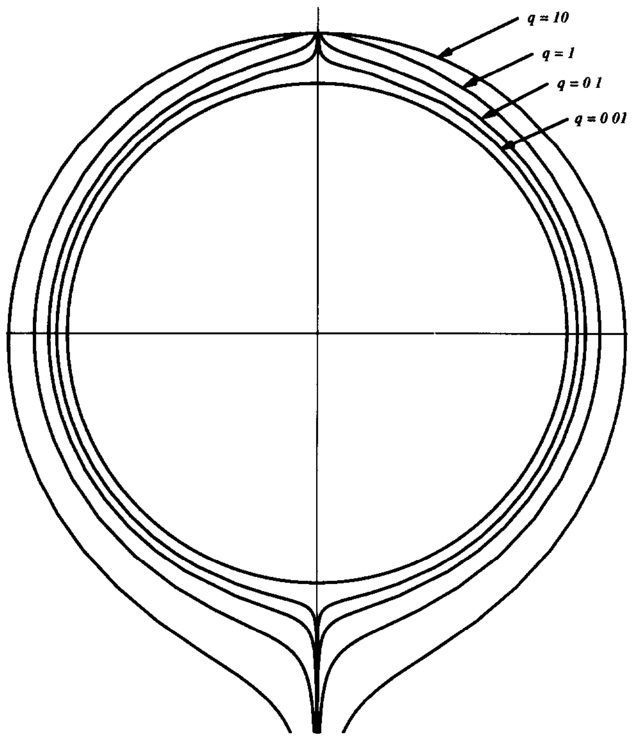

Fig. 5. The curves in Fig. 4 drawn on a circular cylinder, indicating the shape of film predicted.

$$-b(\alpha) \leqslant \tilde{y} \leqslant b(\alpha) \tag{28}$$

Also the nondimensional maximum depth $H(\alpha)$, defined by $H=\tilde{h}(0,\alpha)$, is given by

$$H=\begin{cases}\tanh(\frac{1}{2}B)/(1-\lambda^2)^{1/4} & \text{(i)}\\ \frac{1}{2}b_1 & \text{(ii)}\\ \tan(\frac{1}{2}B)/(1-\lambda^2)^{1/4} & \text{(iii)}\end{cases} \tag{29}$$

with $b_1=(35q/12)^{1/4}$.

Appropriate asymptotic expansions show that

$$b\sim \frac{q}{6\alpha}+\frac{11}{6},\quad H\sim 1+\frac{\alpha^2}{4} \quad \text{as } \alpha\to 0 \tag{30}$$

$$b\sim b_1+\frac{{b_1}^3}{18}\left(\frac{\pi}{2}-\alpha\right),\quad H\sim \frac{b_1}{2}-\frac{{b_1}^3}{72}\left(\frac{\pi}{2}-\alpha\right) \quad \text{as } \alpha\to\frac{\pi}{2} \tag{31}$$

and

$$b\sim \pi-\left(\frac{15\pi(\pi-\alpha)}{q}\right)^{1/3},\quad H\sim\left(\frac{8q}{15\pi(\pi-\alpha)}\right)^{1/3} \quad \text{as } \alpha\to\pi \tag{32}$$

Thus this solution predicts that $b\to\infty$ and $H\to 1$ as $\alpha\to 0$, and that $b\to\pi$ and $H\to\infty$ as $\alpha\to\pi$, for any value of q. Also one can show that $db/d\alpha<0$ for $0<\alpha\leqslant\pi/2$ and that $db/d\alpha>\frac{1}{2}b\tan\alpha$ for $\pi/2<\alpha<\pi$.

The lateral extent of the trickle is specified by the contact lines $y=\pm b(\alpha)$, which vary slowly with s. Figs. 3 and 4 show respectively, for various values of q, the predicted shapes of the bounding curve $b(\alpha)$ and of the maximum depth $H(\alpha)$ for the case of flow round the outer surface of a circular cylinder with horizontal axis (so that $s=\alpha R$, where R is the radius of the cylinder). For q less than about 0.1 the cross-sectional profile is rather uniform around the cylinder, except near $\alpha=0$ and $\alpha=\pi$. For larger q there is more variation. Fig. 5 shows a cross-sectional view of the cylinder and films (the film thickness being exaggerated for clarity), and Fig. 6 shows (for the case $q=1$) examples of the film profiles at various stations α around the cylinder. Note that, unlike Nusselt's solution (17), this solution does *not* have top-to-bottom symmetry, i.e. a profile at a station α on the 'topside' is different from the profile at the corresponding station $(\pi-\alpha)$ on the 'underside'. (This may be seen from the fact that the forms of F for $\alpha<\pi/2$ and $\alpha>\pi/2$ in Eq. (14) are different.)

