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B. R. DUFFY[1] and H. K. MOFFATT[2]

[1] *Department of Mathematics, University of Strathclyde, Livingstone Tower, 26 Richmond Street, Glasgow G1 1XH, UK*
[2] *Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, UK*

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A thin-film approximation is used in an analysis of the flow of a thin trickle of viscous fluid down a near-vertical plane. An approximate similarity solution is obtained, representing essentially a source (or sink) flow. Several interpretations of the solution are discussed.

1 Introduction

Problems concerning the 'draining' of viscous films down inclined surfaces have received much attention in the literature. In this paper we use a thin-film approximation to study the steady spreading or contraction of viscous liquid supplied (at a prescribed rate) on a near-vertical plane; gravity may be considered the 'main' driving force, but surface tension cannot be neglected. The assumption that the flow changes only slowly down the plane then leads to an approximate similarity solution for this three-dimensional viscous free-surface flow. Several interpretations of this solution are discussed.

Only steady flows are considered, so that, in particular, any contact lines are fixed; difficulties associated with moving contact lines are thereby avoided. (Cf. Davis, 1983, and the many references therein.) Unsteady flows have been considered (within a thin-film theory) by, for example, Huppert (1982), Schwartz (1989), Lister (1992) and Moriarty *et al.* (1991).

2 The governing equation, and a similarity solution

If liquid is supplied (at a prescribed volume flux Q) from a point source on an inclined plane, then there will be an adjustment region where the trickle widens (from 'zero' width) until it attains a constant cross section, of non-zero width $2a$ (say). There will be a similar adjustment if the source is of small but non-zero width. If the liquid is supplied from a very wide source, then it is commonly observed that an 'instability' occurs, and the stream splits into two or more narrower trickles (fingers), each of which is eventually a uniform rivulet. Again, there will be adjustment regions where the width of each trickle settles to an appropriate value of a (although determination of these values of a is by no means straightforward; see, for example, Mikielewicz & Moszynski, 1978; Huppert, 1982; Schwartz, 1989).

Smith (1973) has considered the spreading of a trickle from a point source on an inclined plane in the case when surface tension is negligible. However, when the plane is vertical the

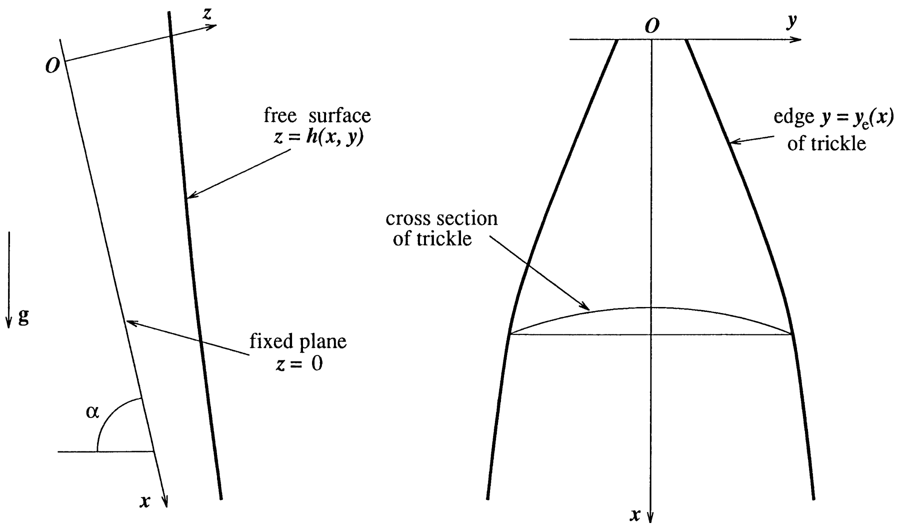

FIGURE 1. A trickle of fluid emitted on a near-vertical plate.

effects of surface tension cannot be ignored; we consider this case here, again taking the film to be thin and the flow to be slow.

Suppose a thin film of Newtonian liquid, of constant density ρ and viscosity μ, is flowing down a plane inclined at an angle α to the horizontal, α being defined such that the liquid is on the 'topside' of the plane when $\alpha < \frac{1}{2}\pi$ and is on the 'underside' of the plane when $\alpha > \frac{1}{2}\pi$ (see Figure 1). We shall take the plane to be vertical or nearly vertical, so that $|\alpha - \frac{1}{2}\pi|$ is zero or is small.

