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Motion and expansion of a viscous vortex ring. Part 1. A higher-order asymptotic formula for the velocity

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A large-Reynolds-number asymptotic solution of the Navier–Stokes equations is sought for the motion of an axisymmetric vortex ring of small cross-section embedded in a viscous incompressible fluid. In order to take account of the influence of elliptical deformation of the core due to the self-induced strain, the method of matched asymptotic expansions is extended to a higher order in a small parameter $\epsilon = (\nu/\Gamma)^{1/2}$, where ν is the kinematic viscosity of fluid and Γ is the circulation. Alternatively, ϵ is regarded as a measure of the ratio of the core radius to the ring radius, and our scheme is applicable also to the steady inviscid dynamics.

We establish a general formula for the translation speed of the ring valid up to third order in ϵ. This is a natural extension of Fraenkel–Saffman's first-order formula, and reduces, if specialized to a particular distribution of vorticity in an inviscid fluid, to Dyson's third-order formula. Moreover, it is demonstrated, for a ring starting from an infinitely thin circular loop of radius R_0, that viscosity acts, at third order, to expand the circles of stagnation points of radii $R_s(t)$ and $\tilde{R}_s(t)$ relative to the laboratory frame and a comoving frame respectively, and that of peak vorticity of radius $R_p(t)$ as $R_s \approx R_0 + [2\log(4R_0/\sqrt{\nu t}) + 1.4743424]\,\nu t/R_0$, $\tilde{R}_s \approx R_0 + 2.5902739\,\nu t/R_0$, and $R_p \approx R_0 + 4.5902739\,\nu t/R_0$. The growth of the radial centroid of vorticity, linear in time, is also deduced. The results are compatible with the experimental results of Sallet & Widmayer (1974) and Weigand & Gharib (1997).

The procedure of pursuing the higher-order asymptotics provides a clear picture of the dynamics of a curved vortex tube; a vortex ring may be locally regarded as a line of dipoles along the core centreline, with their axes in the propagating direction, subjected to the self-induced flow field. The strength of the dipole depends not only on the curvature but also on the location of the core centre, and therefore should be specified at the initial instant. This specification removes an indeterminacy of the first-order theory. We derive a new asymptotic development of the Biot-Savart law for an arbitrary distribution of vorticity, which makes the non-local induction velocity from the dipoles calculable at third order.

1. Introduction

Vortex rings are usually considered as invariant states in an inviscid fluid in which the vortex lines are endowed with curvature which enables the rings to propagate.

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The study of the translation speed of vortex rings may be traced back to Helmholtz (1858) and Kelvin (1867). Extending Kelvin's result, Dyson (1893) (see also Fraenkel 1972) obtained the speed U of a thin axisymmetric vortex ring, steadily translating in an inviscid incompressible fluid of infinite extent, up to third (virtually fourth) order in a small parameter ϵ:

$$U = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8}{\epsilon}\right) - \frac{1}{4} - \frac{3\epsilon^2}{8}\left[\log\left(\frac{8}{\epsilon}\right) - \frac{5}{4}\right] + O(\epsilon^4 \log\epsilon)\right\}, \qquad (1.1)$$

where Γ is the circulation, R_0 is the ring radius, and $\epsilon = \sigma/R_0$ is the ratio of core radius σ to R_0. The assumption is made that $\epsilon \ll 1$. The speed depends on the distribution of vorticity, and, in this treatment, the one for which vorticity is proportional to distance from the symmetric axis is selected. We consider Kelvin's formula (the first two terms) as the first order and the ϵ^2-terms as the third. Kelvin's formula corresponds to a uniform distribution of vorticity, at leading order, in a circular core. The local flow induced by the vortex ring consists not only of a uniform flow but also of a straining field. The latter manifests itself at second order and deforms the core into an ellipse, elongated in the propagating direction (Dyson 1893; Fraenkel 1972):

$$r = \sigma\left\{1 - \frac{3\epsilon^2}{8}\left[\log\left(\frac{8}{\epsilon}\right) - \frac{17}{12}\right]\cos 2\theta + \cdots\right\}, \qquad (1.2)$$

where (r, θ) are local moving cylindrical coordinates about the core centre which will be introduced in §2. It is remarkable that the inclusion of the third-order term in the propagating velocity achieves a significant improvement in approximation; (1.1) compares well even with the exact value for the fat limit of Hill's spherical vortex (Hill 1894). In this limit, $\epsilon = \sqrt{2}$ under a suitable normalization. As pointed out by Fraenkel (1972), the velocity obtained from (1.1) with $\epsilon = \sqrt{2}$ differs from the exact value by an error of only about 5%. It may follow that Dyson's formula for $\epsilon \ll 1$ applies with reasonable accuracy to the whole family of Fraenkel–Norbury vortex rings ($0 < \epsilon \leqslant \sqrt{2}$) (Norbury 1973). This surprising agreement encourages us to explore a higher-order approximation of the velocity in more general circumstances.

For a stagnant core, the $\frac{1}{4}$ in (1.1) is changed into $\frac{1}{2}$ (Hicks 1885). Fraenkel (1970) and Saffman (1970) generalized Kelvin's formula to admit an arbitrary distribution of vorticity:

$$U = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8R_0}{\hat{\sigma}}\right) - \frac{1}{2} + \frac{4\pi^2}{\Gamma^2}\int_0^{\hat{\sigma}} r'\left[v(r')\right]^2 \mathrm{d}r'\right\}, \qquad (1.3)$$

where v is the local azimuthal velocity. The parameter $\hat{\sigma}$ is the core radius for a discrete distribution surrounded by an irrotational flow. For a localized but continuous distribution of vorticity, *the limit of $\hat{\sigma} \to \infty$ is to be taken.*

Viscosity acts to diffuse vorticity, and the motion ceases to be steady. Its influence on the travelling speed, at large Reynolds number, was first addressed by Tung & Ting (1967). For the case where the vorticity is, at a virtual instant, a 'δ-function' concentrated on the circle of radius R_0, Saffman (1970) (see also Callegari & Ting 1978) succeeded in deriving an explicit formula, valid up to $O[(\nu/\Gamma)^{1/2}]$, as

$$U = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8R_0}{2\sqrt{\nu t}}\right) - \tfrac{1}{2}(1 - \gamma + \log 2) + O\left[\left(\frac{\nu t}{R_0^2}\right)^{1/2}\log\left(\frac{\nu t}{R_0^2}\right)\right]\right\} \qquad (1.4a)$$

$$\approx \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{4R_0}{\sqrt{\nu t}}\right) - 0.55796576 + O\left[\left(\frac{\nu t}{R_0^2}\right)^{1/2}\log\left(\frac{\nu t}{R_0^2}\right)\right]\right\}, \qquad (1.4b)$$

where ν is the kinematic viscosity, t is the time measured from the instant at which the core is infinitely thin, and $\gamma = 0.57721566\ldots$ is Euler's constant. To obtain (1.4), it suffices to introduce into (1.3) the local distribution of the Lamb–Oseen diffusing vortex,

$$\omega(r) = \frac{\Gamma}{4\pi\nu t}\mathrm{e}^{-r^2/4\nu t}, \quad v(r) = \frac{\Gamma}{2\pi r}(1 - \mathrm{e}^{-r^2/4\nu t}), \tag{1.5}$$

where $\omega(r)$ is the local distribution of the toroidal vorticity. It is to be born in mind that (1.5) possesses circular symmetry. Saffman paid little attention to the precise order of error. Stanaway, Cantwell & Spalart (1988*a, b*) approached this by conducting a direct numerical simulation of the Navier–Stokes equations for a single axisymmetric vortex ring, at moderate Reynolds numbers, $0.001 \leqslant Re_\Gamma \leqslant 1000$, where $Re_\Gamma = \Gamma/\nu$. They claimed that the error is smaller than indicated by (1.4); their results suggested that it is $O\left[(\nu t/R_0^2)\log(\nu t/R_0^2)\right]$, or third order, in common with the inviscid solution.

The non-existence of the limit $\nu \to 0$ of (1.4) pertains to the specific concentrated distribution of vorticity at $t = 0$. If the initial distribution is replaced by a core of finite thickness, the limit of $\nu \to 0$ recovers the inviscid result for the corresponding steady vortex ring. The inviscid formula is accessible by application of the double limit $\nu \to 0$, $t \to \infty$, in this order, to the solution of the Navier–Stokes equations. The extent to which viscosity affects the motion depends on the time scale of vorticity diffusion, relative to the time in which the ring travels a few diameters, and thus on the Reynolds number. Comprehensive lists of theories of the existence and motion of vortex rings may be found in the textbook by Saffman (1992) and the article by Shariff & Leonard (1992).

Recent direct numerical simulations of fully developed turbulence have revealed that the small-scale structure is dominated by high-vorticity regions concentrated in tube-like structures (Siggia 1981; Kerr 1985; Hosokawa & Yamamoto 1989; She, Jackson & Orszag 1990; Douady, Couder & Brachet 1991; Kida & Ohkitani 1992). Though they occupy a small fraction of the total volume, they are responsible for a much larger fraction of viscous dissipation. Motivated by the intriguing pattern of the dissipation field, Moffatt, Kida & Ohkitani (1994) developed a large-Reynolds-number asymptotic theory to solve the Navier–Stokes equations for a steady stretched vortex tube subjected to uniform non-axisymmetric irrotational strain. The higher-order asymptotics satisfactorily mimic the fine structure of the dissipation field previously obtained by numerical computation (Kida & Ohkitani 1992).

This success provides a convincing demonstration of the advantages that an asymptotic theory offers. An asymptotic solution, usually including only a few parameters, can be efficiently evaluated and is more and more accurate as the Reynolds number is increased. By contrast, numerical simulation is cost- and time-consuming and becomes impractical at large Reynolds numbers. Once the asymptotic solution is obtained, detailed information on the vorticity, strain and pressure field, is immediately accessible.

For the related, though unsteady, planar problem, the asymptotic expansions for a strained diffusing vortex were initiated by Ting & Tung (1965). By an introduction of strained coordinates, combined with higher-order expansions, Jiménez, Moffatt & Vasco (1996) made a substantial improvement in the approximation. Prochazka & Pullin (1998) achieved a further improvement, in the equivalent three-dimensional problem, by refining the strained coordinates. The basic procedure is similar in both two and three dimensions. At leading order, the Burgers and the Lamb–Oseen vortices, both having circular cores, are obtained in the three- and two-dimensional problems respectively. The viscosity plays the part of selecting the realized vorticity profile, Gaussian distributions, as represented by (1.5). At the next order ($O(\nu/\Gamma)$), a quadrupole component emerges, reflecting an elliptical vorticity distribution. The distinguishing feature is that the major axis of the ellipse is aligned at 45° to the principal axis of the external stain (see also Bajer & Moffatt 1998 for a different approach).

In practical flows, the cross-section of the core of a vortex ring is never exactly of circular shape, due to the local straining field that it induces. It is conceivable that the strained cross-section of a propagating vortex ring reflects an equilibrium between self-induced strain and viscous diffusion. The aim of the present investigation is to elucidate the structure of this strained core and its influence on the motion of an axisymmetric vortex ring. As a first step, we develop, in this paper, a framework to handle this problem. We construct a general formula for the third-order correction to the first-order translation speed (1.3). In addition, insight is gained into how viscosity affects the radial drift of vorticity. Our formula is supported by the numerical observation of Stanaway *et al.* (1988*a, b*) that the correction to Saffman's formula is $O[(\nu t/R_0^2)\log(\nu t/R_0^2)]$ rather than Saffman's own estimate (1.4). A similar expansion of ring radius is realized in the experiments of Sallet & Widmayer (1974) and Weigand & Gharib (1997). With this agreement, we have come to a belief that our theory, envisaged as high-Reynolds-number asymptotics, might be in fact applicable at moderate Reynolds numbers.

As long as we stay at first order, the method of Lamb's transformation is quite efficient to calculate the speed of thin axisymmetric vortex rings (Lamb 1932; Saffman 1992). Notably, it allows us to sidestep the solution for the flow field. For a systematic treatment of higher-order fields, however, the method of matched asymptotic expansions, in powers of ϵ, is essential. It has been previously developed to derive the velocity of a slender curved vortex tube in a fluid both with and without viscosity (Tung & Ting 1967; Widnall, Bliss & Zalay 1971; Callegari & Ting 1978; Klein & Majda 1991*a, b*; Ting & Klein 1991). The second-order curvature effect comes into play for a vortex tube with axial flow in the core (Moore & Saffman 1972), and this method has been limited to second order (Fukumoto & Miyazaki 1991). The self-induced strain of a vortex ring makes its appearance at second order, and influences the translation speed only at the third order. We need to extend asymptotic expansions to a higher order to obtain the third-order velocity of a vortex ring. We begin with scrutiny of the first-order theory. A few ambiguous steps are encountered. An understanding of these provides a clue to reaching a better formulation of the perturbation scheme. The previous theories deal exclusively with the inner field. In contrast, the improved formulation rests upon a more precise knowledge of the outer flow, and thus renders the inner and outer expansions systematic.

In § 2, we state the general problem of the matched asymptotic expansions. The existing asymptotic formula for the potential flow associated with a circular vortex loop is not sufficient to carry through our programme. In order to work out the correct inner limit of the outer solution, we devise, in § 3, a technique to produce a systematic asymptotic expression of the Biot-Savart integral accommodating an arbitrary vorticity distribution. In § 4, the inner expansions are re-examined at first order and are extended to second order. Emphasis is put on the boundary conditions and their relevance to the strength of the dipole. Based on these, we demonstrate in § 5 that the radius of the loop consisting of the stagnation points in the core, when viewed from the frame moving with the core, grows linearly in time owing to the action of viscosity. The analysis is extended to expansion laws for the circles of stagnation points relative to the laboratory coordinates, peak vorticity and the radial vorticity centroid. Comments are given on the early-time asymptotic theory developed by Wang, Chu & Chang (1994). We point out that their numerical results on radial expansion seem to be in contradiction with ours. In § 6, satisfactory comparison of the asymptotic solution is made with some experimental measurements and direct numerical simulations of trajectories of a laminar vortex ring. Thereafter, we establish in § 7 a general formula for the translation velocity of a vortex ring. In § 8, an equation governing the temporal evolution of the axisymmetric vorticity at second order is derived from the solvability condition. An integral representation of the exact solution is given, with which the formula of the preceding section is closed. The last section (§ 9) is devoted to a summary and conclusions. Our procedure of pursuing higher-order asymptotics reveals the significance of the dipoles distributed along the core centreline and oriented in the propagating direction. It turns out that their strength needs to be prescribed at an initial instant, which remedies the problem of undetermined constants at $O(\epsilon)$. As a by-product, a clear interpretation is provided of the general mechanism of the self-induced motion of a curved vortex tube.

The present study provides a general treatment. However, it is worthwhile to revisit the specific example of Dyson's inviscid vortex ring, which will be helpful in gaining a deeper insight into the nonlinear mechanism from the viewpoint of our framework. The manipulation of this high-order asymptotic solution also serves as a convincing check on the correctness of our general formulae (7.16) and (7.18) for the translation velocity. The detail will be reported in a companion paper (Fukumoto 2000).

2. Formulation of the matched asymptotic expansions

Consider an axisymmetric vortex ring of circulation Γ moving in an infinite expanse of viscous fluid with kinematic viscosity ν. We suppose that the circulation Reynolds number Re_Γ is very large:

$$Re_\Gamma = \Gamma/\nu \gg 1. \tag{2.1}$$

Two length scales are available, namely measures of the core radius σ and the ring radius R_0.

Suppose that their ratio σ/R_0 is very small. We focus attention on the translational motion of a 'quasi-steady' core. This means that we exclude stable or unstable wavy motion and fast core-area waves. Then, according to (1.1), the time scale in question is of order $R_0/(\Gamma/R_0) = R_0^2/\Gamma$. Denoting the initial value of σ by σ_0, the core spreads over this time to be of order $\sigma \sim \sigma_0 + (\nu t)^{1/2} \sim \sigma_0 + (\nu/\Gamma)^{1/2} R_0$. If the circulation Reynolds number is so large that $Re_\Gamma \gg (R_0/\sigma_0)^2$, then on this time scale viscous diffusion of vorticity may be disregarded and the problem is reduced to steady Euler dynamics. Otherwise, we need to seek the solution of the Navier–Stokes equations with time-dependence retained. Thus our assumption of slenderness requires that the relevant small parameter $\epsilon(\ll 1)$ is

$$\epsilon = \begin{cases} \sigma_0/R_0 & \text{if } Re_\Gamma \gg (R_0/\sigma_0)^2 \\ \sqrt{\nu/\Gamma} & \text{if } Re_\Gamma \lesssim (R_0/\sigma_0)^2. \end{cases} \tag{2.2}$$

The analysis that follows is applicable to inviscid as well as viscous dynamics.

Choose cylindrical coordinates (ρ, ϕ, z) with the z-axis along the axis of symmetry and ϕ along the vortex lines as shown in figure 1. We consider an axisymmetric distribution of vorticity $\boldsymbol{\omega}$ localized about the circle $(\rho, z) = (R(t), Z(t))$:

$$\boldsymbol{\omega} = \zeta(\rho, z)\boldsymbol{e}_\phi, \tag{2.3}$$

where $\boldsymbol{e}_\phi$ is the unit vector in the azimuthal direction. The Stokes streamfunction ψ

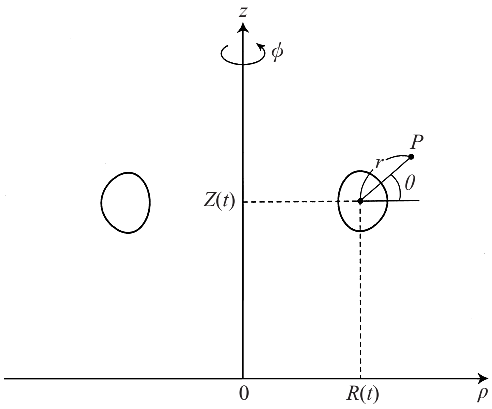

FIGURE 1. The meridional cross-section of a vortex ring and the coordinate systems. The cylindrical coordinates fixed in space are denoted by (ρ, ϕ, z), and (r, θ) are moving cylindrical coordinates centred on $(\rho, z) = (R(t), Z(t))$.

for the flow generated by (2.3) is given, via the Biot-Savart law, in the form

$$\psi(\rho, z) = -\frac{\rho}{4\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{\infty} \frac{\zeta(\rho', z')\rho' \cos\phi' \, \mathrm{d}\rho' \, \mathrm{d}\phi' \, \mathrm{d}z'}{\sqrt{\rho^2 - 2\rho\rho' \cos\phi' + \rho'^2 + (z - z')^2}}. \tag{2.4}$$

The theorems of Kelvin and Helmholtz imply that determination of the ring motion necessitates a knowledge of the flow velocity in the vicinity of the core.

As is well known, the irrotational flow velocity calculated from (2.4) for an infinitely thin core increases without limit primarily in inverse proportion to the distance from the core. In addition, it entails a logarithmic infinity originating from the curvature effect. The singularity may be resolved by matching the outer flow to an inner vortical flow which decays rapidly as the core centre is approached. Thus we are led to inner and outer expansions (Ting & Tung 1965; Tung & Ting 1967). The inner region consists of the core itself and the surrounding toroidal region with thickness of order the core radius σ. There we develop an inner asymptotic expansion, incorporating the effect of curvature, matching at each level to the outer solution (2.4).