This approximate solution breaks down near the top $(\alpha=0)$ and the bottom $(\alpha=\pi)$ of the cylinder, as does Nusselt's solution for the 'multi-tube' case. Nusselt interpreted the infinite values of h in his solution as representing fluid falling onto or falling off the cylinder, at $\alpha=0$ and $\alpha=\pi$, respectively; in our solution, an infinite h occurs at $\alpha=\pi$

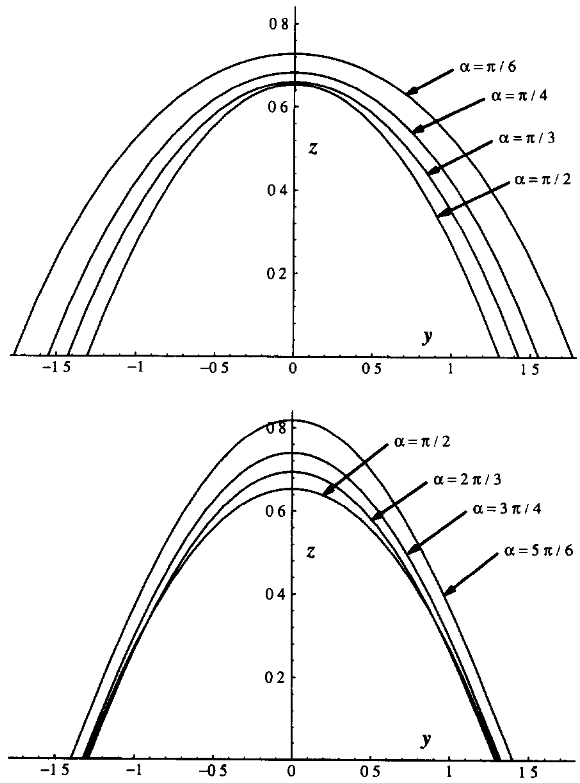

Fig. 6. Examples of film profiles on a circular cylinder, for the case $q=1$ in Fig. 3: plot of $(\rho g/\gamma)^{1/2}h\cot\beta$ as a function of $(\rho g/\gamma)^{1/2}y$ from Eq. (27), at various stations α round the cylinder. For this value of q, the maximum film depth $H(\alpha)$ essentially decreases with α until $\alpha\approx 80°$, and then increases; also the maximum width $2b(\alpha)$ decreases with α until $\alpha\approx 110°$, and then increases.

only, but an infinite b occurs at $\alpha=0$ [Incidentally, for flow down the *inside* surface of the same cylinder, Figs 3 and 4 merely need to be viewed right-to-left, that is, $\alpha=\pi$ is at the top of the cylinder and $\alpha=0$ is at the bottom The singularities in h and in b then occur at the topmost and bottom-most points, respectively]

The asymptotic forms (30)–(32) show that $db/d\alpha=-b_1^3/18$ at $\alpha=\pi/2$ and that $db/d\alpha\rightarrow+\infty$ as $\alpha\rightarrow\pi$, so there must be a point where $db/d\alpha=0$, that is, the trickle always attains its minimum width in $\alpha>\pi/2$ On the other hand H increases from its value 1 near $\alpha=0$, and satisfies $H=b_1/2$ and $dH/d\alpha=b_1^3/72$ at $\alpha=\pi/2$ (and approaches ∞ as $\alpha\rightarrow\pi$) Thus, depending on the value of b_1 (i e of q), the liquid depth in $0<\alpha<\pi/2$ may either increase monotonically with α, or may initially increase, then decrease, then increase again

Some aspects of these flows accord with common experience Simple 'kitchen' experiments were performed, with trickles of syrup (of width ~5–15 mm and depth ~2–8 mm) running round the curved side of a large cooking pot (of diameter ~30 cm), lying on its side, the experiments showed rough agreement with the curves in Figs 3–5 Better experiments would be needed to test whether the agreement is more than superficial

Acknowledgements

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