We use Cartesian coordinates $Oxyz$ as in Figure 1, with Ox down a line of greatest slope and with Oz normal to the plane. A standard thin-film analysis shows that the (generally unsteady) free-surface profile $z = h(x, y, t)$ satisfies

$$3\mu h_t = \nabla \cdot [h^3 \nabla(\rho g h \cos\alpha - \gamma \nabla^2 h)] - 3\rho g h^2 h_x \sin\alpha, \qquad (2.1)$$

where γ denotes the coefficient of surface tension (assumed constant), g denotes the acceleration due to gravity, and suffixes denote partial differentiation.

The case of a rivulet of uniform width is obtained by putting $h = h(y)$ and setting to zero the term in square brackets in (2.1):

$$\rho g h \cos\alpha - \gamma h_{yy} = \text{constant}.$$

(cf. Towell & Rothfeld, 1966; Allen & Biggin, 1974). This is to be integrated subject to the boundary conditions

$$h = 0 \quad \text{and} \quad h' = \mp \tan\beta_0 \quad \text{at} \quad y = \pm a,$$

where a is the semi-width of the trickle, and β_0 is the contact angle at the air–liquid–solid contact line, taken to be constant. For small $|\alpha - \frac{1}{2}\pi|$ the solution is

$$h(y) = \frac{\tan\beta_0}{2a}(a^2 - y^2).$$

The velocity component down the plane is $u = (\rho g \sin\alpha/2\mu)(2hz - z^2)$, and the volume flux of liquid running down the plane is

$$Q = \int_{-a}^{a}\int_{0}^{h(y)} u\,dz\,dy, = \frac{\rho g \sin\alpha}{3\mu}\int_{-a}^{a} h^3\,dy, = \frac{4\rho g a^4 \sin\alpha \tan^3\beta_0}{105\mu},$$

independent of γ (in this approximation).

In general, liquid issuing from a source (whether a point source or a distributed source) will not have this profile, and there will be some sort of adjustment zone before a constant profile can be attained. Smith (1973) has studied this adjustment for the case when α is not near $\frac{1}{2}\pi$, with surface tension neglected (at leading order). We, on the other hand, are interested in a near-vertical surface, so that $\alpha \approx \frac{1}{2}\pi$, and surface tension cannot be neglected. In the spirit of Smith's analysis, we seek a steady solution for which the length scale down the plane is much greater than that across the stream, so that $|h_y| \gg |h_x|$ in general. Then equation (2.1) becomes approximately

$$hh_{yyyy} + 3h_y h_{yyy} + 3(\rho g \sin\alpha/\gamma)\, h_x = 0. \tag{2.2}$$

The self-consistency of this approximation is checked *a posteriori* (see equations (2.12) and (2.13)) by verifying that $1 \gg |h_y| \gg |h_x|$ for the solution obtained; a more formal justification of the approximation is given in an appendix.

Here we shall consider solutions of (2.2) that are symmetrical about $y = 0$, or, to be more precise, solutions for which h is even in y. This means that

$$h_y(x, 0) = 0, \quad h_{yyy}(x, 0) = 0. \tag{2.3}$$

We take the free-surface to meet the plane $z = 0$ in the curves $y = \pm y_e(x)$, which are therefore three-phase contact lines (the 'edges' of the trickle). Then

$$h(x, \pm y_e) = 0. \tag{2.4}$$

Also, the volume flux of liquid down the plane is approximately

$$Q = \frac{\rho g \sin\alpha}{3\mu}\int_{-y_e(x)}^{y_e(x)} h^3\,dy, \tag{2.5}$$

and herein we take Q to be a prescribed constant.

It is expected that the contact angle β at the (anchored) three-phase lines $y = \pm y_e(x)$ will satisfy a restriction of the form $\beta_r \leqslant \beta \leqslant \beta_a$, where β_r and β_a are constants (the receding and advancing contact angles). However, there is no reason to expect β itself to be constant on these contact lines (and indeed, simple observations of film-drainage down an inclined plane suggest strongly that β will vary quite considerably, at least where the film width varies in a downstream direction). Presumably conditions near the lateral contact lines are dominated by viscous stresses, in which case it is legitimate to assume that the contact angle adjusts itself locally to the value dictated by these stresses. There may, of course, be an 'inner region' (perhaps on the scale of intermolecular forces) within which an adjustment to the appropriate contact angle takes place; however, the behaviour in such an inner region lies outside the scope of the present analysis. We therefore leave open the question of the contact-angle, and merely report a family of free-surface profiles that are consistent with (2.2)–(2.5), and for which, as we shall see, β is variable.