To this end, it is advantageous to introduce, in the axial plane, local polar coordinates (r, θ) moving with the core centre† $(R(t), Z(t))$ with $\theta = 0$ in the ρ-direction (figure 1):

$$\rho = R(t) + r\cos\theta, \qquad z = Z(t) + r\sin\theta. \tag{2.5}$$

Let us non-dimensionalize the inner variables. The radial coordinate is normalized by the core radius $\epsilon R_0 (= \sigma)$ and the local velocity $\boldsymbol{v} = (u, v)$, relative to the moving frame, by the maximum velocity $\Gamma/(\epsilon R_0)$. In view of (1.1), the normalization parameter for the ring speed $(\dot{R}(t), \dot{Z}(t))$, the slow dynamics, should be Γ/R_0. The suitable dimensionless inner variables are thus defined as

$$\left.\begin{array}{c} r^* = \dfrac{r}{\epsilon R_0}, \quad t^* = \dfrac{t}{R_0^2/\Gamma}, \quad \psi^* = \dfrac{\psi}{\Gamma R_0}, \quad \zeta^* = \dfrac{\zeta}{\Gamma/R_0^2\epsilon^2}, \\ \boldsymbol{v}^* = \dfrac{\boldsymbol{v}}{\Gamma/(R_0\epsilon)}, \quad (\dot{R}^*, \dot{Z}^*) = \dfrac{(\dot{R}, \dot{Z})}{\Gamma/R_0}. \end{array}\right\} \tag{2.6}$$

The difference in normalization between the last two of (2.6) should be kept in mind.

† The definition of the 'core centre' will be discussed at some length in § 4.2.

In keeping with (2.2), define

$$\hat{\nu} = \begin{cases} 0 & \text{if } Re_\Gamma \gg (R_0/\sigma_0)^2 \\ 1 & \text{if } Re_\Gamma \lesssim (R_0/\sigma_0)^2. \end{cases} \tag{2.7}$$

The equations handled in the inner region are the coupled system of the vorticity equation, obtained by taking the curl of the Navier–Stokes equations, and the subsidiary relation between ζ and ψ. Note that the time derivative, when written in terms of the moving coordinates, is to be transformed as

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_{(z,\rho)} &= \left.\frac{\partial}{\partial t}\right|_{(r,\theta)} - \dot{R}\frac{\partial}{\partial \rho} - \dot{Z}\frac{\partial}{\partial z} \\ &= \left.\frac{\partial}{\partial t}\right|_{(r,\theta)} - (\dot{R}\cos\theta + \dot{Z}\sin\theta)\frac{\partial}{\partial r} - (-\dot{R}\sin\theta + \dot{Z}\cos\theta)\frac{1}{r}\frac{\partial}{\partial \theta}. \end{aligned} \tag{2.8}$$

Dropping the stars, these equations take the following form:

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + \frac{1}{\epsilon^2}\left(u\frac{\partial \zeta}{\partial r} + \frac{v}{r}\frac{\partial \zeta}{\partial \theta}\right) - \frac{1}{\epsilon\rho^2}\left(\frac{\partial \psi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\cos\theta\right)\zeta \\ = \hat{\nu}\left[\Delta\zeta + \frac{\epsilon}{\rho}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\zeta - \frac{\epsilon^2}{\rho^2}\zeta\right], \end{aligned} \tag{2.9}$$

$$\zeta = \frac{1}{\rho}\Delta\psi - \frac{\epsilon}{\rho^2}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\psi, \tag{2.10}$$

where

$$\rho = R + \epsilon r\cos\theta, \tag{2.11}$$

Δ is the two-dimensional Laplacian,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{2.12}$$

and u and v are the r- and θ-components of the relative velocity $\boldsymbol{v}$:

$$u = \frac{1}{r\rho}\frac{\partial \psi}{\partial \theta} - \epsilon(\dot{R}\cos\theta + \dot{Z}\sin\theta), \tag{2.13}$$

$$v = -\frac{1}{\rho}\frac{\partial \psi}{\partial r} - \epsilon(-\dot{R}\sin\theta + \dot{Z}\cos\theta). \tag{2.14}$$

We now postulate the following series expansions of the solution:

$$\zeta = \zeta^{(0)} + \epsilon\zeta^{(1)} + \epsilon^2\zeta^{(2)} + \epsilon^3\zeta^{(3)} + \cdots, \tag{2.15}$$

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \epsilon^3\psi^{(3)} + \cdots, \tag{2.16}$$

$$R = R^{(0)} + \epsilon R^{(1)} + \epsilon^2 R^{(2)} + \cdots, \tag{2.17}$$

$$Z = Z^{(0)} + \epsilon Z^{(1)} + \epsilon^2 Z^{(2)} + \cdots, \tag{2.18}$$

where $\zeta^{(i)}$ and $\psi^{(i)}$ $(i = 0, 1, 2, 3, \ldots)$ are functions of r, θ and sometimes t. There arises $\log\epsilon$ as well, but we conveniently take it to be of order unity, since multiples of $\log\epsilon$ happen to be ruled out at least to the above order. Inserting these expansions into (2.9) and (2.10), supplemented by (2.11)–(2.14), and collecting terms with like powers of ϵ, we obtain the equations to be solved in the inner region. These are written out in full in Appendix A.

The permissible solution must satisfy the condition

$$u \text{ and } v \text{ are finite at } r = 0, \tag{2.19}$$

which is better than the restrictive one that $u = v = 0$ at $r = 0$. The demand that it smoothly match the asymptotic form, valid in the vicinity of the core, of the outer solution will determine the values of $\dot{R}^{(i)}$ and $\dot{Z}^{(i)}$ $(i = 0, 1, 2, \ldots)$.

This procedure was followed by Tung & Ting (1967) and Callegari & Ting (1978) and others, up to first order. Our aim is to explore the second and third orders. Before that, we reconsider the earlier results. A close examination of the low-order fields, the first part of § 4, will highlight the importance of the distribution of vorticity $\zeta(r, \theta)$, especially its dipole structure, which may have so far gone unnoticed.

3. Outer solution

In this section, we inquire into the asymptotic behaviour of the Biot-Savart law (2.4) at small values of r in order to deduce the matching conditions on the inner solution. The existing theories of matched asymptotic expansions for the motion of a slender curved vortex tube ignore the detailed distribution of vorticity in the core and replace the volume integral merely with a line integral. This amounts to a 'δ-function' concentrated core, the vorticity being characterized by the single parameter Γ only.

For a circular vortex loop of unit strength placed at $(\rho, z) = (R, Z)$, $\zeta = \delta(\rho - R)\,\delta\,(z - Z)$ and the Stokes streamfunction (2.4) simplifies to

$$\psi_m(\rho, z\,; R) = -\frac{\rho}{4\pi}\int_0^{2\pi} \frac{R\cos\phi'\,\mathrm{d}\phi'}{\sqrt{\rho^2 - 2\rho R\cos\phi' + R^2 + (z - Z)^2}} \tag{3.1a}$$

$$= -\frac{1}{2\pi}(r + r_2)\left[K\left(\frac{r_2 - r}{r_2 + r}\right) - E\left(\frac{r_2 - r}{r_2 + r}\right)\right], \tag{3.1b}$$

where $r_2 = (4R^2 + r^2 + 4Rr\cos\theta)^{1/2}$ is the greatest distance from the point (ρ, z) to the loop, and K and E are the complete elliptic integrals of the first and second kinds respectively, with modulus $(r_2 - r)/(r_2 + r)$ (Lamb 1932). We call ψ_m the monopole field. The second expression is well known as Maxwell's formula.

With the aid of the asymptotic behaviour of K and E for modulus close to unity, the asymptotic form of ψ_m for $r \ll r_2$ is obtainable at once (Dyson 1893; Tung & Ting 1967) (see (3.7) below). It turns out however that, when going to higher orders, (3.1) is not sufficient to qualify as the outer solution. Investigation of the detailed structure of (2.4) is unavoidable.

For this purpose, it is expedient to adapt Dyson's 'shift operator' technique to an arbitrary distribution of vorticity, and to cast (2.4) in the following form:

$$\psi = -\frac{\rho}{4\pi}\iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}\hat{z}'\zeta(x', \hat{z}')\exp\left(x'\frac{\partial}{\partial R} - \hat{z}'\frac{\partial}{\partial \hat{z}}\right)\int_0^{2\pi}\frac{R\cos\phi'\,\mathrm{d}\phi'}{\sqrt{\rho^2 - 2\rho R\cos\phi' + R^2 + \hat{z}^2}}, \tag{3.2}$$

where $(x, \hat{z}) = (\rho - R, z - Z)$ are local Cartesian coordinates attached to the moving frame, and ζ is rewritten in terms of them. This expression is formally exact.

To substantiate (3.2), we suppose that the vorticity decreases with distance from the local origin $r = 0$ so rapidly that all of the moments are bounded:

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m z^n \zeta(x, z)\,\mathrm{d}x\,\mathrm{d}z\right| < \infty \quad \text{for } m, n = 0, 1, 2, \ldots. \tag{3.3}$$

For simplicity, we use z for $\hat{z}$. In order to manipulate the inner limit of (3.2), the exponential function of the operators is formally expanded in a Taylor series, giving

$$
\begin{aligned}
\psi(\rho,z) = \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}z'\zeta(x',z') \Bigg\{ 1 &+ \left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right) + \frac{1}{2!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^2 \\
&+ \frac{1}{3!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^3 + \frac{1}{4!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^4 + \frac{1}{5!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^5 \\
&+ \frac{1}{6!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^6 + \cdots \Bigg\} \psi_m(\rho,z;R).
\end{aligned} \tag{3.4}
$$

We shall find in §4 that up to $O(\epsilon^3)$ the vorticity distribution has the following dependence on the local polar coordinate θ:

$$
\zeta(x,z) = \zeta_0 + \zeta_{11}\cos\theta + \zeta_{12}\sin\theta + \zeta_{21}\cos 2\theta + \zeta_{31}\cos 3\theta, \tag{3.5}
$$

with $\zeta_0, \zeta_{11}, \zeta_{12}, \zeta_{21}$ and ζ_{31} being functions of r and t. For ζ_{ij}, i labels the Fourier mode, with $j=1$ and 2 corresponding to $\cos i\theta$ and $\sin i\theta$ respectively. Introducing (3.5) into (3.4) and integrating with respect to θ first, we arrive at the result

$$
\begin{aligned}
\psi = \Bigg\{ &\sum_{n=0}^{\infty} \frac{1}{[(2n)!!]^2}\left[2\pi\int_0^{\infty} r^{2n+1}\zeta_0\,\mathrm{d}r\right]\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}\right)^n + \left[\pi\int_0^{\infty} r^2\zeta_{11}\,\mathrm{d}r\right]\frac{\partial}{\partial R} \\
&+ \frac{1}{8}\left[\pi\int_0^{\infty} r^4\zeta_{11}\,\mathrm{d}r\right]\frac{\partial}{\partial R}\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{1}{4!8}\left[\pi\int_0^{\infty} r^6\zeta_{11}\,\mathrm{d}r\right]\frac{\partial}{\partial R}\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}\right)^2 \\
&- \left[\pi\int_0^{\infty} r^2\zeta_{12}\,\mathrm{d}r\right]\frac{\partial}{\partial z} - \frac{1}{8}\left[\pi\int_0^{\infty} r^4\zeta_{12}\,\mathrm{d}r\right]\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}\right) \\
&- \frac{1}{4!8}\left[\pi\int_0^{\infty} r^6\zeta_{12}\,\mathrm{d}r\right]\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}\right)^2 + \frac{1}{4}\left[\pi\int_0^{\infty} r^3\zeta_{21}\,\mathrm{d}r\right]\left(\frac{\partial^2}{\partial R^2} - \frac{\partial^2}{\partial z^2}\right) \\
&+ \frac{1}{4!2}\left[\pi\int_0^{\infty} r^5\zeta_{21}\,\mathrm{d}r\right]\left(\frac{\partial^2}{\partial R^2} - \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}\right) \\
&+ \frac{1}{4!}\left[\pi\int_0^{\infty} r^4\zeta_{31}\,\mathrm{d}r\right]\left(\frac{\partial^3}{\partial R^3} - 3\frac{\partial^3}{\partial R\partial z^2}\right) + \cdots \Bigg\}\psi_m.
\end{aligned} \tag{3.6}
$$

The first term, summation of a series, is obtained by resorting to an ingenious technique developed by Dyson (see Appendix B). By virtue of (B 5) satisfied by ψ_m, the operation of differentiation can be considerably simplified with second-derivatives in z being replaced by derivatives in R as shown in Appendix B. Hereafter we invoke the asymptotic form of ψ_m valid in the neighbourhood of the ring ($\epsilon \ll r/R \ll 1$):

$$
\begin{aligned}
\psi_m = -\frac{\Gamma R}{2\pi}\Bigg\{ &\log\left(\frac{8R}{r}\right) + \frac{r}{2R}\left[\log\left(\frac{8R}{r}\right) - 1\right]\cos\theta \\
&+ \frac{r^2}{2^4R^2}\left(\left[2\log\left(\frac{8R}{r}\right) + 1\right] + \left[-\log\left(\frac{8R}{r}\right) + 2\right]\cos 2\theta\right) \\
&+ \frac{r^3}{2^6R^3}\left(\left[-3\log\left(\frac{8R}{r}\right) + 1\right]\cos\theta + \left[\log\left(\frac{8R}{r}\right) - \frac{7}{3}\right]\cos 3\theta\right)\Bigg\} \\
&+ \cdots .
\end{aligned} \tag{3.7}
$$

The analysis of the inner solution in §4 will tell us that the vorticity, when expressed in powers of ϵ as (2.15), takes the following form:

$$\left.\begin{aligned} \zeta_0 = \zeta^{(0)} + \epsilon^2\zeta_0^{(2)}, \quad \zeta_{11} = \epsilon\zeta_{11}^{(1)} + \epsilon^3\zeta_{11}^{(3)}, \\ \zeta_{12} = \epsilon^3\zeta_{12}^{(3)}, \quad \zeta_{21} = \epsilon^2\zeta_{21}^{(2)}, \quad \zeta_{31} = \epsilon^3\zeta_{31}^{(3)}, \end{aligned}\right\} \tag{3.8}$$

where the superscript stands for the order of perturbation and the $\zeta_{ij}^{(k)}$ are all functions of r and t. With this form, (3.7) and (3.8), along with (2.17), are substituted into (3.6) and the resulting expression is made dimensionless with use of the normalization (2.6). We use, in advance, the results $R^{(1)} = 0$ and

$$2\pi\int_0^\infty r\zeta_0^{(2)}\mathrm{d}r = 0, \tag{3.9}$$

which will be proved in Appendix C and in §8 respectively. Several useful formulae to facilitate the manipulation are included in Appendix B. We eventually arrive at the asymptotic development of the Biot-Savart law, valid to $O(\epsilon^3)$, in a region $\epsilon \ll r/R \ll 1$ surrounding the core:

$$\begin{aligned}
\psi = &-\frac{R^{(0)}\Gamma}{2\pi}\log\left(\frac{8R^{(0)}}{\epsilon r}\right) + \epsilon\left\{-\frac{\Gamma}{4\pi}\left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right) - 1\right] r\cos\theta + d^{(1)}\frac{\cos\theta}{r}\right\} \\
&+\epsilon^2\left\{-\frac{\Gamma}{2^5\pi R^{(0)}}\left(\left[2\log\left(\frac{8R^{(0)}}{\epsilon r}\right) + 1\right] r^2 - \left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right) - 2\right] r^2\cos 2\theta\right)\right. \\
&\qquad \left.+\frac{d^{(1)}}{2R^{(0)}}\left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right) + \frac{\cos 2\theta}{2}\right] - \frac{\Gamma R^{(2)}}{2\pi}\log\left(\frac{8R^{(0)}}{\epsilon r}\right) + q^{(2)}\frac{\cos 2\theta}{r^2}\right\} \\
&+\epsilon^3\left\{\frac{\Gamma}{2^7\pi(R^{(0)})^2}\left(\left[3\log\left(\frac{8R^{(0)}}{\epsilon r}\right) - 1\right] r^3\cos\theta - \left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right) - \frac{7}{3}\right] r^3\cos 3\theta\right)\right. \\
&\qquad -\frac{d^{(1)}}{8(R^{(0)})^2}\left(\left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right) - \frac{7}{4}\right] r\cos\theta + \frac{r\cos 3\theta}{4}\right) - \frac{\Gamma R^{(2)}}{4\pi R^{(0)}} r\cos\theta \\
&\qquad -\frac{1}{2\pi}\left(\frac{1}{4}\left[2\pi\int_0^\infty r^3\zeta_0^{(2)}\mathrm{d}r\right] + R^{(0)}\left[\pi\int_0^\infty r^2\zeta_{11}^{(3)}\mathrm{d}r\right] + \frac{1}{4}\left[\pi\int_0^\infty r^3\zeta_{21}^{(2)}\mathrm{d}r\right]\right)\frac{\cos\theta}{r} \\
&\qquad +\frac{q^{(2)}}{4R^{(0)}r}(\cos\theta + \cos 3\theta) - \frac{1}{\pi R^{(0)}}\left(\frac{1}{3\cdot 2^8}\left[2\pi\int_0^\infty r^7\zeta^{(0)}\mathrm{d}r\right]\right. \\
&\qquad\quad -\frac{R^{(0)}}{8\cdot 4!}\left[\pi\int_0^\infty r^6\zeta_{11}^{(1)}\mathrm{d}r\right] + \frac{(R^{(0)})^2}{4!}\left[\pi\int_0^\infty r^5\zeta_{21}^{(2)}\mathrm{d}r\right] \\
&\qquad\quad \left.\left.+\frac{(R^{(0)})^3}{6}\left[\pi\int_0^\infty r^4\zeta_{31}^{(3)}\mathrm{d}r\right]\right)\frac{\cos 3\theta}{r^3} - \frac{R^{(0)}}{2\pi}\left[\pi\int_0^\infty r^2\zeta_{12}^{(3)}\mathrm{d}r\right]\frac{\sin\theta}{r}\right\} + \cdots,
\end{aligned} \tag{3.10}$$

where

$$\Gamma = 2\pi\int_0^\infty r\zeta^{(0)}\,\mathrm{d}r \tag{3.11}$$

($\Gamma = 1$ when non-dimensionalized), and $d^{(1)}$ and $q^{(2)}$ are the strength of the dipole at

$O(\epsilon)$ and the quadrupole at $O(\epsilon^2)$:

$$d^{(1)} = -\frac{1}{2\pi}\left\{\frac{1}{4}\left[2\pi\int_0^\infty r^3\zeta^{(0)}\mathrm{d}r\right] + R^{(0)}\left[\pi\int_0^\infty r^2\zeta_{11}^{(1)}\mathrm{d}r\right]\right\}, \tag{3.12}$$

$$q^{(2)} = -\frac{1}{2\pi R^{(0)}}\left\{-\frac{1}{2^6}\left[2\pi\int_0^\infty r^5\zeta^{(0)}\mathrm{d}r\right] + \frac{R^{(0)}}{8}\left[\pi\int_0^\infty r^4\zeta_{11}^{(1)}\mathrm{d}r\right] + \frac{(R^{(0)})^2}{2}\left[\pi\int_0^\infty r^3\zeta_{21}^{(2)}\mathrm{d}r\right]\right\}. \tag{3.13}$$

The terms multiplied by Γ stem from $\Gamma\psi_m$, and only these have been previously employed as the outer solution. We now recognize that, at higher orders, the monopole field needs to be corrected by the induction velocity associated with the di-, quadru-, hexa poles ... distributed along the centreline $r = 0$ of the core. In the light of (3.12) and (3.13), the detailed profile of vorticity in the core is necessary to evaluate these multi-pole induction terms. The above formal asymptotic expression will be vital in finding a higher-order general formula for the ring speed in § 7.

Parts of (3.10) supply the matching conditions on the inner solution. The distributions of $\zeta_{11}^{(1)}$, $\zeta_0^{(2)}$, $\zeta_{21}^{(2)}$, $\zeta_{11}^{(3)}$, $\zeta_{12}^{(3)}$ and $\zeta_{31}^{(3)}$ are as yet unknown, but will be determined by the inner expansion and the matching procedure. The present scheme looks sound in that the inner solution influences the outer field, and so, by further reflection, takes part in the determination of the ring motion. It will be clarified that the dipole components $\zeta_{11}^{(1)}$, $\zeta_{11}^{(3)}$, $\zeta_{12}^{(3)}$ are distinctive in that they contain parameters, the strength of the dipole, to be prescribed. On the other hand, the strength of higher-order poles is uniquely determined once the mono- and dipole fields are given. In the subsequent sections, we shall construct the flow field in the inner region order by order.

4. Inner expansions up to second order

In this section, we recall the inner expansions at leading and first orders, developed by Tung & Ting (1967), Widnall *et al.* (1971) and Callegari & Ting (1978), and extend them to second order. In the course of the scrutiny, a few steps that have been incomplete will be identified. These will shed light on some non-trivial aspects of the problem that need to be understood in deriving the third-order velocity formula.

4.1. *Zeroth order*

The equations of motion, expanded in powers of ϵ, are presented in Appendix A. At $O(\epsilon^0)$, the Navier–Stokes equations reduce to the Jacobian form of the Euler equations (A 2):

$$[\zeta^{(0)}, \psi^{(0)}] \equiv \frac{1}{r}\frac{\partial(\zeta^{(0)}, \psi^{(0)})}{\partial(r, \theta)} = 0, \tag{4.1}$$

resulting in $\zeta^{(0)} = \mathscr{F}(\psi^{(0)})$, for some function $\mathscr{F}$.

Suppose that the flow $\psi^{(0)}$ has a single stagnation point at $r = 0$, all the streamlines being closed around that point. Then it is probable that the solution of (4.1), coupled with the ζ–ψ relation (A 7) at $O(\epsilon^0)$, must be 'radial' $\psi^{(0)} = \psi^{(0)}(r)$; the streamlines are then necessarily circles (Moffatt *et al.* 1994). A virtually equivalent statement can be mathematically proved, with rigour, by combining the theorem of Caffarelli & Friedman (1980) with that of Fraenkel (1999) (theorem 4.2 and exercise 4.20).

The functional form of $\psi^{(0)}$ and $\zeta^{(0)}$ remains undetermined at this level of approximation, but is determined through the axisymmetric (or θ-averaged) part of the vorticity equation (A 4) at $O(\epsilon^2)$:

$$\frac{\partial \zeta^{(0)}}{\partial t} = \left(\zeta^{(0)} + \frac{r}{2}\frac{\partial \zeta^{(0)}}{\partial r} \right) \frac{\dot{R}^{(0)}}{R^{(0)}} + \hat{\nu} \left(\frac{\partial^2 \zeta^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta^{(0)}}{\partial r} \right) \tag{4.2}$$

(Tung & Ting 1967). This is to be solved under some initial condition with vorticity localized in a toroidal region, around $r = 0$, of thickness σ_0.

At very large values of the Reynolds number $Re_\Gamma \gg (R_0/\sigma_0)^2$, $\hat{\nu} = 0$ (see (2.7)), and (4.2) admits the general solution

$$\zeta^{(0)} = G\left(r\sqrt{R^{(0)}} \right) R^{(0)}, \tag{4.3}$$

with G an arbitrary function. This is simply a kinematical consequence of the fact that a radial expansion of a ring causes stretching of vortex lines, in proportion to $R^{(0)}$, and, at the same time, contraction of the core area like $(R^{(0)})^{-1}$. In the event that $R^{(0)} = \text{const.}$, which is the case of present concern, (4.3) implies that the vorticity is steady, a result attributable to our assumption of a slow time scale.

On the other hand, if $\hat{\nu} = 1$, viscosity plays the role of selecting the distribution. For instance, we consider a specific initial distribution of a δ-function vorticity concentrated on the circle of radius $R^{(0)}$:

$$\zeta^{(0)} = \delta(\rho - R^{(0)})\delta(z - Z^{(0)}) \quad \text{at} \quad t = 0. \tag{4.4}$$

By a change of variables,

$$\tau = \int_0^t R^{(0)}(t')\mathrm{d}t', \quad \eta = r\sqrt{R^{(0)}}, \tag{4.5}$$

(4.2) is rendered free from the convection term, giving the fundamental solution

$$\zeta^{(0)} = \frac{R^{(0)}}{4\pi\hat{\nu}\tau} \mathrm{e}^{-\eta^2/4\hat{\nu}\tau}. \tag{4.6}$$

When $R^{(0)}$ is constant, (4.6) becomes the Lamb–Oseen diffusing vortex:

$$\zeta^{(0)} = \frac{1}{4\pi\hat{\nu}t} \mathrm{e}^{-r^2/4\hat{\nu}t} \tag{4.7}$$

(Tung & Ting 1967; Saffman 1970).

In view of (2.13), (2.14) and (A 7), the leading-order variables are related to each other through

$$u^{(0)} = \frac{1}{R^{(0)}r}\frac{\partial \psi^{(0)}}{\partial \theta}, \quad v^{(0)} = -\frac{1}{R^{(0)}}\frac{\partial \psi^{(0)}}{\partial r}, \quad \zeta^{(0)} = \frac{1}{R^{(0)}}\Delta\psi^{(0)} = -\frac{1}{r}\frac{\partial}{\partial r}(rv^{(0)}). \tag{4.8}$$

These are integrated to provide a general solution:

$$u^{(0)} = 0, \quad v^{(0)} = -\frac{1}{r}\int_0^r r'\zeta^{(0)}(r', t)\mathrm{d}r', \quad \psi^{(0)} = -R^{(0)}\int_0^r v^{(0)}(r', t)\mathrm{d}r'. \tag{4.9}$$

In the case of the Lamb–Oseen vortex, they are specialized to

$$v^{(0)} = -\frac{1}{2\pi r}(1 - \mathrm{e}^{-r^2/4\hat{\nu}t}), \quad \psi^{(0)} = \frac{R^{(0)}}{2\pi}\int_0^r \frac{1}{r'}(1 - \mathrm{e}^{-r'^2/4\hat{\nu}t})\mathrm{d}r'. \tag{4.10}$$

This solution automatically fulfils the matching condition, the leading-order part of (3.10),

$$\psi^{(0)} \sim \frac{R^{(0)}}{2\pi}\log r + \text{const.} \quad \text{as } r \to \infty. \tag{4.11}$$

This parallels the asymptotic analysis of a non-axisymmetric Burgers vortex developed by Moffatt *et al.* (1994); (4.1) at leading order is nothing other than the steady Euler equation. If $(1 \ll)\ Re_\Gamma \lesssim (R_0/\sigma_0)^2$, viscosity has a residual effect, at $O(\nu/\Gamma)$, of selecting the distribution of vorticity. The inviscid solution (4.3) for $\hat{\nu} = 0$ is reproduced by taking the limits $\nu \to 0,\ t \to \infty$ in that order. In this case, viscosity is effective on a time scale much longer than that relevant to the ring motion. The alternative ordered limiting process $t \to \infty,\ \nu \to 0$ is described by the viscous solution (4.7); this situation is reminiscent of the type of situation to which the Prandtl–Batchelor theorem applies (Batchelor 1956) when the effect of viscosity, although small, determines the appropriate leading-order solution.

The assumption (4.4) of a 'δ-function' core at $t = 0$ is of course idealized. In reality, vortex rings are formed through a succession of complicated process, involving impulsive ejection of a fluid volume and the resulting generation of a tangential discontinuity, or a very large gradient in velocity, the Kelvin–Helmholtz instability, roll-up of a vortex sheet and formation of a spiral structure (see, for example, Maxworthy 1977; Didden 1979; Pullin 1979; Nitsche & Krasny 1994; James & Madnia 1996; Wakelin & Riley 1997; Gharib, Rambod & Shariff 1998). At an initial stage, the resulting distribution of vorticity depends upon this process (cf. Moore & Saffman 1973). The relevance of our solution for practical flows generated by such fast processes may well be questionable. Experimental evidence that supports this relevance will be collected in § 6.

4.2. *First order*

Combining (A 3) with (A 8), we see that the first-order perturbation $\psi^{(1)}$ satisfies

$$(\Delta - a)\,\psi^{(1)} = -\cos\theta v^{(0)} + R^{(0)} r a(\dot{Z}^{(0)}\cos\theta - \dot{R}^{(0)}\sin\theta) + 2r\zeta^{(0)}\cos\theta, \tag{4.12}$$

where

$$a(r,t) = -\frac{1}{v^{(0)}}\frac{\partial\zeta^{(0)}}{\partial r}. \tag{4.13}$$

Here we have anticipated that $\zeta_0^{(1)} = 0$, from analysis of the vorticity equation at $O(\epsilon^3)$. The proof of this is relegated to Appendix C.

The solution satisfying the condition (2.19) that the relative velocities $(u^{(1)}, v^{(1)})$ are finite at $r = 0$ is explicitly written down in the following way. The θ-dependence is

$$\psi^{(1)} = \psi_{11}^{(1)}\cos\theta + \psi_{12}^{(1)}\sin\theta. \tag{4.14}$$

The Fourier coefficients are conveniently decomposed into two parts:

$$\psi_{11}^{(1)} = \tilde{\psi}_{11}^{(1)} - R^{(0)} r\dot{Z}^{(0)}, \quad \psi_{12}^{(1)} = \tilde{\psi}_{12}^{(1)} + R^{(0)} r\dot{R}^{(0)}. \tag{4.15}$$

The equations for $\tilde{\psi}_{11}^{(1)}$ and $\tilde{\psi}_{12}^{(1)}$ then become

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{1}{r^2} + a\right)\right]\tilde{\psi}_{11}^{(1)} = -v^{(0)} + 2r\zeta^{(0)}, \tag{4.16}$$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{1}{r^2} + a\right)\right]\tilde{\psi}_{12}^{(1)} = 0. \tag{4.17}$$

By inspection, we find that $v^{(0)}$ is a solution of (4.17) (Widnall *et al.* 1971; Callegari & Ting 1978). The general solution is then immediately obtainable as

$$\tilde{\psi}_{11}^{(1)} = \Psi_{11}^{(1)} + c_{11}^{(1)} v^{(0)}, \quad \tilde{\psi}_{12}^{(1)} = c_{12}^{(1)} v^{(0)}, \tag{4.18}$$

where $\Psi_{11}^{(1)}$ is a particular integral of (4.16):

$$\Psi_{11}^{(1)} = v^{(0)} \int_0^r \frac{\mathrm{d}r'}{r'[v^{(0)}(r')]^2} \left\{ \int_0^{r'} r'' v^{(0)}(r'')[-v^{(0)}(r'') + 2r''\zeta^{(0)}(r'')]\mathrm{d}r'' \right\}$$

$$= -v^{(0)} \left\{ \frac{r^2}{2} + \int_0^r \frac{\mathrm{d}r'}{r'[v^{(0)}(r')]^2} \int_0^{r'} r'' \left[v^{(0)}(r'')\right]^2 \mathrm{d}r'' \right\}, \tag{4.19}$$

and $c_{11}^{(1)}$ and $c_{12}^{(1)}$ are constants (which may depend on t).

The matching condition, a part of the $O(\epsilon)$-terms of (3.10), that has been employed in previous work,

$$\psi^{(1)} \sim -\frac{1}{4\pi} \left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right) - 1 \right] r\cos\theta \quad \text{as } r \to \infty, \tag{4.20}$$

gives rise to

$$\dot{R}^{(0)} = 0, \tag{4.21}$$

and

$$\dot{Z}^{(0)} = \frac{1}{4\pi R_0} \left[\log\left(\frac{8R_0}{\epsilon}\right) - \tfrac{1}{2} + A \right], \tag{4.22}$$

where

$$A = \lim_{r\to\infty} \left\{ 4\pi^2 \int_0^r r'[v^{(0)}(r')]^2\,\mathrm{d}r' - \log r \right\}. \tag{4.23}$$

In (4.22) and henceforth, we use, with some abuse of notation, R_0 in place of $R^{(0)}$. The translation velocity (4.22), along with (4.23), is identical with (1.3). In the case of the Lamb–Oseen vortex (4.7) or (4.10),

$$A = \log\left(\frac{1}{2\sqrt{\hat{\nu} t}}\right) + \tfrac{1}{2}(\gamma - \log 2), \tag{4.24}$$

and (4.22) reduces to (1.4). Unfortunately the two parameters $c_{11}^{(1)}$ and $c_{12}^{(1)}$ remain undetermined. Setting this aside, once the streamfunction is known, the distribution of vorticity is calculable through

$$\zeta^{(1)} = \frac{1}{R_0}(a\tilde{\psi}_{11}^{(1)} + r\zeta^{(0)})\cos\theta + \frac{a}{R_0}\tilde{\psi}_{12}^{(1)}\sin\theta. \tag{4.25}$$

To have an idea on this indeterminacy, we revisit the discrete model in an inviscid flow initially studied by Kelvin (1867) and Dyson (1893). This is the case amenable to analysis in explicit form. The leading-order flow is 'the Rankine vortex', that is, a straight circular vortex tube of unit radius surrounded by an irrotational flow:

$$\zeta^{(0)} = \begin{cases} \dfrac{1}{\pi} \\ 0, \end{cases} \qquad v^{(0)} = \begin{cases} -\dfrac{r}{2\pi} & (r \leqslant 1) \\ -\dfrac{1}{2\pi r} & (r > 1). \end{cases} \tag{4.26}$$

The choice

$$c_{11}^{(1)} = \tfrac{5}{8}, \quad c_{12}^{(1)} = 0, \tag{4.27}$$

ensures continuity of the relative velocity across the core boundary ($r = 1$) to $O(\epsilon)$ (Widnall *et al.* 1971). Still the difficulty remains unresolved when a continuous distribution of vorticity takes the place of the discrete one, because the continuity condition on velocity is no longer of help. To make matters worse, both $c_{11}^{(1)}$ and $c_{12}^{(1)}$ admit arbitrary time dependence as long as we apply the matching condition (4.20).

Observe now that, with spatial translation of the origin of the moving frame by $\epsilon\alpha$ in the ρ-direction and $\epsilon\beta$ in the z-direction, the streamfunction, when redefined as a function of the relative coordinates, is altered to

$$\begin{aligned}
&\psi\left(\rho-(R_0+\epsilon^2\alpha), z-(Z_0+\epsilon^2\beta)\right)\\
&\quad=\psi^{(0)}(\rho-R_0, z-Z_0)+\epsilon\left[\psi^{(1)}(\rho-R_0, z-Z_0)-\alpha\,\epsilon\frac{\partial\psi^{(0)}}{\partial\rho}(\rho-R_0, z-Z_0)\right.\\
&\qquad\left.-\beta\,\epsilon\frac{\partial\psi^{(0)}}{\partial z}(\rho-R_0, z-Z_0)\right]+O(\epsilon^2)\\
&\quad=\psi^{(0)}+\epsilon\left(\psi^{(1)}+\alpha R_0 v^{(0)}\cos\theta+\beta R_0 v^{(0)}\sin\theta\right)+O(\epsilon^2).
\end{aligned} \tag{4.28}$$

Comparison of (4.28) with (4.15) and (4.18) suggests that, when measured in terms of the inner variable, $c_{11}^{(1)}$ is tied in with shift of the moving frame radially outward by $\epsilon c_{11}^{(1)}/R_0$ and $c_{12}^{(1)}$ with shift in the axial direction by $\epsilon c_{12}^{(1)}/R_0$. Alternatively, $c_{11}^{(1)}$ and $c_{12}^{(1)}$ may be thought of as the parameters placing the circular core in a given moving frame, to an accuracy of $O(\epsilon)$, in terms of the inner spatial scale. In § 5, we shall demonstrate that this is indeed the case, though the proof is given only for the radial displacement. The axial displacement is trivial, whereas the radial one is necessarily accompanied by the stretching or contraction of vortex lines as a whole. Accordingly, the latter cannot be expressible solely by the Taylor expansions.

Without loss of generality, we may put

$$c_{12}^{(1)}=0, \tag{4.29}$$

which avoids unnecessary complication of the subsequent analysis. Still, a freedom of choice of radial location of the centre is at our disposal. There is a link between the location of the core centre and the strength of the dipole; in conjunction with the outward displacement by $\epsilon\alpha$ of the circular core in the axial plane, a crescent region arises, with thickness of order $\epsilon\alpha$ just outside the original core, where vorticity is gained. In compensation, a crescent region on the inner side of the core loses vorticity, or equivalently gains equal negative vorticity. It follows that a shift of the core gives rise to a vortex pair which induces, when viewed locally, a dipole field. Fixing the initial location of the core or $c_{11}^{(1)}(0)$ is equivalent to specification of the initial strength $d^{(1)}(0)$ of the dipole defined by (3.12). Accordingly, (4.20) should be superseded by

$$\psi^{(1)}\sim\left\{-\frac{1}{4\pi}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-1\right]r+\frac{d^{(1)}(t)}{r}\right\}\cos\theta \quad \text{as } r\to\infty, \tag{4.30}$$

with which a proper formulation of the initial-value problem (or the steady problem if $v=0$) is completed.

Yet, we have no way of finding the temporal evolution of $d^{(1)}(t)$ for $t>0$. In effect, it may be arbitrary. We can verify that whatever the evolution of $d^{(1)}(t)$ or $c_{11}^{(1)}(t)$ for $t>0$ may be, this arbitrariness is consistently absorbed at third order, producing the same radial velocity $\dot{R}^{(2)}+\dot{c}_{11}^{(1)}/R_0$ of the ring (Fukumoto 2000). This implies that the asymptotic solution of the Navier–Stokes equations is unique but that its representation entails some freedom. The ring, a real entity, is ignorant of the artificially allocated coordinates. The speed of the ring is expressed, in an infinite variety of ways, as the sum of the speed of the moving coordinates and that of the ring in this frame. This situation is reminiscent of the well-known fact that the gravitational potential of a compact solid contains a dipole term unless the origin of coordinates is placed at the centroid (Jeffreys & Jeffreys 1956).

It is illustrative to draw the streamlines of the steady inviscid solution perturbing the Rankine vortex (4.26). With the choice of (4.27), we have the streamfunctions, relative to the frame moving with the speed given by (4.22) with $A = 1/4$, to first order:

$$\psi^{(0)} = \begin{cases} \dfrac{R_0}{4\pi}\left[r^2 - 1 - 2\log\left(\dfrac{8R_0}{\epsilon}\right)\right] & (r \leqslant 1) \\ -\dfrac{R_0}{2\pi}\log\left(\dfrac{8R_0}{\epsilon r}\right) & (r > 1), \end{cases} \tag{4.31}$$

$$\tilde{\psi}_{11}^{(1)} = \begin{cases} \dfrac{5}{16\pi}(r^3 - r) & (r \leqslant 1) \\ \dfrac{1}{4\pi}\left[r\log r + \dfrac{3}{4}\left(r - \dfrac{1}{r}\right)\right] & (r > 1), \end{cases} \tag{4.32}$$

where we have used

$$a = -2\delta(r - 1). \tag{4.33}$$

Figure 2(*a*) displays the streamlines $\psi^{(0)} = \text{const.}$, figure 2(*b*) $\tilde{\psi}_{11}^{(1)}\cos\theta = \text{const.}$, and figure 2(*c*) $\psi^{(0)} + \epsilon\tilde{\psi}_{11}^{(1)}\cos\theta = \text{const.}$ For clarity, the rather large value $\epsilon = 0.5$ is chosen. As expected, the origin $r = 0$ coincides with the centre of the circular core. More remarkably, figure 2(*b*) exhibits a dipole flow associated with a pair of antiparallel vortices. Its source is twofold. One is the 'apparent' dipole due to displacement of the centre as discussed above. The other has a kinematical origin. When a columnar vortex is bent to form a torus, vortex lines are stretched on the convex side, and contracted on the concave side. As a consequence, the vorticity is enhanced on the convex side but is diminished on the concave side, implying the creation of a vortex pair at $O(\epsilon)$. Indeed, (4.25) gives

$$\zeta^{(1)} = \begin{cases} \dfrac{r}{\pi R_0}\cos\theta & (r \leqslant 1) \\ 0 & (r > 1). \end{cases} \tag{4.34}$$

We speculate that the dipole is a key property of a curved vortex tube. A vortex ring may be locally considered as a line of dipoles based at the core centreline embedded in the flow field induced by the circular line vortex. The driving mechanism of the self-propulsion is not only convection due to the flow of the circular vortex but also induction due to the effective vortex pair. The dipole strength depends upon the distribution of vorticity in the core, and this is one of the reasons why we are concerned with the inner field.