We seek a similarity solution to (2.2) of the form

$$h(x,y) = f(x)\,G(\eta), \quad \eta = y/y_e(x),$$

so that

$$G(\pm 1) = 0.$$

Substitution of this form of h into (2.2) shows that appropriate f and y_e are

$$f(x) = A(cx)^m, \quad y_e(x) = (cx)^n, \quad m - 4n + 1 = 0,$$

where c, A, m and n are constants (a shift of origin having been made, to eliminate any additive constant in x). The constancy of Q further implies that $3m + n = 0$, so that $m = -1/13$ and $n = 3/13$. The constant $A(\neq 0)$ may be chosen freely, and if we make the choice $A = c\rho g \sin\alpha / 104\gamma$ (so that c and A have physical dimensions $(\text{length})^{10/3}$ and $(\text{length})^{4/3}$, respectively) then the equation for $G(\eta)$ is

$$GG'''' + 3G'G''' - 24(G + 3\eta G') = 0,$$

i.e.

$$G^3(G''' - 24\eta) = \text{constant}.$$

For the symmetrical solutions that we are considering here, we have $G''' = 0$ when $\eta = 0$, so that the constant of integration is 0. Thus $G''' = 24\eta$, and, trivially, the solution satisfying $G(\pm 1) = 0$ is

$$G(\eta) = (\eta^4 - 1) - \lambda(\eta^2 - 1), \tag{2.6}$$

where λ is some constant. The flux condition gives a relation between c and λ:

$$Q = \frac{(\rho g \sin\alpha)^4 c^3}{2^9 \times 3 \times 13^3 \times \gamma^3 \mu} \int_{-1}^{1} G^3 \, d\eta, \tag{2.7}$$

which reduces to

$$c^3 P(\lambda) = \frac{N \gamma^3 \mu}{(\rho g \sin\alpha)^4} Q, \tag{2.8}$$

where $N = 2^4 \times 3 \times 5 \times 7 \times 13^3 = 3\,690\,960$ and

$$P(\lambda) := \lambda^3 - \tfrac{10}{3}\lambda^2 + \tfrac{124}{33}\lambda - \tfrac{56}{39}. \tag{2.9}$$

This cubic polynomial $P(\lambda)$ is monotonic in λ, with real root $\lambda_0 (\approx 1.184)$.

Thus, overall we have

$$h(x,y) = \frac{c\rho g \sin\alpha}{104\gamma} (cx)^{-1/13} G(\eta), \tag{2.10}$$

where

$$\eta = y/y_e(x), \quad y_e(x) = (cx)^{3/13}, \tag{2.11}$$

with $G(\eta)$ as in (2.6) and with (2.8) holding. The quartic function $G(\eta)$ essentially gives the cross-sectional shape of the trickle at each station $x = \text{constant}$; we shall find interpretations of the solution for both positive and negative values of the constant c, valid for $G(\eta) \geqslant 0$

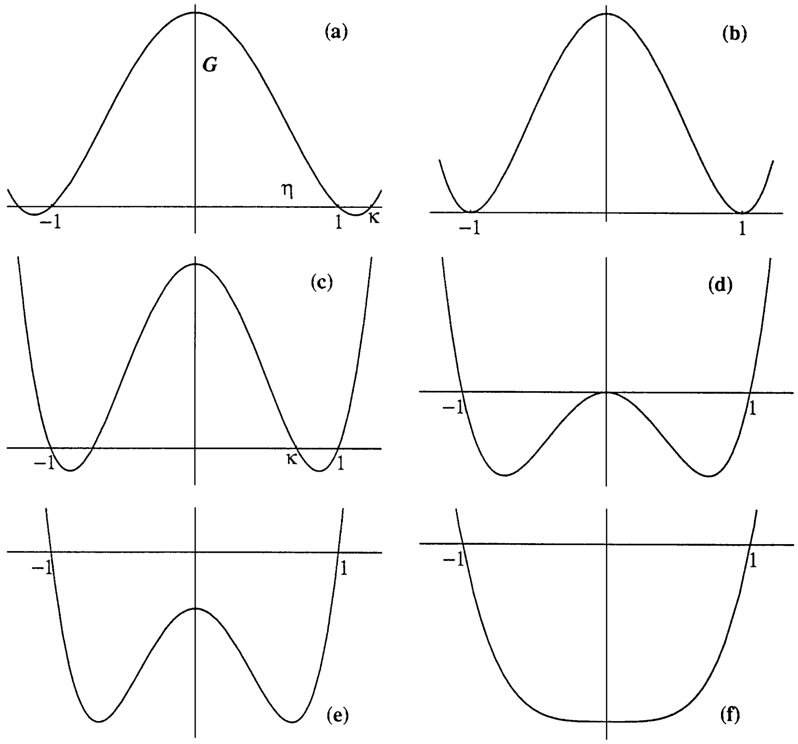

FIGURE 2. Cross-sectional profiles given by the function $G(\eta)$ in equation (2.6), for (a) $\lambda > 2$, (b) $\lambda = 2$, (c) $1 < \lambda < 2$, (d) $\lambda = 1$, (e) $0 < \lambda < 1$, (f) $\lambda \leqslant 0$. In (a) and (c) $\kappa := (\lambda-1)^{1/2}$.