We end this subsection with an alternative derivation of the translation velocity (4.22). The following procedure allows us to skip the solution of (4.12), and will actually be crucial in getting a general formula for the third-order correction to the translation velocity in § 7.

By substitution from (A 8), the $\sin\theta$-part of (A 3) takes the following form, a step before (4.12):

$$\frac{1}{r}\left\{\frac{\partial\zeta^{(0)}}{\partial r}\psi_{11}^{(1)} + v^{(0)}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi_{11}^{(1)}\right)\right]\right\} - 2\zeta^{(0)}v^{(0)} + \frac{[v^{(0)}]^2}{r} + R_0\dot{Z}^{(0)}\frac{\partial\zeta^{(0)}}{\partial r} = 0. \tag{4.35}$$

Multiplying by r^2 and integrating with respect to r from 0 to some large value, say L, we have

$$\dot{Z}^{(0)} = \frac{\pi}{R_0\left[2\pi\int_0^L r\zeta^{(0)}\,\mathrm{d}r\right]}\left\{ v^{(0)}\frac{\partial}{\partial r}\left(r\psi_{11}^{(1)}\right)\bigg|_{r=L} + \int_0^L v^{(0)}\left(-2\zeta^{(0)}r^2 + v^{(0)}r\right)\mathrm{d}r\right\}. \quad (4.36)$$

Remembering that $2\pi\int_0^\infty r\zeta^{(0)}\,\mathrm{d}r = \Gamma = 1$ and $v^{(0)} \sim -1/(2\pi r)$ as $r \to \infty$ from (4.9) and using the matching condition (4.30), the limiting form of (4.36) as $L \to \infty$ recovers (4.22) and (4.23).

4.3. *Second order*

The second-order equations (A 4), cleared of the axisymmetric part, and (A 9) are coupled to yield

$$\Delta\psi^{(2)} = R_0\zeta^{(2)} + \frac{ra}{R_0}\psi^{(1)}\cos\theta + \left[\frac{ar^2\dot{Z}^{(0)}}{2} + \frac{1}{2R_0}(rv^{(0)} + r^2\zeta^{(0)})\right](1+\cos 2\theta) + \frac{1}{2R_0}\left(\frac{\partial\psi_{11}^{(1)}}{\partial r} + \frac{\psi_{11}^{(1)}}{r}\right) + \frac{1}{2R_0}\left(\frac{\partial\psi_{11}^{(1)}}{\partial r} - \frac{\psi_{11}^{(1)}}{r}\right)\cos 2\theta, \quad (4.37)$$

where

$$\zeta^{(2)} = \zeta_0^{(2)} + \left\{\frac{a}{R_0}\psi_{21}^{(2)} + \frac{b}{4R_0^2}(\psi_{11}^{(1)})^2 + \frac{r}{2R_0^2}(a + R_0 b\dot{Z}^{(0)})\psi_{11}^{(1)} + \frac{1}{4}\left[b(\dot{Z}^{(0)})^2 + \frac{3}{R_0}a\dot{Z}^{(0)}\right]r^2\right\}\cos 2\theta, \quad (4.38)$$

and

$$b(r,t) = -\frac{1}{v^{(0)}}\frac{\partial a}{\partial r}. \quad (4.39)$$

No condition is available, to this order, to determine the distribution $\zeta_0^{(2)}$ of the axisymmetric vorticity component at $O(\epsilon^2)$. Its functional form will be found by solving the convection–diffusion equation for $\zeta_0^{(2)}$ obtained from the solvability condition for (A 6), the fourth-order equation. This procedure is deferred to § 8.

The above equations reveal that the second-order perturbation $\psi^{(2)}$ comprises mono- and quadrupole terms:

$$\psi^{(2)} = \psi_0^{(2)} + \psi_{21}^{(2)}\cos 2\theta. \quad (4.40)$$

The latter reflects an elliptical core deformation. The governing equation for the monopole is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\psi_0^{(2)} = R_0\zeta_0^{(2)} + \frac{ra}{2R_0}\tilde{\psi}_{11}^{(1)} + \frac{1}{2R_0}\left[rv^{(0)} + r^2\zeta^{(0)} + \frac{\partial\psi_{11}^{(1)}}{\partial r} + \frac{\psi_{11}^{(1)}}{r}\right] + R^{(2)}\zeta^{(0)}. \quad (4.41)$$

The boundary conditions are the finiteness of velocity at the origin (2.19),

$$\psi_0^{(2)} \propto r \quad \text{as } r \to 0, \quad (4.42)$$

and the matching condition, a part of (3.10) at $O(\epsilon^2)$,

$$\psi_0^{(2)} \sim -\frac{1}{2^4\pi R_0}\left[\log\left(\frac{8R_0}{\epsilon r}\right)+\frac{1}{2}\right]r^2+\left(\frac{d^{(1)}}{2R_0}-\frac{R^{(2)}}{2\pi}\right)\log\left(\frac{8R_0}{\epsilon r}\right) \quad \text{as } r\to\infty. \tag{4.43}$$

Under these conditions, (4.41) is readily integrated once to give

$$\frac{\partial \psi_0^{(2)}}{\partial r} = \frac{R_0}{r}\int_0^r r'\zeta_0^{(2)}\mathrm{d}r' + \frac{1}{2R_0 r}\int_0^r r'^2 a\tilde{\psi}_{11}^{(1)}\mathrm{d}r' + \frac{\psi_{11}^{(1)}}{2R_0} + \left(\frac{r^2}{4R_0}-R^{(2)}\right)v^{(0)} + \frac{3}{4R_0 r}\int_0^r r'^3\zeta^{(0)}\mathrm{d}r', \tag{4.44}$$

where (4.30) has been used. Integrating (4.16), after multiplication by r^2, we have

$$\int_0^r r'^2 a\tilde{\psi}_{11}^{(1)}\mathrm{d}r' = r^2\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r} - r\tilde{\psi}_{11}^{(1)} + \frac{r^3}{2}v^{(0)} - \frac{3}{2}\int_0^r r'^3\zeta^{(0)}\mathrm{d}r'. \tag{4.45}$$

This helps to further reduce (4.44) to

$$\frac{\partial \psi_0^{(2)}}{\partial r} = \frac{R_0}{r}\int_0^r r'\zeta_0^{(2)}\mathrm{d}r' + \frac{r}{2R_0}\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r} + \left(\frac{r^2}{2R_0}-R^{(2)}\right)v^{(0)} - \frac{r}{2}\dot{Z}^{(0)}. \tag{4.46}$$

Next we turn to the quadrupole $\psi_{21}^{(2)}\cos 2\theta$. It is simpler to deal with the streamfunction $\tilde{\psi}^{(2)}$ for the flow relative to the moving coordinate frame. This is available by subtracting $-\dot{Z}^{(0)}\rho^2/2$, with ρ provided by (2.11), corresponding to the uniform flow, and thus defining

$$\tilde{\psi}^{(2)} = \tilde{\psi}_0^{(2)} + \tilde{\psi}_{21}^{(2)}\cos 2\theta, \tag{4.47}$$

where

$$\psi_0^{(2)} = \tilde{\psi}_0^{(2)} - \tfrac{1}{4}\dot{Z}^{(0)}r^2, \tag{4.48}$$

$$\psi_{21}^{(2)} = \tilde{\psi}_{21}^{(2)} - \tfrac{1}{4}\dot{Z}^{(0)}r^2. \tag{4.49}$$

Then (4.37) and (4.38) simplify for $\tilde{\psi}_{21}^{(2)}$ to

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}-a\right)\tilde{\psi}_{21}^{(2)} = \frac{b}{4R_0}(\tilde{\psi}_{11}^{(1)})^2 + \frac{ra}{R_0}\tilde{\psi}_{11}^{(1)} + \frac{1}{2R_0}\left(rv^{(0)} + r^2\zeta^{(0)} + \frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r} - \frac{\tilde{\psi}_{11}^{(1)}}{r}\right). \tag{4.50}$$

The boundary conditions demand that

$$\tilde{\psi}_{21}^{(2)} \propto r^2 \quad \text{as } r\to 0, \tag{4.51}$$

$$\tilde{\psi}_{21}^{(2)} \sim \frac{r^2}{4}\left\{\dot{Z}^{(0)} + \frac{1}{8\pi R_0}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-2\right]\right\} + \frac{d^{(1)}}{4R_0} \quad \text{as } r\to\infty. \tag{4.52}$$

Once the streamfunctions are available, the vorticity distribution is calculable from (4.38), or equivalently,

$$\zeta^{(2)} = \zeta_0^{(2)} + \left[\frac{a}{R_0}\tilde{\psi}_{21}^{(2)} + \frac{b}{4R_0^2}(\tilde{\psi}_{11}^{(1)})^2 + \frac{ra}{2R_0^2}\tilde{\psi}_{11}^{(1)}\right]\cos 2\theta. \tag{4.53}$$

This distribution is sensitive to the displacement of the core centre, as represented by (4.28). It is noteworthy that (4.50) is a natural extension of the quadrupole equation encountered by Moffatt *et al.* (1994) and Jiménez *et al.* (1996) for non-axisymmetric Burgers and Lamb–Oseen vortices respectively.

In a general case, it is unlikely that (4.50) can be further integrated analytically. The numerical computation is postponed to a subsequent paper, but we content ourselves with an explicit solution for Dyson's model. The axisymmetric part of the vorticity, along with $R^{(2)}$, is shown to be suppressed: $\zeta_0^{(2)} = R^{(2)} = 0$. Then the monopole field, the integral of (4.46), becomes

$$\tilde{\psi}_0^{(2)} = \frac{1}{\pi R_0} \begin{cases} \dfrac{7}{128}r^4 - \dfrac{5}{64}r^2 - \dfrac{3}{128} & (r < 1) \\ \dfrac{1}{16}\left(\log r - \dfrac{3}{4}\right) r^2 + \dfrac{3}{32}\log r & (r > 1), \end{cases} \tag{4.54}$$

and the quadrupole field is

$$\tilde{\psi}_{21}^{(2)} = \frac{1}{\pi R_0} \begin{cases} \dfrac{3}{64}r^4 + \dfrac{1}{16}\left[3\log\left(\dfrac{8R_0}{\epsilon}\right) - 5\right] r^2 & (r < 1) \\ \dfrac{1}{32}\left[3\log\left(\dfrac{8R_0}{\epsilon}\right) - \log r - \dfrac{5}{2}\right] r^2 - \dfrac{3}{64} & \\ \qquad + \dfrac{3}{32}\left[\log\left(\dfrac{8R_0}{\epsilon}\right) - \dfrac{3}{2}\right]\dfrac{1}{r^2} & (r > 1). \end{cases} \tag{4.55}$$

These reproduce the velocity field at $O(\epsilon^2)$ obtained, in a different manner, by Widnall & Tsai (1977). The streamlines $\tilde{\psi}^{(2)} = \text{const.}$ are shown in figure 3. This resembles the pattern of a pure strain.

We notice that the right-hand sides of (4.41) and (4.50), along with the boundary conditions, accommodate no terms proportional to $(\log \epsilon)^2$. This fact lends some support to our perturbation scheme that dispenses with additional expansions in powers of $\log \epsilon$.

5. Radial expansion

We are now at a stage where we can tackle the third-order problem. An analysis of (A 5) and (A 10) leads us to the following form of the streamfunction $\psi^{(3)}$:

$$\psi^{(3)} = \psi_{11}^{(3)} \cos\theta + \psi_{12}^{(3)} \sin\theta + \psi_{31}^{(3)} \cos 3\theta. \tag{5.1}$$

A dipole field again appears as the result of nonlinear interactions among the mono-, di- and quadrupoles of lower orders. It is the $\cos\theta$ component that takes part in the correction to the ring speed at $O(\epsilon^3)$. The $\sin\theta$ component is responsible for the radial velocity. We begin with the derivation of the radial velocity, or radial expansion of the ring, and follow with the derivation of a general formula for the translation velocity in § 7.

It is not difficult to get $\dot{R}^{(2)}$ from equation for $\psi_{12}^{(3)}$ in much the same way as getting $\dot{Z}^{(0)}$ from (4.35) as expounded toward the end of § 4.2. However, by appeal to a fundamental conservation law, we can determine it without having to calculate the third-order flow field. Recall that the hydrodynamic impulse $\boldsymbol{P}$ is conserved, regardless of the inviscid or viscous character of the flow (Lamb 1932; Saffman 1992):

$$\boldsymbol{P} = \frac{1}{2}\iiint \boldsymbol{x} \times \boldsymbol{\omega}\, \mathrm{d}V = \text{const.} \tag{5.2}$$

In the present axisymmetric problem, only the axial component P_z is non-zero, and

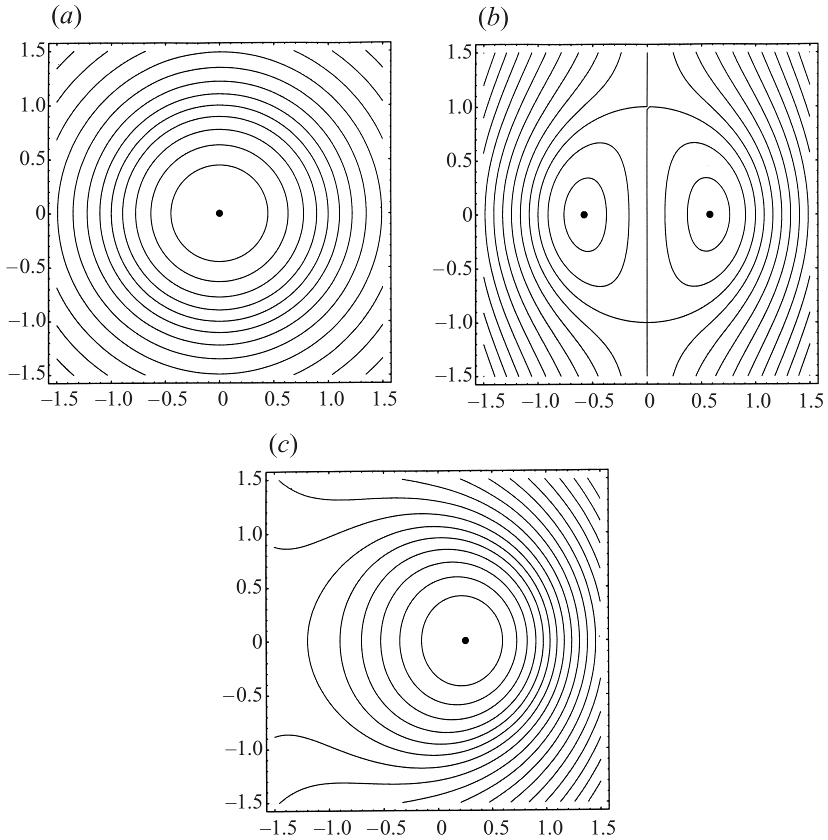

FIGURE 2. Streamline patterns of the inviscid vortex ring with uniform ζ/ρ, relative to the frame moving at the speed U given by (1.1), up to $O(\epsilon)$, as obtained by (4.31) and (4.32). The small parameter $\epsilon = 0.5$. (*a*) $\psi^{(0)} = \text{const.}$, (*b*) $\tilde{\psi}^{(1)}_{11}\cos\theta = \text{const.}$, (*c*) $\psi^{(0)} + \epsilon\tilde{\psi}^{(1)}_{11}\cos\theta = \text{const.}$

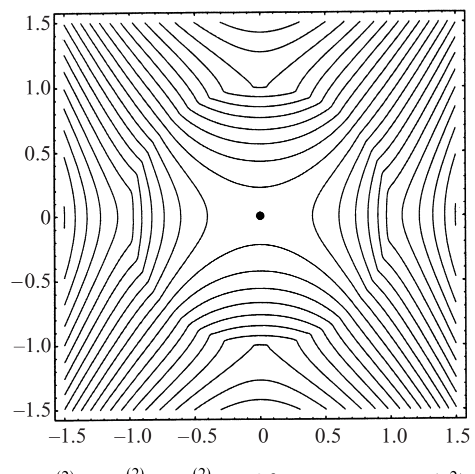

FIGURE 3. Streamline pattern $\tilde{\psi}^{(2)} = \tilde{\psi}^{(2)}_0 + \tilde{\psi}^{(2)}_{21}\cos 2\theta = \text{const.}$, at $O(\epsilon^2)$, of the inviscid vortex ring with uniform ζ/ρ, relative to the frame moving with the speed (1.1). The components $\tilde{\psi}^{(2)}_0$ and $\tilde{\psi}^{(2)}_{21}$ are given by (4.54) and (4.55), respectively.

upon substitution from (2.3), (2.15) and (2.17), it becomes, after normalization by ΓR_0^2,

$$P_z = \pi R_0^2 + \epsilon^2 P^{(2)} + \cdots, \tag{5.3}$$

where, assuming (3.9) (see § 8),

$$P^{(2)} \equiv \pi \left(2R_0 R^{(2)}(t) - 4\pi d^{(1)}(t)\right), \tag{5.4}$$

which should be constant throughout the time evolution. An appropriate initial condition for $R^{(2)}$ is

$$R^{(2)} = 0 \quad \text{at } t = 0. \tag{5.5}$$

It follows that $P^{(2)} = -4\pi^2 d^{(1)}(0)$, and the radial motion is completely ruled by the evolution of the dipole strength $d^{(1)}(t)$:

$$R^{(2)}(t) = \frac{2\pi}{R_0}[d^{(1)}(t) - d^{(1)}(0)]. \tag{5.6}$$

This derivation shows that our formulation of the initial-value problem relates to the fundamental laws of conservation of not merely circulation but also hydrodynamic impulse.

Now we can confirm that increase of $c_{11}^{(1)}$, the coefficient of the homogeneous solution (4.18), by $R_0\alpha(t)$, amounts to shift of the core centre radially outwards by $\alpha(t)$ (see (4.28)). In view of (4.25) and (3.12), $\zeta_{11}^{(1)}$ is increased by $\alpha a v^{(0)}$, resulting in a change in $d^{(1)}$ as

$$d^{(1)}(t) \to d^{(1)}(t) - \frac{\alpha(t)R_0}{2\pi}. \tag{5.7}$$

In accordance, (5.6) gives way to

$$\left[R^{(2)}(t) + \alpha(t)\right] - \alpha(0) = \frac{2\pi}{R_0}[d^{(1)}(t) - d^{(1)}(0)], \tag{5.8}$$

showing the expected increase in the radial position of the centre.

The radial motion is explicitly deducible in specific cases. In the inviscid case ($\hat{\nu} = 0$), $\zeta^{(0)}$ is steady so that the time dependence of $\psi_{11}^{(1)}$ occurs only through $c_{11}^{(1)}$, say, $c_{11}^{(1)}(t) = c_{11}^{(1)}(0) + R_0\alpha(t)$ with $\alpha(0) = 0$. Then $d^{(1)}(t)$ in (5.8), which has nothing to do with $\alpha(t)$, is a constant. Hence

$$R^{(2)}(t) + \alpha(t) = 0 \quad \text{for } t \geqslant 0. \tag{5.9}$$

The vortex ring undergoes no radial expansion.