and $G(\eta) \leqslant 0$, respectively (see equation (3.3)). Figure 2 shows the forms of $G(\eta)$ for the cases (a) $\lambda > 2$, (b) $\lambda = 2$, (c) $1 < \lambda < 2$, (d) $\lambda = 1$, (e) $0 < \lambda < 1$, (f) $\lambda \leqslant 0$. The depth of the trickle varies like $x^{-1/13}$, and y_e gives its 'spread' as it runs down the plane. (Of course, as this is a similarity solution, the profile is essentially the same at each x-station, merely scaled in depth by $x^{-1/13}$ and in width by $x^{3/13}$. Also at a given x-station the depth and width (for a given λ) are proportional to $Q^{4/13}$ and $Q^{1/13}$.)

We expect the solution to be valid if $1 \gg |h_y| \gg |h_x|$ in general; evaluation of these inequalities at the 'typical' value $\eta = 1$ shows that

$$cx \gg \left|\frac{(\lambda-2)\rho g c \sin\alpha}{52\gamma}\right|^{13/4} \quad \text{and} \quad cx \gg \left|\frac{3c}{13}\right|^{13/10}, \tag{2.12}$$

which imply essentially that the solution can be valid only well away from the origin. Also, with the above form for h the neglected term $\rho g h \cos\alpha$ in (2.1) (associated with lateral spreading due to gravity) is much smaller than the retained term $\gamma \nabla^2 h$ if

$$cx \ll \left|\frac{12\gamma}{\rho g \lambda \cos\alpha}\right|^{13/6} \tag{2.13}$$

(which in fact imposes no restriction in the case when α is exactly $\pi/2$).

The governing equations (2.2) and (2.4) involve the two length scales

$$L_1 = (\gamma/\rho g \sin\alpha)^{1/2} \quad \text{and} \quad L_2 = (\mu Q/\rho g \sin\alpha)^{1/4} \tag{2.14}$$

(which are prescribed quantities). There is therefore some freedom in choosing non-dimensional variables. With the scheme

$$x = kL_1\tilde{x}, \quad y = kL_1\tilde{y}, \quad y_e = kL_1\tilde{y}_e, \quad h = k^3 L_1 \tilde{h}/104, \quad c = (kL_1)^{10/3}\tilde{c}, \tag{2.15}$$

where $k = N^{1/10}(L_2/L_1)^{2/5}$, the solution may be written in the form

$$\tilde{h}(\tilde{x},\tilde{y}) = \frac{G(\eta)}{\tilde{x}^{1/13}[P(\lambda)]^{4/13}}, \quad \eta = \frac{\tilde{y}}{\tilde{y}_e}, \quad \tilde{y}_e = \left(\frac{\tilde{x}^3}{P(\lambda)}\right)^{1/13}, \tag{2.16}$$

with $G(\eta)$ and $P(\lambda)$ as in (2.6) and (2.9). The solution involves one free parameter, here taken to be λ; thus (2.16) represents a family of solutions, with parameter λ.

The analysis is valid for both $c < 0$ and $c > 0$, provided that y_e and h are positive; this means that $cx > 0$, that $cG \geqslant 0$, and that the η-domain must be chosen such that $G(\eta)$ is of one sign.

3 Interpretation of the solution

(i) For all cases except case (c) we take $|\eta| \leqslant 1$ for the moment. Case (c) is different in that the corresponding interval for η is $|\eta| \leqslant \kappa$, where $\kappa := (\lambda-1)^{1/2}$ (with $1 < \lambda < 2$); thus, for example, the integration limits in equation (2.7) for case (c) should be $\pm\kappa$, rather than ± 1. However, one can show easily that under the change of variables

$$\eta = \eta^*/(\lambda^*-1)^{1/2}, \quad (\lambda-1)(\lambda^*-1) = 1, \quad c = c^*(\lambda^*-1)^{13/6}, \quad y_e = (\lambda^*-1)^{1/2} y_e^*,$$

the solution for case (c) becomes the same as that for case (a), with η, λ, c and y_e replaced by η^*, λ^*, c^* and y_e^*, and with $|\eta^*| \leqslant 1$. This means that case (c) can be subsumed into case (a), and need not be treated separately.

In all cases, then, the contact angle $\beta(x)$ at $y = \pm y_e$ and the centre-line depth $h_0(x)$ of the liquid are given approximately by

$$\tan\beta \approx -\left.\frac{\partial h}{\partial y}\right|_{y=y_e} = \frac{(\lambda-2)\,c\rho g \sin\alpha}{52\gamma(cx)^{4/13}} \tag{3.1}$$

and

$$h_0 := h(x,0) = \frac{(\lambda-1)\,c\rho g\sin\alpha}{104\gamma(cx)^{1/13}}. \tag{3.2}$$

Note that in this solution β varies round the contact line (except when $\lambda = 2$), and cannot be prescribed. (Smith's solution has a similar feature.) It is expected that (3.1) can hold only where $\beta_r \leqslant \beta \leqslant \beta_a$.

The condition $Q > 0$ requires that $\lambda > \lambda_0$ if $c > 0$ and $\lambda < \lambda_0$ if $c < 0$. In addition the conditions $h_0 > 0$ and $0 < \tan\beta$ ($\ll 1$) require that

$$\left.\begin{array}{llllll} \text{either} & \text{I}: c > 0, & x > 0, & \lambda \geqslant 2, & G(\eta) \geqslant 0 & \text{(cases (a), (b))}, \\ \text{or} & \text{II}: c < 0, & x < 0, & \lambda \leqslant 1, & G(\eta) \leqslant 0 & \text{(cases (d), (e), (f))} \end{array}\right\} \tag{3.3}$$

(both of which give $Q > 0$ and $h(x,y) \geqslant 0$, as required).

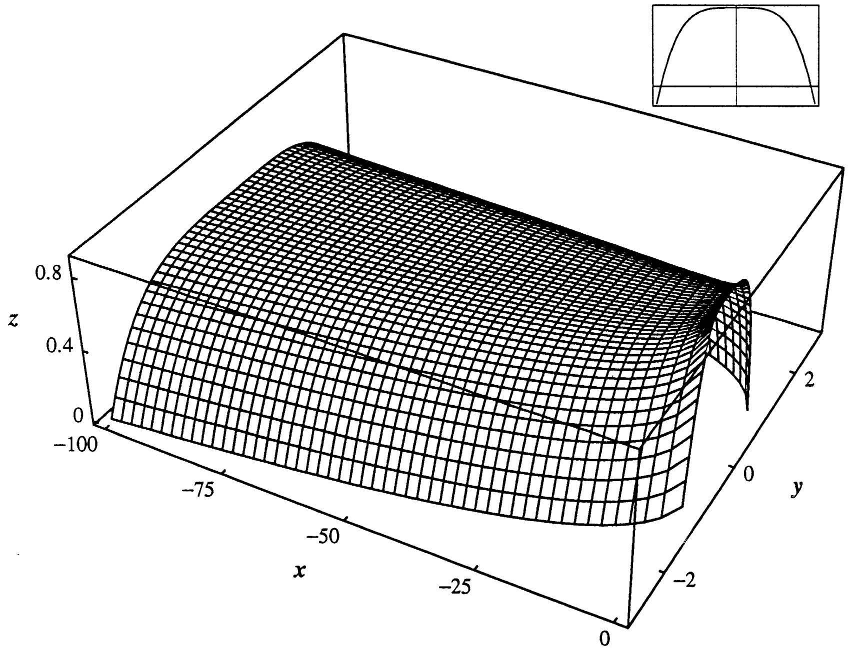

FIGURE 3. The free-surface shape predicted by the similarity solution (2.16) in the case $\lambda = 0$. This is of type II, i.e. getting narrower and deeper downstream. The inset shows the cross-sectional profile (cf. Figure 2).

A solution of type I describes flow (in $x > 0$) from a source at $x = 0$, the trickle getting wider (like $x^{3/13}$) and shallower (like $x^{-1/13}$) with distance x down the plane. A solution of type II describes flow (in $x < 0$) into a sink at $x = 0$, the trickle getting narrower and deeper down the plane. In type-II solutions the lateral contraction can continue only until the depth becomes so great (and the contact angle becomes so big) that the fluid 'spills over' and runs down the plane as a uniform trickle.

Both types of solution are symmetric about the mid-line $\eta = 0$. As may be seen in Figure 2, in all cases except (e) the cross section of the trickle (at a fixed x-station) has a single maximum, at $y = 0$. In case (e) (for which h has the sign of $-G$) the cross section is 'double-humped', with maxima at $\eta = \pm(\lambda/2)^{1/2}$ and a minimum at $\eta = 0$. Figures 3, 4 and 5 show examples of free-surface profiles for three representative values of the parameter λ; the first two are of type II and the latter is of type I.