For a viscous vortex ring ($\hat{\nu} = 1$), we concentrate on the vorticity distribution (4.7) starting from a δ-function core. In this case, $P_z = \pi R_0^2$ identically and hence the $O(\epsilon^2)$ correction term is absent: $P^{(2)}(t) = 0$ for $t \geqslant 0$. The initial condition (5.5) requires that $d^{(1)}(0) = 0$. The numerical evaluation of the behaviour of $\Psi_{11}^{(1)}$, for large r, is carried out with ease to yield

$$\Psi_{11}^{(1)} = \frac{r}{4\pi}\left[\log r + \lim_{r\to\infty}\left(4\pi^2\int_0^r r'[v^{(0)}(r')]^2 \mathrm{d}r' - \log r\right) + \frac{1}{2}\right] + \frac{D^{(1)}}{r} + \cdots, \tag{5.10}$$

where

$$D^{(1)}(t) \approx 0.41225489 \times t. \tag{5.11}$$

The first-order streamfunction with $c_{11}^{(1)} = 0$ corresponds to a dipole field whose stagnation point is permanently located at $r = 0$ (Klein & Knio 1995). By identifying $D^{(1)}(t)$ with $d^{(1)}(t)$ in (5.6) and restoring dimensional variables, we conclude that, given initially a circular line vortex of radius R_0, the stagnation point $\rho = \tilde{R}_s(t)$ in the core, viewed from the comoving coordinates, drifts outwards linearly in time owing to the action of viscosity as

$$\tilde{R}_s \approx R_0 + 2.5902739\,\frac{\nu t}{R_0}. \tag{5.12}$$

We have shown that, by appealing to conservation of impulse, the first-order streamfunction directly gives the answer. In Appendix D, we show how to derive (5.6) via the third-order equation for $\psi_{12}^{(3)}$, to confirm the above result.

The analysis is extended to related expansion laws. The first-order streamfunction for the flow viewed from the laboratory coordinate system in which the flow is at rest at infinity is augmented by $-R_0 r\dot{Z}^{(0)}\cos\theta$ as given by (4.14) with (4.15) and $\psi_{12}^{(1)} = 0$. Substitution from (4.10) yields the Taylor expansion of (4.19) for small values of r as

$$\Psi_{11}^{(1)} = \frac{5r^3}{64\pi t} + o(r^3), \tag{5.13}$$

resulting in

$$\psi_{11}^{(1)} = -\left(\frac{c_{11}^{(1)}}{8\pi t} + R_0\dot{Z}^{(0)}\right) r + \frac{5r^3}{64\pi t} + \cdots. \tag{5.14}$$

The choice

$$c_{11}^{(1)}(t) = -8\pi R_0\dot{Z}^{(0)} t, \tag{5.15}$$

dictates that the stagnation point be maintained at $r = 0$. In the light of (4.18), with $v^{(0)}$ provided by (4.10), and (5.10), the strength of the dipole is augmented by $-c_{11}^{(1)}/2\pi$ and is given, in the present case, by

$$d_s^{(1)}(t) = D^{(1)}(t) + 4R_0\dot{Z}^{(0)} t. \tag{5.16}$$

The second-order correction $R_s^{(2)}(t)$ to the radius of the stagnation circle is found from (5.6) with $d^{(1)}$ provided by (5.16). Substitution from Saffman's formula (1.4) tells us that the radius $R_s(t)$ of the stagnation circle grows as

$$R_s \approx R_0 + \left[2\log\left(\frac{4R_0}{\sqrt{\nu t}}\right) + 1.4743424\right]\frac{\nu t}{R_0}. \tag{5.17}$$

Unlike (5.12), (5.17) includes a term proportional to $t\log t$.

The temporal evolution of the radius $R_p(t)$ of the loop of peak vorticity is deducible by choosing $c_{11}^{(1)}(t)$ in (4.18) so that the local origin $r = 0$ of the moving frame is maintained at the maximum of $\zeta^{(0)} + \epsilon\zeta^{(1)}$. Inserting the Gaussian distribution (4.7) and (4.10) into (4.13) and using (5.13), we manipulate the Taylor expansion of (4.25) for small r as

$$\zeta^{(1)} = \frac{1}{R_0}\left[\left(\frac{c_{11}^{(1)}}{8\pi t^2} + \frac{1}{4\pi t}\right)x + O(r^2 x)\right], \tag{5.18}$$

where $x = r\cos\theta$ and $x^2 + z^2 = r^2$. The condition of maximum vorticity $\zeta^{(0)} + \epsilon\zeta^{(1)}$ at $r = 0$ (or $x = z = 0$) implies that

$$c_{11}^{(1)}(t) = -2t. \tag{5.19}$$

With this choice, the dipole strength $d_p^{(1)}(t)$ evolves as

$$d_p^{(1)}(t) \approx 4.5902739 \times \frac{t}{2\pi}. \tag{5.20}$$

Relying upon the formula (5.6), we obtain the expansion law of the peak-vorticity circle of radius $R_p(t)$, which is expressed, in terms of the dimensional variables, as

$$R_p \approx R_0 + 4.5902739\,\frac{\nu t}{R_0}. \tag{5.21}$$

We point out that the asymptotic value 4.5902739... at large Reynolds number is different to 1.65... at $Re_\Gamma = 10^4$ calculated numerically by Wang *et al.* (1994). In spite of this, at the same value of Re_Γ, their numerical value 0.5779... in the formula of the translation velocity is in close agreement with Saffman's asymptotic value 0.57796576.... They took $t\Gamma/R_0^2$ as a small parameter and developed matched asymptotic expansions valid at early times. In their analysis, the second-order non-linear effect, the elliptical deformation of the core, is left out, while the effect of viscosity is retained to $O(\epsilon^3)$. They restricted the initial condition to a δ-function core (4.4) and obtained, through a similar procedure to that in Appendix D, the radial velocity $\dot{R}_p^{(2)}$, a step before reaching (5.6). Their formula depends entirely on the specific distribution (4.7), while ours, (5.6), with the time-derivative removed, holds true for an arbitrary initial condition, and furthermore, being connected with the invariance property of the Navier–Stokes equations, is more convincing. At present, we cannot account for this discrepancy.

In an effort to determine the translation speed of a vortex ring using the so-called Lamb's transformation method, Helmholtz (1858) and Saffman (1970) for convenience introduced the definition of vorticity centroid in the axial direction. The counterpart in the radial direction is

$$R_c \equiv \frac{1}{2}\iiint \frac{(\boldsymbol{x}\times\boldsymbol{\omega})\cdot\boldsymbol{P}}{P^2}\rho\,\mathrm{d}V = \frac{\pi}{P_z}\iint \rho^3\zeta r\mathrm{d}r\mathrm{d}\theta. \tag{5.22}$$

This is decomposed for convenience as

$$\begin{aligned} R_c &= \frac{\pi}{P_z}\iint (R+\epsilon r\cos\theta)\rho^2\zeta r\,\mathrm{d}r\,\mathrm{d}\theta \\ &\approx R_0 + \epsilon^2\left\{R^{(2)} + \frac{\pi^2}{P_z}\int_0^\infty \left(2R_0 r^3\zeta^{(0)} + R_0^2 r^2\zeta_{11}^{(1)}\right)\mathrm{d}r\right\}. \end{aligned} \tag{5.23}$$

With the help of (3.12) and (5.6), (5.23) is reduced to

$$R_c(t) \approx R_0 - \frac{2\pi\epsilon^2}{R_0}d^{(1)}(0) + \frac{3\epsilon^2}{4R_0}\left[2\pi\int_0^\infty r^3\zeta^{(0)}\,\mathrm{d}r\right]. \tag{5.24}$$

For the initial 'δ-function' core, (4.7) is inserted into (5.24), yielding

$$R_c \approx R_0 + \frac{3\nu t}{R_0}, \tag{5.25}$$

again contradicting Wang *et al.*'s claim that R_c is a constant.

An alternative choice for the position of the radial centroid is

$$R_{c1} \equiv \frac{1}{\Gamma}\iint \rho\zeta\,\mathrm{d}S = \frac{1}{\Gamma}\iint \rho\zeta r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{5.26}$$

Stanaway *et al.* (1988*a,b*) gave the temporal evolution of $R_{c1}(t)$ without proof. For completeness, we give it. Repeating the same procedure as above, we have, corresponding to (5.24),

$$R_{c1}(t) \approx R_0 - \frac{2\pi\epsilon^2}{R_0}d^{(1)}(0) - \frac{\epsilon^2}{4R_0}\left[2\pi\int_0^\infty r^3\zeta^{(0)}\mathrm{d}r\right]. \tag{5.27}$$

For the Lamb–Oseen vortex at $O(\epsilon^0)$, this becomes

$$R_{c1} \approx R_0 - \frac{\nu t}{R_0}. \tag{5.28}$$

This 'shrinking' law was satisfactorily observed in the full numerical simulations of Stanaway *et al.* We see that the radial drift of vorticity distribution is sensitive to the definition. The first one $R_c(t)$ moves outwards in conformity with the circles of relative and non-relative stagnation points and maximum vorticity, but the opposite is true for $R_{c1}(t)$.

Plainly, our asymptotic theory is invalidated after a long time when the core is swollen and vorticity cancellation between the core regions of opposite-signed vorticity is caused. Besides the velocity formula (1.4), Saffman (1970) obtained an estimate of ring radius $R(t)$, together with the translation velocity $U(t)$, at a mature stage when νt is comparable to R_0^2, up to coefficients of order unity, based largely on judicious dimensional argument. In our notation, these estimates were

$$R \approx (R_0^2 + k'\nu t)^{1/2} \approx R_0 + \frac{k'}{2}\frac{\nu t}{R_0}, \tag{5.29}$$

$$U \approx \frac{\Gamma \pi R_0^2}{k}(R_0^2 + k'\nu t)^{-3/2}. \tag{5.30}$$

Here the constants k and k' remain indeterminate and the definition of R is obscure. We have succeeded in determining k', with the introduction of a precise definition of ring radius. It is to be noted that none of the expansion (or shrinking) laws of radial positions $\tilde{R}_s(t)$, $R_s(t)$, $R_p(t)$, $R_c(t)$ and $R_{c1}(t)$ involve the parameter Γ. They are all attributable to the effect of viscous diffusion of curved vortex lines, which is linear in ν.

6. Comparison with experiments

We can now make a comparison with some practical flows. Fortunately, there exist a few experimental results that address the distribution of vorticity, the trajectory and translation velocity of laminar vortex rings.

Recall the discussion at the end of § 4.1 wherein the Gaussian distribution (4.7) of vorticity resulted from the idealized condition (4.4). However this was realized in some experiments. Sullivan, Widnall & Ezekiel (1973) measured the velocity distribution around laminar vortex rings in air, at $Re_\Gamma = 7780$ and 37900, using a laser Doppler velocimeter. The vorticity distribution, obtained by differentiating the data, resembles the Gaussian distribution, though they did not make this explicit (see also Sallet & Widmayer 1974). Recently, Weigand & Gharib (1997) made refined measurements of various quantities, at Reynolds numbers Re_Γ ranging from 830 to 1650. Their digital-particle-image-velocimetry data of laminar vortex rings in water decisively demonstrated that the vorticity profile relaxes, at a few nozzle diameters downstream from the exit, to the Gaussian distribution (4.7). On the other hand, there are arguments in favour of a different distribution (Maxworthy 1972, 1977; Saffman 1978). Reconciling of the latter with ours calls for further study.

Weigand & Gharib (1997) measured ring trajectories, in the Reynolds-number range $830 \leqslant Re_\Gamma \leqslant 1650$, using a calibrated video system. At early times in the range $6\times 10^{-3} \lesssim \nu t/R^2 \lesssim 3\times 10^{-2}$, the data of translation velocity are well fitted with (1.4). At longer times, say $\nu t/R^2 \lesssim 10^{-2}$, the data cross over to (5.30). They found that the best fit is achieved with $k = 14.4$ and $k' = 7.8$. The evolution (5.29) of ring radius with $k'/2 = 3.9$ exhibits a remarkable agreement with the asymptotic form (5.21) for the peak-vorticity circle.

Yet, this is an indirect evidence. Sallet & Widmayer (1974) detected directly the growth of ring radius of laminar, as well as turbulent, vortex rings in air in the Reynolds-number range $4 \times 10^4 \lesssim Re_\Gamma \lesssim 6 \times 10^4$. They appealed to hot-wire measurements. The identified core centre is looked upon as being close to the stagnation circle viewed from the laboratory frame. They expressed their data in terms of the ring growth rate α_s along the trajectory defined by

$$\alpha_s \approx \tan \alpha_s \equiv \frac{\Delta \rho_s}{\Delta z}, \tag{6.1}$$

where $\Delta\rho_s(t) = R_s(t) - R_0$ is the change of radius of the stagnation circle and $\Delta z(t)$ is the downstream distance travelled by the ring. The distance travelled during the time τ from the virtual initial instant is evaluated from (1.4) as $\int_0^\tau U(t)\,\mathrm{d}t$. On combining with (5.17), (6.1) becomes

$$\alpha_s \approx 8\pi \left\{ 1 + \frac{0.79513694}{\log\left(4R_0/\sqrt{\nu\tau}\right) - 0.057796576} \right\} \frac{1}{Re_\Gamma}. \tag{6.2}$$

Thus for example, the choice $Re_\Gamma = 4 \times 10^4$ and $R_0/(\nu\tau)^{1/2} = 10$ produces $\alpha_s \approx 8 \times 10^{-4}$. This value lies in the range $0.7 \times 10^{-3} \leqslant \alpha_s \leqslant 1.5 \times 10^{-3}$ read off from figures 11 and 14 of Sallet & Widmayer (1974).†

Oshima (1972) visualized vortex rings in water by use of fine grains of tin made by electrolysis. Figure 8 of her paper could capture the qualitative feature of our theory in that the radial velocity of the ring does not depend on the circulation though the translation velocity does so. However the quantitative agreement was not attained; a crude estimate from figure 8 tells us that typically $\mathrm{d}R/\mathrm{d}t \approx 0.005\ \mathrm{cm\,s^{-1}}$. Identifying R_0 with radius 0.95 cm of the orifice, (5.12) and (5.21) leads to $\mathrm{d}\tilde{R}_s/\mathrm{d}t \approx 0.027\,\mathrm{cm\,s^{-1}}$ and $\mathrm{d}R_p/\mathrm{d}t \approx 0.048\ \mathrm{cm\,s^{-1}}$, respectively. The *eye* of the vortex ring, the visualized centre, may not correspond to either of these points.

The linear law and its variant were not easy to identify for a high-Reynolds-number vortex ring produced by Watanabe *et al.* (1995), though the trend of expansion was mentioned. In a direct numerical simulation at high Reynolds numbers $Re_\Gamma = O(10^5)$, Wakelin & Riley (1997) tracked the trajectory of the circle of maximum vorticity. In their figures, a faint tendency of expansion of this circle is visible.

The radial expansion of the peak-vorticity circle can be indicated by drawing the vorticity distribution to $O(\epsilon)$. Figure 4 depicts the temporal variation of $\zeta^{(0)} + \epsilon\zeta^{(1)}$ given by (4.7) and (4.25) with $\epsilon = 0.3$ or $Re_\Gamma \approx 11.111111$. We set $c_{11}^{(1)} = 2\pi D^{(1)}(t) \approx 2.5902739\,t$ and thereby keep the origin $r = 0$ at the relative stagnation point within an order of accuracy $O(\epsilon^3)$. The radial movement of $R_p(t)$ faster than $\tilde{R}_s(t)$ is clearly seen. As mentioned above, Sullivan *et al.* (1973) and Weigand & Gharib (1997) showed that the Lamb–Oseen vortex gives a good description of the measured vorticity distribution. In addition, their refined data showed a slight asymmetric deviation from the Gaussian distribution. Weigand & Gharib attributed the origin of asymmetry to vorticity diffusion between the opposite-sign vortices. From the standpoint of asymptotic expansions, the dipole structure or the effective vortex pair induced at $O(\epsilon)$, due to the curvature effect, underpins the outward shift of the point of maximum vorticity, accompanied by the asymmetric deformation of the distribution.

† Maxworthy (1977) referred to the value $\alpha \approx 1.5 \times 10^{-5}$ in § 5 of his paper. This may be a typographical error and the correct value should be 1.5×10^{-3}.

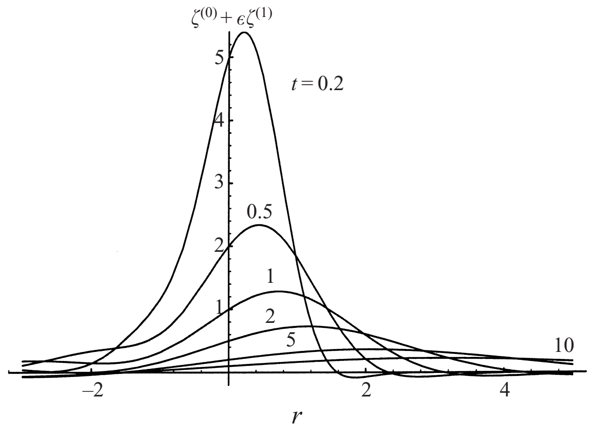

FIGURE 4. Development of vorticity distribution $\zeta^{(0)} + \epsilon\zeta^{(1)}$ of a viscous vortex ring, at indicated times, starting from a δ-function profile at $t = 0$. The small parameter $\epsilon = 0.3$, and vorticity is calculated from (4.7) and (4.25).

Notice that a region of negative vorticity appears, on the convex side of ring, for the radial distance ρ greater than a certain value. Though this overshoot is physically unrealistic, it does not necessarily imply a flaw in our theory. The same type of difficulty, non-uniformity of validity of asymptotic expansions around a Gaussian vorticity distribution, was recognized by Moffatt *et al.* (1994), Jiménez *et al.* (1996) and Prochazka & Pullin (1998), and the problem of a viscous vortex ring is no exception. In reality, (4.7) and (4.25) tend, as $r \to \infty$, to

$$\frac{\epsilon\zeta^{(1)}}{\zeta^{(0)}} \sim \frac{r\cos\theta}{R_0}\left\{1 - \frac{1}{2t}\left(-c_{11}^{(1)} + 2\pi D^{(1)} + \tfrac{1}{2}\left[\log\left(\frac{r}{2\sqrt{\nu t}}\right) + \tfrac{1}{2}(1+\gamma-\log 2)\right]r^2\right)\right\}. \tag{6.3}$$

Whatever value we adopt for the disposable parameter $c_{11}^{(1)}$, the inversion of sign of vorticity is unavoidable on the convex side ($-\pi/2 \leqslant \theta \leqslant \pi/2$) with

$$r \gtrsim \left(\frac{4R_0 t}{\epsilon}\right)^{1/3}, \quad \text{or} \quad \left[\frac{3}{\log(R_0/2\epsilon\sqrt{t})}\frac{4R_0 t}{\epsilon}\right]^{1/3} \tag{6.4}$$

in dimensionless form.

7. Formula for the third-order correction to the translation velocity

The equations to be manipulated to obtain $\dot{Z}^{(2)}$ are the coupled system of the vorticity equation (A 5) and ζ–ψ relation (A 10). For convenience, let us divide these into three parts: the terms including $\dot{Z}^{(2)}$, $\zeta_0^{(2)}$, $\psi_0^{(2)}$ and $R^{(2)}$ only which are designated by subscript a; those including $\zeta_{21}^{(2)}$ and $\psi_{21}^{(2)}$, except for $[\zeta^{(2)}, \psi^{(0)}] + [\zeta^{(0)}, \psi^{(2)}]$, designated by subscript b; and the remaining terms with subscript c.

By substitution from (4.14) with $\psi_{12}^{(1)}$ discarded, (4.40) and the corresponding representations (4.25) and (4.53) for $\zeta^{(1)}$ and $\zeta^{(2)}$, the $\sin\theta$-terms of (A 5) give

$$\frac{1}{r}\left(\frac{\partial\psi^{(0)}}{\partial r}\zeta_{11}^{(3)} - \frac{\partial\zeta^{(0)}}{\partial r}\psi_{11}^{(3)}\right) + N_a + N_b + N_c = 0. \tag{7.1}$$

Here N_a, N_b and N_c are defined and reduced in the following way:

$$
\begin{aligned}
N_a \equiv & -R_0\left(\dot{Z}^{(2)}\frac{\partial\zeta^{(0)}}{\partial r}+\dot{Z}^{(0)}\frac{\partial\zeta_0^{(2)}}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial\psi_0^{(2)}}{\partial r}\zeta_{11}^{(1)}-\frac{\partial\zeta_0^{(2)}}{\partial r}\psi_{11}^{(1)}\right)\\
&-\frac{R^{(2)}}{R_0 r}\left(\frac{\partial\psi^{(0)}}{\partial r}\zeta_{11}^{(1)}-\frac{\partial\zeta^{(0)}}{\partial r}\psi_{11}^{(1)}\right)-\frac{1}{R_0}\left(\frac{\partial\psi^{(0)}}{\partial r}\zeta_0^{(2)}+\frac{\partial\psi_0^{(2)}}{\partial r}\zeta^{(0)}-\frac{2R^{(2)}}{R_0}\frac{\partial\psi^{(0)}}{\partial r}\zeta^{(0)}\right)\\
= & -R_0\left(\dot{Z}^{(2)}\frac{\partial\zeta^{(0)}}{\partial r}+\dot{Z}^{(0)}\frac{\partial\zeta_0^{(2)}}{\partial r}\right)-R^{(2)}\left(\dot{Z}^{(0)}\frac{\partial\zeta^{(0)}}{\partial r}+\frac{v^{(0)}\zeta^{(0)}}{R_0}\right)\\
&+\frac{1}{R_0 r}\frac{\partial\psi_0^{(2)}}{\partial r}a\tilde{\psi}_{11}^{(1)}+\zeta_0^{(2)}v^{(0)}-\frac{1}{r}\frac{\partial\zeta_0^{(2)}}{\partial r}\psi_{11}^{(1)},
\end{aligned}
\tag{7.2}
$$

$$
\begin{aligned}
N_b \equiv & \frac{1}{r}\left[\frac{1}{2}\left(\frac{\partial\zeta_{21}^{(2)}}{\partial r}\psi_{11}^{(1)}-\zeta_{11}^{(1)}\frac{\partial\psi_{21}^{(2)}}{\partial r}\right)+\zeta_{21}^{(2)}\frac{\partial\psi_{11}^{(1)}}{\partial r}-\frac{\partial\zeta_{11}^{(1)}}{\partial r}\psi_{21}^{(2)}\right]\\
&+R_0\dot{Z}^{(0)}\left(\frac{1}{2}\frac{\partial\zeta_{21}^{(2)}}{\partial r}+\frac{\zeta_{21}^{(2)}}{r}\right)+\left[-\frac{v^{(0)}}{2}\zeta_{21}^{(2)}+\frac{1}{R_0}\left(\frac{1}{2}\frac{\partial\psi_{21}^{(2)}}{\partial r}+\frac{\psi_{21}^{(2)}}{r}\right)\zeta^{(0)}\right]\\
= & \frac{1}{2R_0}\left(\frac{b}{r}\tilde{\psi}_{11}^{(1)}+a\right)v^{(0)}\tilde{\psi}_{21}^{(2)}+\frac{1}{4R_0^2}\left[\frac{1}{2r}\frac{\partial b}{\partial r}\left(\tilde{\psi}_{11}^{(1)}\right)^3+\frac{2b}{r}\left(\tilde{\psi}_{11}^{(1)}\right)^2\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right.\\
&\left.+3a\tilde{\psi}_{11}^{(1)}\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}+\left(\frac{a}{r}-\frac{3bv^{(0)}}{2}\right)\left(\tilde{\psi}_{11}^{(1)}\right)^2\right]\\
&-\frac{rv^{(0)}}{4R_0^2}a\tilde{\psi}_{11}^{(1)}+\frac{\dot{Z}^{(0)}}{4R_0}\left(a\tilde{\psi}_{11}^{(1)}+r\frac{\partial a}{\partial r}\tilde{\psi}_{11}^{(1)}+ra\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}+r^2\frac{\partial\zeta^{(0)}}{\partial r}\right),
\end{aligned}
\tag{7.3}
$$

$$
\begin{aligned}
N_c \equiv & -\frac{r}{R_0}\cos\theta\left([\zeta^{(0)},\psi^{(2)}]+[\zeta^{(1)},\psi^{(1)}]+[\zeta^{(2)},\psi^{(0)}]\right)\\
&+\frac{r^2}{2R_0^2}(1+\cos\theta)\left([\zeta^{(0)},\psi^{(1)}]+[\zeta^{(1)},\psi^{(0)}]\right)-R_0\dot{Z}^{(2)}\frac{\partial\zeta^{(0)}}{\partial r}\sin\theta\\
&-\frac{1}{R_0}\left(\frac{\partial\psi^{(1)}}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}\cos\theta-\frac{r}{R_0}\frac{\partial\psi^{(0)}}{\partial r}\sin 2\theta\right)\zeta^{(1)}\\
&+\left\{\frac{r}{R_0^2}\left[\frac{\partial\psi^{(1)}}{\partial r}\sin 2\theta+\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}(1+\cos 2\theta)\right]-\frac{3r^2}{4R_0^3}\frac{\partial\psi^{(0)}}{\partial r}(\sin 3\theta+\sin\theta)\right\}\zeta^{(0)}\\
= & -\frac{r}{4R_0}\dot{Z}^{(0)}\left[\frac{\partial}{\partial r}\left(a\tilde{\psi}_{11}^{(1)}\right)-\frac{a\tilde{\psi}_{11}^{(1)}}{r}+r\frac{\partial\zeta^{(0)}}{\partial r}\right]-\frac{1}{4R_0^2}\left(\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}-\frac{\tilde{\psi}_{11}^{(1)}}{r}+rv^{(0)}\right)a\tilde{\psi}_{11}^{(1)}.
\end{aligned}
\tag{7.4}
$$

To deduce the second expressions of (7.2) and (7.3), we have made use of (A 3) and (A 4).

In an analogous fashion, the $\cos\theta$-component of (A 10) is conveniently decomposed as follows:

$$
\zeta_{11}^{(3)}=\frac{1}{R_0}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\psi_{11}^{(3)}+\zeta_{11a}^{(3)}+\zeta_{11b}^{(3)}+\zeta_{11c}^{(3)},
\tag{7.5}
$$

where

$$\zeta_{11a}^{(3)} \equiv \frac{1}{R_0}\left\{-\frac{r}{R_0}\Delta\psi_0^{(2)} - \frac{R^{(2)}}{R_0}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi_{11}^{(1)}\right)\right] + \frac{2R^{(2)}}{R_0^2}r\Delta\psi^{(0)}\right\}$$

$$-\frac{1}{R_0^2}\left(\frac{\partial\psi_0^{(2)}}{\partial r} - \frac{2R^{(2)}}{R_0}\frac{\partial\psi^{(0)}}{\partial r}\right)$$

$$= -\frac{r}{R_0}\left(\zeta_0^{(2)} + \frac{ra}{2R_0^2}\tilde{\psi}_{11}^{(1)}\right) - \frac{1}{R_0^2}\frac{\partial\psi_0^{(2)}}{\partial r} - \frac{r}{2R_0^3}$$

$$\times\left(\frac{\partial\psi_{11}^{(1)}}{\partial r} + \frac{\psi_{11}^{(1)}}{r} + rv^{(0)} + r^2\zeta^{(0)}\right) - \frac{R^{(2)}}{R_0}(a\tilde{\psi}_{11}^{(1)} + v^{(0)} + r\zeta^{(0)}), \qquad (7.6)$$

$$\zeta_{11b}^{(3)} \equiv -\frac{r}{2R_0^2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\psi_{21}^{(2)} - \frac{1}{R_0^2}\left(\frac{1}{2}\frac{\partial\psi_{21}^{(2)}}{\partial r} + \frac{\psi_{21}^{(2)}}{r}\right), \qquad (7.7)$$

$$\zeta_{11c}^{(3)} \equiv \frac{3r^2}{4R_0^2}\zeta_{11}^{(1)} + \frac{r}{2R_0^3}\left(3\frac{\partial\psi_{11}^{(1)}}{\partial r} + \frac{\psi_{11}^{(1)}}{r}\right) + \frac{3r^2}{2R_0^3}v^{(0)}$$

$$= \frac{3r^2}{4R_0^3}(a\tilde{\psi}_{11}^{(1)} + r\zeta^{(0)}) + \frac{r}{2R_0^3}\left(3\frac{\partial\psi_{11}^{(1)}}{\partial r} + \frac{\psi_{11}^{(1)}}{r}\right) + \frac{3r^2}{2R_0^3}v^{(0)}. \qquad (7.8)$$

Equations (7.1) and (7.5) are combined to yield

$$\frac{1}{r}\left\{v^{(0)}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi_{11}^{(3)}\right)\right] + \frac{\partial\zeta^{(0)}}{\partial r}\psi_{11}^{(3)}\right\} + \left(\frac{R_0v^{(0)}}{r}\zeta_{11a}^{(3)} - N_a\right)$$

$$+\left(\frac{R_0v^{(0)}}{r}\zeta_{11b}^{(3)} - N_b\right) + \left(\frac{R_0v^{(0)}}{r}\zeta_{11c}^{(3)} - N_c\right) = 0. \qquad (7.9)$$

In this expression,

$$\frac{R_0v^{(0)}}{r}\zeta_{11a}^{(3)} - N_a = R^{(2)}\dot{Z}^{(0)}\frac{\partial\zeta^{(0)}}{\partial r} + R_0\left(\dot{Z}^{(2)}\frac{\partial\zeta^{(0)}}{\partial r} + \dot{Z}^{(0)}\frac{\partial\zeta_0^{(2)}}{\partial r}\right)$$

$$-\frac{1}{r}\left\{\frac{1}{r}\left(\int_0^r r'\zeta_0^{(2)}\mathrm{d}r'\right)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi_{11}^{(1)}\right)\right] - \frac{\partial\zeta_0^{(2)}}{\partial r}\psi_{11}^{(1)}\right\}$$

$$-\frac{2}{r^2}v^{(0)}\left(\int_0^r r'\zeta_0^{(2)}\mathrm{d}r'\right)$$

$$+\frac{2}{R_0}\left(\frac{\partial\psi^{(0)}}{\partial r}\frac{1}{R_0}\Delta\psi_0^{(2)} + \frac{\partial\psi_0^{(2)}}{\partial r}\zeta^{(0)} + 2R^{(2)}v^{(0)}\zeta^{(0)}\right)$$

$$-\frac{1}{2R_0^2}\left[(a\tilde{\psi}_{11}^{(1)} + 2r\zeta^{(0)})\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r} - \frac{v^{(0)}}{r}\tilde{\psi}_{11}^{(1)} + r^2\zeta^{(0)}v^{(0)}\right]$$

$$+\frac{\dot{Z}^{(0)}}{2R_0}(a\tilde{\psi}_{11}^{(1)} - v^{(0)} + 2r\zeta^{(0)}), \qquad (7.10)$$

$$
\begin{aligned}
\frac{R_0 v^{(0)}}{r}\zeta^{(3)}_{11b} - N_b = &-\frac{v^{(0)}}{R_0 r}\left(\frac{1}{2}\frac{\partial\tilde{\psi}^{(2)}_{21}}{\partial r} + \frac{\tilde{\psi}^{(2)}_{21}}{r}\right)\\
&+\frac{1}{r}\left\{\frac{1}{2}\left(-\frac{\partial\zeta^{(2)}_{21}}{\partial r}\tilde{\psi}^{(1)}_{11} + \frac{1}{R_0}a\tilde{\psi}^{(1)}_{11}\frac{\partial\tilde{\psi}^{(2)}_{21}}{\partial r}\right) - \zeta^{(2)}_{21}\frac{\partial\tilde{\psi}^{(1)}_{11}}{\partial r}\right.\\
&\qquad\left.+\frac{1}{R_0}\left[\frac{\partial}{\partial r}(a\tilde{\psi}^{(1)}_{11}) + r\frac{\partial\zeta^{(0)}}{\partial r}\right]\tilde{\psi}^{(2)}_{21}\right\} - \frac{r v^{(0)}}{4R_0^2}a\tilde{\psi}^{(1)}_{11}\\
&-\frac{v^{(0)}}{4R_0^2}\left(r v^{(0)} + r^2\zeta^{(0)} + \frac{\partial\tilde{\psi}^{(1)}_{11}}{\partial r} - \frac{\tilde{\psi}^{(1)}_{11}}{r}\right)\\
&-\frac{\dot{Z}^{(0)}}{4R_0}\left[\left(a + r\frac{\partial a}{\partial r}\right)\tilde{\psi}^{(1)}_{11} + ra\frac{\partial\tilde{\psi}^{(1)}_{11}}{\partial r} + r^2\frac{\partial\zeta^{(0)}}{\partial r} - 2v^{(0)}\right],
\end{aligned}
\tag{7.11}
$$

$$
\begin{aligned}
\frac{R_0 v^{(0)}}{r}\zeta^{(3)}_{11c} - N_c = &\frac{1}{4R_0^2}\left(\frac{\partial\tilde{\psi}^{(1)}_{11}}{\partial r} - \frac{\tilde{\psi}^{(1)}_{11}}{r} + 4r v^{(0)}\right)a\tilde{\psi}^{(1)}_{11}\\
&+\frac{1}{2R_0^2}\left(3\frac{\partial\tilde{\psi}^{(1)}_{11}}{\partial r} + \frac{\tilde{\psi}^{(1)}_{11}}{r}\right)v^{(0)} + \frac{3}{4R_0^2}[r^2\zeta^{(0)}v^{(0)} + 2r(v^{(0)})^2]\\
&+\frac{\dot{Z}^{(0)}}{4R_0}\left[\left(r\frac{\partial a}{\partial r} - a\right)\tilde{\psi}^{(1)}_{11} + ar\frac{\partial\tilde{\psi}^{(1)}_{11}}{\partial r} + r^2\frac{\partial\zeta^{(0)}}{\partial r} - 8v^{(0)}\right].
\end{aligned}
\tag{7.12}
$$

In (7.10), (4.46) has been invoked to simplify the original expression.

With these forms, (7.10)–(7.12) are summed, and (7.9) becomes, after numerous cancellations,

$$
\begin{aligned}
&\frac{1}{r}\left\{v^{(0)}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi^{(3)}_{11}\right)\right] + \frac{\partial\zeta^{(0)}}{\partial r}\psi^{(3)}_{11}\right\}\\
&+R^{(2)}\dot{Z}^{(0)}\frac{\partial\zeta^{(0)}}{\partial r} + R_0\left(\dot{Z}^{(2)}\frac{\partial\zeta^{(0)}}{\partial r} + \dot{Z}^{(0)}\frac{\partial\zeta^{(2)}_0}{\partial r}\right)\\
&-\frac{1}{r}\left\{\frac{1}{r}\left(\int_0^r r'\zeta^{(2)}_0\mathrm{d}r'\right)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi^{(1)}_{11}\right)\right] - \frac{\partial\zeta^{(2)}_0}{\partial r}\psi^{(1)}_{11}\right\} - \frac{2v^{(0)}}{r^2}\left(\int_0^r r'\zeta^{(2)}_0\mathrm{d}r'\right)\\
&+\frac{2}{R_0}\left(\frac{1}{R_0}\frac{\partial\psi^{(0)}}{\partial r}\Delta\psi^{(2)}_0 + \frac{\partial\psi^{(2)}_0}{\partial r}\zeta^{(0)} + 2R^{(2)}v^{(0)}\zeta^{(0)}\right) - \frac{v^{(0)}}{R_0 r}\left(\frac{1}{2}\frac{\partial\tilde{\psi}^{(2)}_{21}}{\partial r} + \frac{\tilde{\psi}^{(2)}_{21}}{r}\right)\\
&+\frac{1}{r}\left\{\frac{1}{2}\left(-\frac{\partial\zeta^{(2)}_{21}}{\partial r}\tilde{\psi}^{(1)}_{11} + \frac{1}{R_0}a\tilde{\psi}^{(1)}_{11}\frac{\partial\tilde{\psi}^{(2)}_{21}}{\partial r}\right) - \zeta^{(2)}_{21}\frac{\partial\tilde{\psi}^{(1)}_{11}}{\partial r}\right.\\
&\qquad\left.+\frac{1}{R_0}\left[\frac{\partial}{\partial r}\left(a\tilde{\psi}^{(1)}_{11}\right) + r\frac{\partial\zeta^{(0)}}{\partial r}\right]\tilde{\psi}^{(2)}_{21}\right\} - \frac{1}{4R_0^2}\left[a\tilde{\psi}^{(1)}_{11}\frac{\partial\tilde{\psi}^{(1)}_{11}}{\partial r} + \frac{a}{r}\left(\tilde{\psi}^{(1)}_{11}\right)^2\right]\\
&+\frac{5v^{(0)}}{4R_0^2}\left(\frac{\partial\tilde{\psi}^{(1)}_{11}}{\partial r} + \frac{\tilde{\psi}^{(1)}_{11}}{r}\right) - \frac{r\zeta^{(0)}}{R_0^2}\frac{\partial\tilde{\psi}^{(1)}_{11}}{\partial r}\\
&+\frac{3rv^{(0)}}{4R_0^2}a\tilde{\psi}^{(1)}_{11} + \frac{5}{4R_0^2}r\left(v^{(0)}\right)^2 + \frac{\dot{Z}^{(0)}}{R_0}\left(r\zeta^{(0)} - 2v^{(0)}\right) = 0.
\end{aligned}
\tag{7.13}
$$

The non-singular condition on the relative velocity at $r = 0$ is

$$\psi_{11}^{(3)} \propto r \quad \text{as } r \to 0, \tag{7.14}$$

and the matching condition (3.10) reads

$$\psi_{11}^{(3)} \sim \frac{3}{2^7\pi R_0^2}\left[\log\left(\frac{8R_0}{\epsilon r}\right) - \frac{1}{3}\right] r^3 - \frac{d^{(1)}}{8R_0^2}\left[\log\left(\frac{8R_0}{\epsilon r}\right) - \frac{7}{4}\right] r - \frac{R^{(2)}}{4\pi R_0} r + \frac{d_1^{(3)}}{r} \quad \text{as } r \to \infty. \tag{7.15}$$

The coefficient $d_1^{(3)}$ signifies the strength of the $\cos\theta$-component of the dipole at $O(\epsilon^3)$, which is irrelevant to the determination of $\dot{Z}^{(2)}$. It deserves emphasis that the terms multiplied by $d^{(1)}$ represent the flow induced by a line of dipoles arranged on the core centreline, which has been overlooked in previous studies. This contribution may be ignored at low orders, but is indispensable for finding the correct value of the translation speed at $O(\epsilon^3)$.