(ii) Consider now case (c) with $1 < \lambda < \lambda_0$. In the interval $\kappa \leqslant \eta \leqslant 1$ we have $G(\eta) \leqslant 0$, $c < 0$ and $x < 0$; also the limits in the integral in (2.7) become $\eta = \kappa$ to $\eta = 1$. The solution in this interval represents flow (into a sink at $x = 0$) of a 'meandering' trickle occupying $\kappa y_e \leqslant y \leqslant y_e$ (for $x < 0$); the trickle veers away from the line of greatest slope, and gets narrower and deeper as the sink is approached. Thus overall the trickle is highly asymmetrical in y, even though G is even in η. (There is a similar interpretation for the solution in $-1 \leqslant \eta \leqslant -\kappa$.)

(iii) Consider cases (a) and (b) with $|\eta| \geqslant \kappa$, so that $G(\eta) \geqslant 0$, $c > 0$, $\lambda \geqslant 2$ and $x > 0$. Roughly speaking, these correspond to flow of a liquid sheet around the *outside* of the region $-y_e \leqslant y \leqslant y_e$; thus this region (which, since $x > 0$, gets wider with increasing x) is

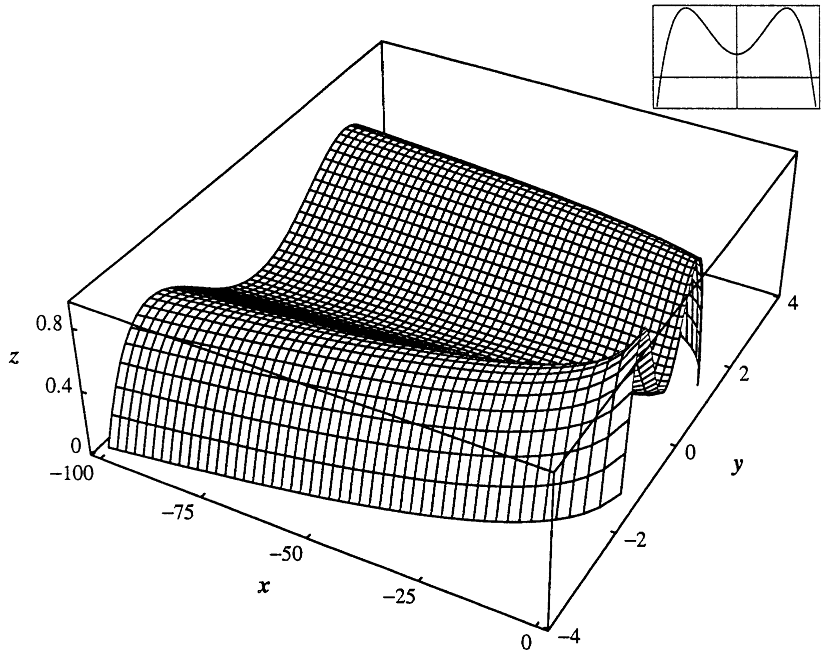

FIGURE 4. The free-surface shape in the case $\lambda = 0.9$. This is of type II.

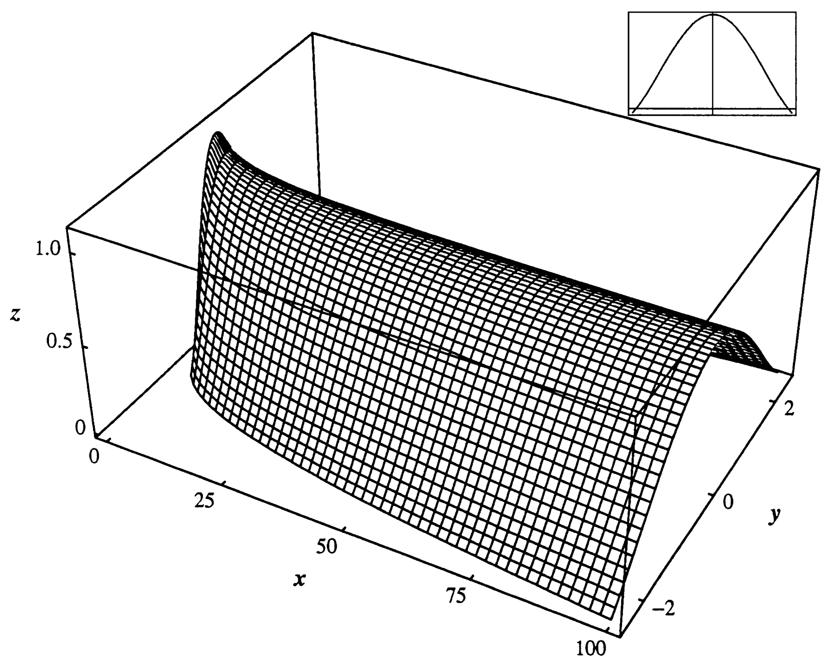

FIGURE 5. The free-surface shape in the case $\lambda = 3$. This is of type I, i.e. getting wider and shallower downstream.