The remaining task is to multiply (7.13) by r^2, integrate with respect to r, and seek the limiting form as $r \to \infty$. We assume again that (3.9) is true. Other necessary formulae are collected in Appendix E. Taking the limit $r \to \infty$, we eventually arrive at the desired formula:

$$\begin{aligned}
\dot{Z}^{(2)} = &-\frac{3d^{(1)}}{8R_0^3}\left[\log\left(\frac{8R_0}{\epsilon}\right) + \frac{4}{3}A - \frac{7}{6}\right] - \frac{P^{(2)}}{8\pi^2 R_0^3}\left[\log\left(\frac{8R_0}{\epsilon}\right) + A - \frac{3}{2}\right] \\
&-\frac{\pi}{4R_0^3}\left(B - \frac{13}{8}\int_0^\infty r^4 \zeta^{(0)} v^{(0)} \mathrm{d}r\right) \\
&-\frac{2\pi}{R_0}\int_0^\infty \left[\int_0^r r'\zeta_0^{(2)}(r')\mathrm{d}r'\right] v^{(0)}(r)\mathrm{d}r - \frac{\pi}{2R_0^2}\int_0^\infty \left(2ra + b\tilde{\psi}_{11}^{(1)}\right) r v^{(0)} \tilde{\psi}_{21}^{(2)} \mathrm{d}r \\
&+\frac{\pi}{8R_0^3}\int_0^\infty \left[ra\left(\tilde{\psi}_{11}^{(1)} - 3r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right)\tilde{\psi}_{11}^{(1)} + b\left(\tilde{\psi}_{11}^{(1)} - r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right)\left(\tilde{\psi}_{11}^{(1)}\right)^2\right] \mathrm{d}r,
\end{aligned} \tag{7.16}$$

where A and $P^{(2)}$ are given by (4.23) and (5.4), and

$$B = \lim_{r\to\infty}\left[\int_0^r r' v^{(0)}(r')\tilde{\psi}_{11}^{(1)}(r')\mathrm{d}r' + \frac{1}{16\pi^2}(\log r + A) r^2 + \frac{d^{(1)}}{2\pi}\log r\right] \tag{7.17}$$

is either a constant or some function of t. We repeat that $P^{(2)}$ does not depend on t. For instance, if a viscous vortex ($\hat{\nu} = 1$) has initially a concentrated distribution of vorticity and thus $\zeta^{(0)}$ evolves as (4.7), then $P^{(2)} = 0$ and (7.16) reduces to

$$\begin{aligned}
\dot{Z}^{(2)} = &-\frac{3d^{(1)}}{8R_0^3}\left\{\log\left(\frac{8R_0}{\epsilon}\right) + \frac{4}{3}\log\left(\frac{1}{4\sqrt{t}}\right) + \frac{2}{3}\gamma - \frac{7}{6}\right\} - \frac{\pi B}{4R_0^3} - \frac{39}{2^7\pi R_0^3}\hat{\nu} t \\
&-\frac{2\pi}{R_0}\int_0^\infty \left[\int_0^r r'\zeta_0^{(2)}(r')\mathrm{d}r'\right] v^{(0)}(r)\mathrm{d}r - \frac{\pi}{2R_0^2}\int_0^\infty \left(2ra + b\tilde{\psi}_{11}^{(1)}\right) r v^{(0)} \tilde{\psi}_{21}^{(2)} \mathrm{d}r \\
&+\frac{\pi}{8R_0^3}\int_0^\infty \left[ra\left(\tilde{\psi}_{11}^{(1)} - 3r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right)\tilde{\psi}_{11}^{(1)} + b\left(\tilde{\psi}_{11}^{(1)} - r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right)\left(\tilde{\psi}_{11}^{(1)}\right)^2\right] \mathrm{d}r.
\end{aligned} \tag{7.18}$$

The term involving the integral of $\zeta_0^{(2)}(r)$ admits an interpretation as being inherited from the first-order formula (4.22) and (4.23). In (4.46), we discard the terms unrelated to $R^{(2)}$, leaving

$$\frac{\partial \psi_0^{(2)}}{\partial r} = \frac{R_0}{r}\int_0^r r'\zeta_0^{(2)}\mathrm{d}r' - R^{(2)}v^{(0)}. \tag{7.19}$$

In (4.23), $v^{(0)}$ is generalized to the axisymmetric part v_0 of the azimuthal velocity valid to $O(\epsilon^2)$:

$$v_0 = \frac{1}{R}\frac{\partial \psi_0}{\partial r} \sim \frac{1}{R_0+\epsilon^2 R^{(2)}}\left(\frac{\partial \psi^{(0)}}{\partial r} + \epsilon^2\frac{\partial \psi_0^{(2)}}{\partial r}\right). \tag{7.20}$$

Hence

$$\begin{aligned} 4\pi^2 &\int_0^r r'[v_0(r')]^2\mathrm{d}r' \\ &\sim 4\pi^2\int_0^r r'[v^{(0)}(r')]^2\mathrm{d}r' - 8\pi^2\epsilon^2\int_0^r \left[\int_0^{r'} r''\zeta_0^{(2)}(r'')\mathrm{d}r''\right]v^{(0)}(r')\mathrm{d}r', \end{aligned} \tag{7.21}$$

giving rise to the third-order contribution.

Our formula is completed with construction of a formal solution for $\zeta_0^{(2)}$, which is the topic of the following section. If $\hat{\nu} = 0$, $\dot{Z}^{(2)}$ is constant, whereas, if $\hat{\nu} = 1$, it varies with time. In either case, numerical computation of $\tilde{\psi}_{21}^{(2)}$ and $\zeta_0^{(2)}$ and numerical integration in the above formula are required to evaluate $\dot{Z}^{(2)}$.

Dyson's vortex ring is analytically tractable. The leading-order solution is (4.26) or (4.31). The first- and second-order solutions are given by (4.32) and (4.55). With these expressions substituted into (4.23) and (3.12), $A = \frac{1}{4}$ and $d^{(1)} = -3/(16\pi)$. The latter is consistent with the form of (4.32) at large values of r. From (7.17), $B = 11/(3\times 2^7\pi^2)$. We may put $R^{(2)} = 0$ and $\zeta_0^{(2)} = 0$, and thence $P^{(2)} = 3\pi/4$ from (5.4). Introduction of these values, (4.26), (4.32) and (4.55) into the formula (7.16) yields the third-order correction in Dyson's formula (1.1). We remark that the rapid convergence of the functional form of the higher-order flow field may provide an explanation of the unexpected fact that Dyson's formula approximates even the speed of Hill's spherical vortex, the fattest limit of a family (Dyson 1893; Fraenkel 1972). At the same time, by handling this explicit solution in the integrals of (7.16), we can appreciate a potential difficulty in numerical integration which originates from a delicate cancellation of large numbers.

Stanaway *et al.* (1988*a*, *b*) performed a full numerical simulation of the Navier–Stokes equations for initial-value problems of a single vortex ring over a wide range of Reynolds numbers, $0.001 \leqslant Re_\Gamma \leqslant 1000$. The computed translation velocity agreed well with Saffman's formula. Further, they undertook an assessment of the error estimate of Saffman. They found that the error is $O[(\nu t/R_0^2)\log(\nu t/R_0^2)]$, being smaller than that displayed in (1.4) by an order. This observation indicates that the correction manifests itself at $O(\epsilon^3)$, favouring our theory. To put it other way round, the range of Reynolds number over which the present high-Reynolds-number asymptotic theory is validated may be extended to a lower Reynolds number, say a few hundreds, in the same way as Dyson's theory can be applied to fat cores. This expectation is reinforced by the agreement between the present theory and experiments as discussed in § 6, as regards the expansion of ring radius.

We reiterate that a uniformly valid description of vorticity distribution is not attained (figure 4). Jiménez *et al.* (1996) and Prochazka & Pullin (1998) illustrated that the same type of difficulty in problems of strained vortices is remedied by introduction of strained coordinates. Their ingenious prescription amounts to taking the summation of an infinite sequence of specific terms among ϵ-expansions. If this is the case for a vortex ring also, the translation velocity $\dot{Z}^{(2)}$ obtained by substituting the 'renormalized' vorticity, such as $\zeta^{(2)}$, remains unaltered, when it is expanded in powers of ϵ in the integrand.

8. Axisymmetric part of the second-order vorticity

The last remaining hurdle is to specify the second-order correction $\zeta_0^{(2)}(r,t)$ to the distribution of axisymmetric vorticity, whereby closure of (7.16) is achieved. For this, we have to extract the axisymmetric part from the vorticity equation (A 6) at $O(\epsilon^4)$.

Substituting $\psi^{(1)} = \psi_{11}^{(1)}\cos\theta$, (4.40), (5.1) and the associated vorticity distribution (4.25) and (4.53) into (A 6), we get, after some manipulation, a somewhat simple convection–diffusion equation for $\zeta_0^{(2)}$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{2R_0}\left(\zeta_{11}^{(1)}\tilde{\psi}_{12}^{(3)} - \zeta_{12}^{(3)}\tilde{\psi}_{11}^{(1)}\right) - \frac{r}{2R_0^2}\zeta^{(0)}\tilde{\psi}_{12}^{(3)} - \frac{\dot{R}^{(2)}r^2}{2R_0}\zeta^{(0)}\right]$$
$$= -\frac{\partial\zeta_0^{(2)}}{\partial t} + \hat{\nu}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\zeta_0^{(2)}}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{2R_0^2}a\tilde{\psi}_{11}^{(1)}\right)\right], \tag{8.1}$$

where we have defined

$$\psi_{12}^{(3)} = \tilde{\psi}_{12}^{(3)} + R_0 r\dot{R}^{(2)}, \tag{8.2}$$

and (4.18) has been used. The $\cos\theta$-component of (A 5) is solved for $\zeta_{12}^{(3)}$, giving

$$\zeta_{12}^{(3)} = \frac{a}{R_0}\tilde{\psi}_{12}^{(3)} + \frac{r}{v^{(0)}}\left\{-\frac{\partial\zeta_{11}^{(1)}}{\partial t} + \hat{\nu}\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\zeta_{11}^{(1)} + \frac{1}{R_0}\frac{\partial\zeta^{(0)}}{\partial r}\right]\right\}. \tag{8.3}$$

Thus (8.1) is further simplified to

$$\frac{\partial\zeta_0^{(2)}}{\partial t} - \hat{\nu}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\zeta_0^{(2)}}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left\{-\frac{r}{2R_0v^{(0)}}\left[\frac{\partial\zeta_{11}^{(1)}}{\partial t}\right.\right.$$
$$\left.\left.-\hat{\nu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\zeta_{11}^{(1)}\right]\tilde{\psi}_{11}^{(1)} + \frac{\dot{R}^{(2)}r^2}{2R_0}\zeta^{(0)}\right\}. \tag{8.4}$$

The appropriate boundary condition is

$$|\zeta_0^{(2)}(r,t)| \text{ decays rapidly with } r, \tag{8.5}$$

and the initial condition is

$$\zeta_0^{(2)}(r,0) = 0 \quad \text{for } 0 \leqslant r < \infty. \tag{8.6}$$

A glance at (8.4) shows that, whether viscosity is present or not,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty r\zeta_0^{(2)}(r,t)\mathrm{d}r = 0. \tag{8.7}$$

Use of the initial condition (8.6) proves (3.9).

By taking advantage of Green's function, the unique solution of (8.4) for $\hat{\nu} = 1$ may be written out. If in particular a δ-function core is assumed at the initial instant, or if $\zeta^{(0)}$ evolves according to (4.7), then (8.4) admits the following solution:

$$\zeta_0^{(2)}(r,t) = \frac{1}{2R_0 t}\left(\int_0^t t'\dot{R}^{(2)}(t')\mathrm{d}t'\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\zeta^{(0)}\right) \\ -\frac{1}{4R_0\hat{\nu}}\int_0^t \mathrm{d}t'\frac{\exp\{-r^2/4\hat{\nu}(t-t')\}}{t-t'}\int_0^\infty \mathrm{d}r' \exp\left\{-\frac{r'^2}{4\hat{\nu}(t-t')}\right\} I_0\left(\frac{rr'}{2\hat{\nu}(t-t')}\right) \\ \times\frac{\partial}{\partial r'}\left\{\frac{r'\tilde{\psi}_{11}^{(1)}(r',t')}{v^{(0)}(r',t')}\left[\frac{\partial}{\partial t'} - \hat{\nu}\left(\frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} - \frac{1}{r'^2}\right)\right]\zeta_{11}^{(1)}(r',t')\right\}, \tag{8.8}$$

where I_0 is the modified Bessel function of zeroth order of the first kind.

On the other hand, if the action of viscous diffusion is switched off, the uniqueness of the solution is lost. In this case, the time dependence is attributable to that of $c_{11}^{(1)}$ in $\tilde{\psi}_{11}^{(1)}$ given by (4.18), so we distinguish this term by a change of notation:

$$\tilde{\psi}_{11}^{(1)} \to \tilde{\psi}_{11}^{(1)} + \alpha(t)R_0 v^{(0)}, \tag{8.9a}$$

$$\zeta_{11}^{(1)} \to \zeta_{11}^{(1)} - \alpha(t)\frac{\partial\zeta^{(0)}}{\partial r}. \tag{8.9b}$$

Then (8.4), with the viscous term dropped, becomes

$$\frac{\partial\zeta_0^{(2)}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\dot{R}^{(2)}r^2}{2R_0}\zeta^{(0)} + \frac{r}{2R_0 v^{(0)}}\frac{\partial\zeta^{(0)}}{\partial r}\left(\dot{\alpha}\tilde{\psi}_{11}^{(1)} + \alpha\dot{\alpha}R_0 v^{(0)}\right)\right]. \tag{8.10}$$

Integration of (8.10) in time gives $\zeta_0^{(2)}$, within an arbitrary additive function $f(r)$, as

$$\zeta_0^{(2)} = \frac{\left[R^{(2)}(t) - R^{(2)}(0)\right]}{2R_0 r}\frac{\partial}{\partial r}\left(r^2\zeta^{(0)}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left\{-\frac{[\alpha(t)-\alpha(0)]}{2R_0}ra\tilde{\psi}_{11}^{(1)} \right. \\ \left. + \frac{\left[\alpha^2(t)-\alpha^2(0)\right]}{4}r\frac{\partial\zeta^{(0)}}{\partial r}\right\} + f(r). \tag{8.11}$$

For the three-dimensional motion of a vortex filament with axial flow, Moore & Saffman (1972), Callegari & Ting (1978) and subsequent papers turn, *ad hoc*, to the law of conservation of mass flux in order to close the system of filament equations. The above procedure shows that, in the presence of small viscosity, the higher-order equations are capable of fixing the distribution of vorticity and thus of uniquely determining the motion. The inviscid dynamics is compatible with an arbitrary function $\zeta^{(0)}(r)$, and the appearance of an undetermined function $f(r)$ in (8.11) is in harmony with this.

9. Conclusion

In this paper, we have extended the method of matched asymptotic expansions to a higher order to calculate higher-order translation and expansion speeds of a viscous vortex ring in an incompressible fluid. We have restricted attention to a simple geometry, a thin axisymmetric vortex ring. By thin, we mean that the ratio ϵ of core to curvature radii is much smaller than unity. In the event that viscous diffusion is called into play, or the circulation Reynolds number Re_Γ is not very large, an appropriate definition is $\epsilon = (\nu/\Gamma)^{1/2}$. An extension of the first-order general formula for the speed due to Fraenkel and Saffman, is achieved to $O(\epsilon^3)$, which incorporates the influence of elliptical core deformation caused by the self-induced local strain. This includes Dyson's classical result for an inviscid vortex ring of uniform distribution of ζ/ρ. Our new formula is expected to have some practical bearing; it has a potential applicability to fat vortex rings as is inferred from the surprising fact that Dyson's asymptotic solution applies, with reasonably high accuracy, even to Hill's spherical vortex, the fat limit. The rapid decrease in magnitude of the streamfunction increments with order in ϵ lies behind this. This expectation receives support from the estimate of deviation from Saffman's velocity extracted from the full numerical simulation, at moderate Reynolds numbers $Re_\Gamma \leqslant 1000$, performed by Stanaway *et al.* (1988*a, b*).

We have revealed that viscosity acts, at $O(\epsilon^3)$, to enlarge ring radius. In particular, the linear growth in time of radii of circles of relative stagnation point and peak vorticity is explicitly demonstrated for a vortex ring starting from a circle of zero thickness. A similar growth law is obtained for the stagnation circle viewed from the laboratory coordinates. These results compare well with the experimental measurements of Sallet & Widmayer (1974) and Weigand & Gharib (1997). We may sidestep the third-order solution: by invoking conservation of hydrodynamic impulse, we can deduce the radial expansion directly from the first-order solution. Our formulation thus relies upon the fundamental conservation laws of both circulation and impulse. This expansion law delimits the time during which the validity of the present asymptotic solution is guaranteed: for the 'δ-function' circular vortex loop of radius R_0 placed at $t = 0$, the core thickness grows as $\sigma \sim \sqrt{\nu t}$, whereas the elliptical stagnation point and peak-vorticity point in the axial plane drift outwards as $R^{(2)} \sim \nu t/R_0$. It follows that they overshoot the outermost point of the core after $t \sim R_0^2/\nu$, an obvious time limit. Interestingly, at this time, the core becomes as fat as the initial ring radius, namely $\sigma \sim R_0$ at $t \sim R_0^2/\nu$, and, around this time, the toroidal core comes into contact with itself.

A few points of indeterminacy of the low-order theories are pointed out. In the course of obviating these difficulties, a correct formulation of an initial-value problem is found, which is crucial to obtain the higher-order formula. The key ingredient is a line of dipoles based at the core centreline oriented in the propagation direction. The origin of the dipole at $O(\epsilon)$ is twofold. One is the vortex pair effectively produced, when the straight vortex tube is bent, by stretching the outer vortex lines and contracting the inner ones simultaneously. The other is an apparent one arising when the core centre is displaced in a given moving frame in the axial plane. This observation leads us to an improvement in the formulation, namely that the strength $d^{(1)}$ of the dipole should be prescribed as an initial condition, though the temporal evolution might be arbitrary, and that the matching condition on $\psi^{(1)}$ should include this dipole term as well. However the connection between the radial position of the core centre and the strength of the dipole is non-trivial. This is because radial expansion of the ring necessarily accompanies differential stretching of vortex lines. The indistinguishability of these two factors carries over to the three-dimensional dynamics of vortex tubes in general.

The streamline pattern of the first-order flow field exposes the mechanism of self-induced propulsion. To $O(\epsilon)$, a vortex ring is locally regarded as a line of dipoles, or a vortex pair, embedded in the flow field induced by the concentrated circular vortex loop. The vortex ring is not a passive entity. The self-induced flow, around the core centre, is not sufficient to determine the speed. It must be corrected by the active motion driven by the vortex pair. The strength of the dipole depends upon the distribution of leading-order vorticity, and this is why we have to look into the flow field inside the core. At $O(\epsilon^3)$, dipoles are again generated as a result of interplay among lower-order mono-, di-, and quadru-poles. In addition, a non-local induction from the first-order dipoles should not be forgotten.

We have derived, in a systematic manner, a new asymptotic development of the Biot-Savart law, the inner limit of the outer solution, allowing for an arbitrary, but localized, distribution of vorticity. The evaluation of non-local induction from the di- and quadru-poles and thus the derivation of $\dot{Z}^{(2)}$ rests on this formula. It includes vorticity distribution through unknown functions that are to be determined from the matching procedure. With this, we have put the formulation of the matched asymptotic expansions on a sound basis in that the inner and outer expansions cooperate with each other to proceed step-by-step. Derivation of the asymptotic formula of the Biot-Savart law for three-dimensional vortex tubes would be a useful generalization. Here a further question arises as to whether the restriction of geometry to axisymmetry can be lifted or not. Under the localized induction approximation, the three-dimensional motion of a vortex filament having an axial flow inside the core is known to be modelled, up to $O(\epsilon^2)$, by a completely integrable evolution equation (Fukumoto & Miyazaki 1991). This equation is obtained from the first two equations of an integrable hierarchy called the 'localized induction hierarchy' derived by Langer & Perline (1991). The persistence of the relation of this hierarchy with the Euler equations at higher orders would be worth pursuing (cf. Fukumoto & Miyajima 1996).

Our assumption of slow dynamics ignores a diversity of modes associated with wavy motion, which are invariably observed in nature. Inclusion of the formation process and the waves in our scheme calls for special treatment. The short bending wave on a vortex ring has a non-trivial lateral nodal structure, and the wave amplitude is localized inside the core as noted, for example, by Widnall & Tsai (1977). The short waves and similarly the fast core-area varying waves, of little long-range influence, is probably not very important for the translation speed. Numerical computation of the flow field and the third-order correction $\dot{Z}^{(2)}$ to the ring speed is currently underway, and will be reported in a subsequent paper. A comparison will be made with experimental measurements and numerical simulations of decelerating motion of vortex rings.