a 'dry patch', with liquid on either side. However, the liquid sheet extends to $y = \pm\infty$ outside the dry patch, and the volume flux Q of liquid down the plane is infinite.

4 Comments

(i) Simple experiments were performed with large 'drops' of syrup on a perspex sheet. With the sheet horizontal a drop essentially takes up a static circular equilibrium shape (typically of radius ~ 50 mm and depth ~ 5 mm). The sheet is then supported in a vertical position, and the liquid allowed to run down under gravity. Of course, the setup does not correspond precisely to a constant supply flux, but at least it has the feature that the trickle has a lateral contraction as it runs down the plane (somewhat similar to case II(e)) until eventually 'fingers' form, which become nearly-uniform rivulets running down the line of greatest slope. In the contracting zone the flow is effectively steady for a long time, and the trickle cross sections were found to have clear 'two-humped' profiles, roughly as in case (e) of Figure 2, except that the humps are rather closer to the edges $y = \pm y_e(\mathrm{x})$ of the trickle than the figure would suggest; this may imply that the quartic profile in (2.6) is roughly of the right form, but is not correct in detail. Also, in the experiments the edges $y = \pm y_e(x)$ are only slightly curved (and indeed, in many runs were remarkably straight); in terms of the theory, this would mean that the parameter c in (2.11) was very small in the experiments.

(ii) The question arises as to the physical significance of the parameter λ, which is left undetermined even when conditions (2.3)–(2.5) are satisfied. Different choices for λ can lead to very different profiles; perhaps a detailed modelling of the contact-line regions would determine which of these (if any) is physically sensible.

(iii) The solution herein may be contrasted with that of Smith (1973), namely

$$h(x,y) = c_1 x^{-1/7} G_s(\eta), \quad \eta = y/y_s(x), \quad y_s(x) = c_2 x^{3/7}, \quad G_s(\eta) = 1-\eta^2, \tag{4.1}$$

where
$$c_1 = \left(\frac{4725\mu^2 Q^2}{1024\rho^2 g^2 \sin 2\alpha}\right)^{1/7}, \quad c_2 = \left(\frac{12\,005\mu Q \cos^3\alpha}{36\rho g \sin^4\alpha}\right)^{1/7}.$$

This solution was obtained by neglecting the term $\gamma\nabla^2 h$ in comparison with the term $\rho g h \cos\alpha$ in (2.1); clearly, such an approximation is not appropriate when $\alpha = \pi/2$, and the solution would break down in that case. Note that there is no free parameter (equivalent to λ) in (4.1).

Smith interpreted his solution as representing the steady spreading and shallowing of a viscous trickle emitted from a 'point' source Q at the origin. The solution (4.1) compares favourably with results of Schwartz & Michaelides (1988), who solved (2.1) numerically (with $\gamma = 0$), as an initial value problem. (We may also note that, with the choice $\pi/2 < \alpha < \pi$ and $x < 0$, Smith's solution may instead be interpreted as representing a *sink* flow down the *underside* of an inclined plane, the trickle contracting and deepening towards a sink Q at the origin; such a flow would presumably be difficult to establish experimentally!)

Appendix: Derivation of equation (2.2)

The development here parallels that of Smith (1973).

The natural scaling for the problem follows from the definition of the volume flux, and from the assumptions that there is a viscous-gravitational balance in the downstream direction, and that the pressure variations in the transverse direction scale with surface tension:

$$(x, y, z) = X_s(\bar{x}, \delta\bar{y}, \epsilon\delta\bar{z}), \quad (u, v, w) = U_s(\bar{u}, \delta\bar{v}, \epsilon\delta\bar{w}), \quad p = \Pi_s \bar{p}, \quad h = \epsilon\delta X_s \bar{h},$$

where

$$\delta = \left(\frac{\mu Q \gamma^3}{(\rho g \sin\alpha)^4 X_s^{10}}\right)^{1/13}, \quad \epsilon = \left(\frac{(\rho g \sin\alpha)\,\mu^3 Q^3}{\gamma^4 X_s^4}\right)^{1/13},$$

$$U_s = \left(\frac{(\rho g \sin\alpha)^7 Q^8}{\mu^5 \gamma^2 X_s^2}\right)^{1/13}, \quad \Pi_s = \left(\frac{(\rho g \sin\alpha)^5 \mu^2 Q^2 \gamma^6}{X_s^7}\right)^{1/13}.$$

Here u, v and w denote the Cartesian velocity components, p denotes the fluid pressure relative to atmospheric pressure, and X_s is a measure of distance downstream from the source. X_s need not be defined too precisely, since any length scale selected will be artificial in the sense that it will not appear in the solution (cf. Smith, 1973, p. 276).

The continuity and Navier–Stokes equations then lead to

$$u_x + v_y + w_z = 0,$$

$$R(uu_x + vu_y + wu_z) = -\delta^2 p_x + 1 + \delta^2 \epsilon^2 u_{xx} + \epsilon^2 u_{yy} + u_{zz},$$

$$R(uv_x + vv_y + wv_z) = -p_y + \delta^2 \epsilon^2 v_{xx} + \epsilon^2 v_{yy} + v_{zz},$$

$$R\epsilon^2(uw_x + vw_y + ww_z) = -p_z + \delta^2 \epsilon^4 w_{xx} + \epsilon^4 w_{yy} + \epsilon^2 w_{zz} - (\rho g \cos\alpha/\gamma)\,\delta^2 X_s^2,$$

overbars having dropped for clarity; here

$$R = \frac{\rho U_s X_s \epsilon^2 \delta^2}{\mu} = \left(\frac{\rho^{14} g Q^{16}}{\mu^{10} \gamma^{10} X_s^{17}}\right)^{1/13}.$$

We now take $\delta^2 \ll 1$, $\epsilon^2 \ll 1$ and $R \ll 1$ (all of which should be valid for sufficiently large X_s). Then for sufficiently small $|\alpha - \frac{1}{2}\pi|$ we have, at leading order,

$$u_{zz} = -1, \quad v_{zz} = p_y, \quad 0 = p_z.$$

This system is to be integrated subject to the boundary conditions

$$u = v = 0 \quad \text{on} \quad z = 0,$$

and, at leading order,

$$u_z = v_z = 0 \quad \text{and} \quad p = -h_{yy} \quad \text{on} \quad z = h.$$

We thus find

$$p = -h_{yy}, \quad u = hz - \tfrac{1}{2}z^2, \quad v = h_{yyy}(hz - \tfrac{1}{2}z^2).$$

Then, for example, integration of the continuity equation and use of the conditions

$$w = 0 \quad \text{on} \quad z = 0, \quad w = uh_x + vh_y \quad \text{on} \quad z = h$$

leads to

$$3h^2 h_x = -(h^3 h_{yyy})_y,$$

which is (2.2) is non-dimensional form.

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References

ALLEN, R. F. & BIGGIN, C. M. (1974) Longitudinal flow of a lenticular liquid filament down an inclined plane. *Phys. Fluids* **17**, 287–291.

DAVIS, S. H. (1983) Contact line problems in fluid mechanics. *Trans. ASME, J. Appl. Mechanics* **50**, 977–982.

HUPPERT, H. E. (1982) Flow and instability of a viscous current down a slope. *Nature* **300**, 427–429.

LISTER, J. R. (1992) Viscous flows down an inlined plane from point and line sources. *J. Fluid Mechanics* **242**, 631–653.

MIKIELEWICZ, J. & MOSZYNSKI, J. R. (1978) An improved analysis of breakdown of thin liquid films. *Arch. Mech. Stos.* **30**, 489–500.

MORIARTY, J. A., SCHWARTZ, L. W. & TUCK, E. O. (1991) Unsteady spreading of thin liquid films with small surface tension. *Phys. Fluids A* **3**, 733–742.

SMITH, P. C. (1973) A similarity solution for slow viscous flow down an inclined plane. *J. Fluid Mechanics* **58**, 275–288.

SCHWARTZ, L. W. (1989) Viscous flows down an inclined plane: Instability and finger formation. *Phys. Fluids A* **1**, 443–445.

SCHWARTZ, L. W. & MICHAELIDES, E. E. (1988) Gravity flow of a viscous liquid down a slope with injection. *Phys. Fluids* **31**, 2739–2741.

TOWELL, G. D. & ROTHFELD, L. B. (1966) Hydrodynamics of rivulet flow. *J. Am. Inst. Chem. Eng* **12**, 972–980.