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Appendix A. Equations of motion expanded in powers of ϵ

This Appendix presents a list of equations for the expansions (2.15) and (2.16) of ζ and ψ. Tung & Ting (1967) derived the first three of them for the vorticity equation (2.9) and the first two for the ζ–ψ relation (2.10). It will be shown in Appendix C that $\dot{R}^{(1)} = \dot{Z}^{(1)} = 0$, so the terms including $R^{(1)}$ and $Z^{(1)}$ are removed from the expressions below.

Let us begin with the vorticity equation (2.9). The Jacobian is represented in terms of a bracket:

$$[f, g] \equiv \frac{1}{r}\frac{\partial(f, g)}{\partial(r, \theta)}. \tag{A 1}$$

The terms of ϵ^{-2} constitute what we call the equation at leading order $O(\epsilon^0)$:

$$\frac{1}{R^{(0)}}[\zeta^{(0)}, \psi^{(0)}] = 0. \tag{A 2}$$

Repeating the same procedure, we successively obtain the equations at higher orders as follows.

At $O(\epsilon^1)$,

$$\frac{1}{R^{(0)}}\left([\zeta^{(1)}, \psi^{(0)}] + [\zeta^{(0)}, \psi^{(1)}]\right) - \frac{1}{(R^{(0)})^2}\left(\frac{\partial \psi^{(0)}}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial \psi^{(0)}}{\partial \theta}\cos\theta\right)\zeta^{(0)}$$
$$- \left[\left(\dot{Z}^{(0)}\sin\theta + \dot{R}^{(0)}\cos\theta\right)\frac{\partial \zeta^{(0)}}{\partial r} + \left(\dot{Z}^{(0)}\cos\theta - \dot{R}^{(0)}\sin\theta\right)\frac{1}{r}\frac{\partial \zeta^{(0)}}{\partial \theta}\right] = 0, \tag{A 3}$$

where a dot stands for differentiation with respect to time t. Here and hereafter (A 2) is used to simplify the expressions.

At $O(\epsilon^2)$,

$$\frac{1}{R^{(0)}}\left\{[\zeta^{(2)}, \psi^{(0)}] + [\zeta^{(0)}, \psi^{(2)}] + [\zeta^{(1)}, \psi^{(1)}] - \frac{r}{R^{(0)}}\cos\theta\left([\zeta^{(1)}, \psi^{(0)}] + [\zeta^{(0)}, \psi^{(1)}]\right)\right\}$$
$$-\left[\left(\dot{Z}^{(0)}\sin\theta + \dot{R}^{(0)}\cos\theta\right)\frac{\partial \zeta^{(1)}}{\partial r} + \left(\dot{Z}^{(0)}\cos\theta - \dot{R}^{(0)}\sin\theta\right)\frac{1}{r}\frac{\partial \zeta^{(1)}}{\partial \theta}\right]$$
$$-\frac{1}{(R^{(0)})^2}\left\{\left(\frac{\partial \psi^{(0)}}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial \psi^{(0)}}{\partial \theta}\cos\theta\right)\zeta^{(1)}\right.$$
$$+\left(\frac{\partial \psi^{(1)}}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial \psi^{(1)}}{\partial \theta}\cos\theta\right.$$
$$\left.\left.-\frac{r}{R^{(0)}}\left[\frac{\partial \psi^{(0)}}{\partial r}\sin 2\theta + \frac{1}{r}\frac{\partial \psi^{(0)}}{\partial \theta}(1+\cos 2\theta)\right]\right)\zeta^{(0)}\right\}$$
$$= -\frac{\partial \zeta^{(0)}}{\partial t} + \hat{\nu}\Delta\zeta^{(0)}, \tag{A 4}$$

where Δ is defined by (2.12).

At $O(\epsilon^3)$,

$$\frac{1}{R^{(0)}}\left\{[\zeta^{(3)}, \psi^{(0)}] + [\zeta^{(0)}, \psi^{(3)}] + [\zeta^{(2)}, \psi^{(1)}] + [\zeta^{(1)}, \psi^{(2)}]\right.$$
$$-\frac{r}{R^{(0)}}\cos\theta\left([\zeta^{(2)}, \psi^{(0)}] + [\zeta^{(1)}, \psi^{(1)}] + [\zeta^{(0)}, \psi^{(2)}]\right)$$
$$\left.+\left[\frac{r^2}{2(R^{(0)})^2}(1+\cos 2\theta) - \frac{R^{(2)}}{R^{(0)}}\right]\left([\zeta^{(1)}, \psi^{(0)}] + [\zeta^{(0)}, \psi^{(1)}]\right)\right\}$$
$$-\left[\left(\dot{Z}^{(2)}\sin\theta + \dot{R}^{(2)}\cos\theta\right)\frac{\partial \zeta^{(0)}}{\partial r} + \left(\dot{Z}^{(2)}\cos\theta - \dot{R}^{(2)}\sin\theta\right)\frac{1}{r}\frac{\partial \zeta^{(0)}}{\partial \theta}\right.$$
$$\left.+\left(\dot{Z}^{(0)}\sin\theta + \dot{R}^{(0)}\cos\theta\right)\frac{\partial \zeta^{(2)}}{\partial r} + \left(\dot{Z}^{(0)}\cos\theta - \dot{R}^{(0)}\sin\theta\right)\frac{1}{r}\frac{\partial \zeta^{(2)}}{\partial \theta}\right]$$

$$
\begin{aligned}
&-\frac{1}{(R^{(0)})^2}\Bigg\{\left(\frac{\partial\psi^{(0)}}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial\psi^{(0)}}{\partial\theta}\cos\theta\right)\zeta^{(2)}\\
&+\left(\frac{\partial\psi^{(1)}}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}\cos\theta-\frac{r}{R^{(0)}}\left[\frac{\partial\psi^{(0)}}{\partial r}\sin 2\theta+\frac{1}{r}\frac{\partial\psi^{(0)}}{\partial\theta}(1+\cos 2\theta)\right]\right)\zeta^{(1)}\\
&+\Bigg(\frac{\partial\psi^{(2)}}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta}\cos\theta-\frac{r}{R^{(0)}}\left[\frac{\partial\psi^{(1)}}{\partial r}\sin 2\theta+\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}(1+\cos 2\theta)\right]\\
&\quad+\frac{3r^2}{4(R^{(0)})^2}\left[\frac{\partial\psi^{(0)}}{\partial r}(\sin 3\theta+\sin\theta)+\frac{1}{r}\frac{\partial\psi^{(0)}}{\partial\theta}(\cos 3\theta+\cos 3\theta)\right]\\
&\quad-\frac{2R^{(2)}}{R^{(0)}}\left[\frac{\partial\psi^{(0)}}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial\psi^{(0)}}{\partial\theta}\cos\theta\right]\Bigg)\zeta^{(0)}\Bigg\}\\
&=-\frac{\partial\zeta^{(1)}}{\partial t}+\hat{\nu}\left[\Delta\zeta^{(1)}+\frac{1}{R^{(0)}}\left(\frac{\partial\zeta^{(0)}}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial\zeta^{(0)}}{\partial\theta}\sin\theta\right)\right].
\end{aligned}
\tag{A 5}
$$

Finding the axisymmetric part $\zeta_0^{(2)}$ of the second-order vorticity necessitates a further expansion. To avoid complexity, we use $\partial\psi^{(0)}/\partial\theta=\partial\zeta^{(0)}/\partial\theta=0$. The result is then, at $O(\epsilon^4)$,

$$
\begin{aligned}
&\frac{1}{R^{(0)}}\Bigg\{[\zeta^{(4)},\psi^{(0)}]+[\zeta^{(0)},\psi^{(4)}]+[\zeta^{(3)},\psi^{(1)}]+[\zeta^{(2)},\psi^{(2)}]+[\zeta^{(1)},\psi^{(3)}]\\
&\quad-\frac{r}{R^{(0)}}\cos\theta\left([\zeta^{(3)},\psi^{(0)}]+[\zeta^{(2)},\psi^{(1)}]+[\zeta^{(1)},\psi^{(2)}]+[\zeta^{(0)},\psi^{(3)}]\right)\\
&\quad+\left[\frac{r^2}{2(R^{(0)})^2}(1+\cos 2\theta)-\frac{R^{(2)}}{R^{(0)}}\right]\left([\zeta^{(2)},\psi^{(0)}]+[\zeta^{(1)},\psi^{(1)}]+[\zeta^{(0)},\psi^{(2)}]\right)\\
&\quad-\left[\frac{r^3}{4(R^{(0)})^3}(\cos 3\theta+3\cos\theta)-\frac{2R^{(2)}}{(R^{(0)})^2}r\cos\theta+\frac{R^{(3)}}{R^{(0)}}\right]\left([\zeta^{(1)},\psi^{(0)}]+[\zeta^{(0)},\psi^{(1)}]\right)\Bigg\}\\
&-\Bigg[\left(\dot{Z}^{(3)}\sin\theta+\dot{R}^{(3)}\cos\theta\right)\frac{\partial\zeta^{(0)}}{\partial r}\\
&\quad+\left(\dot{Z}^{(2)}\sin\theta+\dot{R}^{(2)}\cos\theta\right)\frac{\partial\zeta^{(1)}}{\partial r}+\left(\dot{Z}^{(2)}\cos\theta-\dot{R}^{(2)}\sin\theta\right)\frac{1}{r}\frac{\partial\zeta^{(1)}}{\partial\theta}\\
&\quad+\left(\dot{Z}^{(0)}\sin\theta+\dot{R}^{(0)}\cos\theta\right)\frac{\partial\zeta^{(3)}}{\partial r}+\left(\dot{Z}^{(0)}\cos\theta-\dot{R}^{(0)}\sin\theta\right)\frac{1}{r}\frac{\partial\zeta^{(3)}}{\partial\theta}\Bigg]\\
&-\frac{1}{(R^{(0)})^2}\Bigg\{\frac{\partial\psi^{(0)}}{\partial r}\zeta^{(3)}\sin\theta+\left(\frac{\partial\psi^{(1)}}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}\cos\theta-\frac{r}{R^{(0)}}\frac{\partial\psi^{(0)}}{\partial r}\sin 2\theta\right)\zeta^{(2)}\\
&\quad+\Bigg(\frac{\partial\psi^{(2)}}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta}\cos\theta-\frac{r}{R^{(0)}}\left[\frac{\partial\psi^{(1)}}{\partial r}\sin 2\theta+\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}(1+\cos 2\theta)\right]\\
&\qquad+\frac{3r^2}{4(R^{(0)})^2}\frac{\partial\psi^{(0)}}{\partial r}(\sin 3\theta+\sin\theta)-\frac{2R^{(2)}}{R^{(0)}}\frac{\partial\psi^{(0)}}{\partial r}\sin\theta\Bigg)\zeta^{(1)}\\
&\quad+\Bigg(\frac{\partial\psi^{(3)}}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial\psi^{(3)}}{\partial\theta}\cos\theta-\frac{r}{R^{(0)}}\left[\frac{\partial\psi^{(2)}}{\partial r}\sin 2\theta+\frac{1}{r}\frac{\partial\psi^{(2)}}{\partial\theta}(1+\cos 2\theta)\right]\\
&\qquad+\frac{3r^2}{4(R^{(0)})^2}\left[\frac{\partial\psi^{(1)}}{\partial r}(\sin 3\theta+\sin\theta)+\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}(\cos 3\theta+3\cos\theta)\right]
\end{aligned}
$$

$$
-\frac{2R^{(2)}}{R^{(0)}}\left[\frac{\partial\psi^{(1)}}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta}\cos\theta\right]-\frac{r^3}{2(R^{(0)})^3}\frac{\partial\psi^{(0)}}{\partial r}(2\sin 2\theta+\sin 4\theta)
$$

$$
\left.\left.+\frac{3R^{(2)}}{(R^{(0)})^2}r\frac{\partial\psi^{(0)}}{\partial r}\sin 2\theta-\frac{2R^{(3)}}{R^{(0)}}\frac{\partial\psi^{(0)}}{\partial r}\sin\theta\right)\zeta^{(0)}\right\}
$$

$$
=-\frac{\partial\zeta^{(2)}}{\partial t}+\hat{\nu}\left[\Delta\zeta^{(2)}+\frac{1}{R^{(0)}}\left(\frac{\partial\zeta^{(1)}}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial\zeta^{(1)}}{\partial\theta}\sin\theta\right)\right.
$$

$$
\left.-\frac{r}{2(R^{(0)})^2}\frac{\partial\zeta^{(0)}}{\partial r}(1+\cos 2\theta)-\frac{1}{(R^{(0)})^2}\zeta^{(0)}\right]. \tag{A 6}
$$

Subsequently we write down the first four expansions of the ζ–ψ relation (2.10). At $O(\epsilon^0)$,

$$
\zeta^{(0)}=\frac{1}{R^{(0)}}\Delta\psi^{(0)}. \tag{A 7}
$$

At $O(\epsilon^1)$,

$$
\zeta^{(1)}=\frac{1}{R^{(0)}}\left(\Delta\psi^{(1)}-\frac{r\cos\theta}{R^{(0)}}\Delta\psi^{(0)}\right)-\frac{1}{(R^{(0)})^2}\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\psi^{(0)}. \tag{A 8}
$$

At $O(\epsilon^2)$,

$$
\zeta^{(2)}=\frac{1}{R^{(0)}}\left\{\Delta\psi^{(2)}-\frac{r\cos\theta}{R^{(0)}}\Delta\psi^{(1)}+\left[\frac{r^2}{2(R^{(0)})^2}(1+\cos 2\theta)-\frac{R^{(2)}}{R^{(0)}}\right]\Delta\psi^{(0)}\right\}
$$

$$
-\frac{1}{(R^{(0)})^2}\left\{\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\psi^{(1)}-\frac{r}{R^{(0)}}\left[(1+\cos 2\theta)\frac{\partial}{\partial r}-\frac{\sin 2\theta}{r}\frac{\partial}{\partial\theta}\right]\psi^{(0)}\right\}. \tag{A 9}
$$

At $O(\epsilon^3)$,

$$
\zeta^{(3)}=\frac{1}{R^{(0)}}\left\{\Delta\psi^{(3)}-\frac{r\cos\theta}{R^{(0)}}\Delta\psi^{(2)}+\left[\frac{r^2}{2(R^{(0)})^2}(1+\cos 2\theta)-\frac{R^{(2)}}{R^{(0)}}\right]\Delta\psi^{(1)}\right.
$$

$$
\left.-\left[\frac{r^3}{4(R^{(0)})^3}(\cos 3\theta+3\cos\theta)-\left(\frac{2R^{(2)}}{(R^{(0)})^2}r\cos\theta-\frac{R^{(3)}}{R^{(0)}}\right)\right]\Delta\psi^{(0)}\right\}
$$

$$
-\frac{1}{(R^{(0)})^2}\left\{\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\psi^{(2)}-\frac{r}{R^{(0)}}\left[(1+\cos 2\theta)\frac{\partial}{\partial r}-\frac{\sin 2\theta}{r}\frac{\partial}{\partial\theta}\right]\psi^{(1)}\right.
$$

$$
+\frac{3r^2}{4(R^{(0)})^2}\left[(\cos 3\theta+3\cos\theta)\frac{\partial}{\partial r}-\frac{\sin 3\theta+\sin\theta}{r}\frac{\partial}{\partial\theta}\right]\psi^{(0)}
$$

$$
\left.-\frac{2R^{(2)}}{R^{(0)}}\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\psi^{(0)}\right\}. \tag{A 10}
$$

Appendix B. Useful formulae for deriving the asymptotic development of the Biot-Savart law

This Appendix collects several formulae that help to derive (3.10)–(3.13). The first term on the right-hand side of (3.6), expressed in the form of a series, is deduced by adapting Dyson's ingenious technique (Dyson 1893). Let us introduce a two-dimensional differential operator

$$\nabla_R^{(-)} \equiv \left(\frac{\partial}{\partial R}, -\frac{\partial}{\partial z}\right), \tag{B 1}$$

and think of this as an ordinary vector. Here z is the relative coordinate in the axial direction, denoted by $\hat{z}$ in § 3. Let the angle between (x', z') and $\nabla_R^{(-)}$ be Θ. Then we may write

$$x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z} = |\nabla_R^{(-)}| r' \cos\Theta. \tag{B 2}$$

This operator facilitates the expansions of (3.2), when the vorticity ζ consists only of the axisymmetric component, as

$$\begin{aligned}\iint_{-\infty}^{\infty} \mathrm{d}x'\, \mathrm{d}z' \zeta_0(r') \exp\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right) &= \int_0^{\infty} \mathrm{d}r\, r\zeta_0(r) \int_0^{2\pi} \mathrm{d}\Theta\ \exp(|\nabla_R^{(-)}| r\cos\Theta) \\ &= \int_0^{\infty} \mathrm{d}r\, r\zeta_0(r) \int_0^{2\pi} \mathrm{d}\Theta \sum_{n=0}^{\infty} \frac{1}{n!} |\nabla_R^{(-)}|^n r^n \cos^n\Theta \\ &= \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!(2m)!!} \left(2\pi \int_0^{\infty} r^{2m+1}\zeta_0 \mathrm{d}r\right) \nabla_R^{2m}.\end{aligned} \tag{B 3}$$

Here $\nabla_R \equiv (\partial/\partial R, \partial/\partial z)$ and

$$\nabla_R^2 = (\nabla_R^{(-)})^2 = \frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}, \tag{B 4}$$

completing the proof.

A further simplification of (3.6) is accomplished by using the following identity for the monopole field ψ_m:

$$\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}\right)\psi_m = \frac{1}{R}\frac{\partial \psi_m}{\partial R} \tag{B 5}$$

(Lamb 1932). Repeated use of (B 5), with the help of the identity met by an arbitrary function f of R,

$$\frac{\partial^2}{\partial R^2}\left(\frac{f}{R}\right) = \frac{1}{R}\frac{\partial^2 f}{\partial R^2} - \frac{2}{R}\frac{\partial}{\partial R}\left(\frac{f}{R}\right), \tag{B 6}$$

enables us to replace the z-derivatives of even orders with R-derivatives:

$$\nabla_R^{2n}\psi_m = (-1)^{n-1}(2n-3)!!\left(\frac{1}{R}\frac{\partial}{\partial R}\right)^n \psi_m \quad \text{for } n \geqslant 2, \tag{B 7}$$

$$\left(\frac{\partial^2}{\partial R^2} - \frac{\partial^2}{\partial z^2}\right)\psi_m = 2R^2\left(\frac{1}{R}\frac{\partial}{\partial R}\right)^2 \psi_m + \frac{1}{R}\frac{\partial \psi_m}{\partial R}, \tag{B 8}$$

$$\left(\frac{\partial^3}{\partial R^3} - 3\frac{\partial^3}{\partial R \partial z^2}\right)\psi_m = 4R^3\left(\frac{1}{R}\frac{\partial}{\partial R}\right)^3 \psi_m + 9R\left(\frac{1}{R}\frac{\partial}{\partial R}\right)^2 \psi_m, \tag{B 9}$$

$$\frac{\partial}{\partial R}\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}\right)^2 \psi_m = -R\left(\frac{1}{R}\frac{\partial}{\partial R}\right)^3 \psi_m, \tag{B 10}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}\right)^2 \psi_m = -\left(\frac{1}{R}\frac{\partial}{\partial R}\right)^2 \frac{\partial \psi_m}{\partial z}, \tag{B 11}$$

$$\left(\frac{\partial^2}{\partial R^2} - \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial z^2}\right)\psi_m = 2R\left(\frac{1}{R}\frac{\partial}{\partial R}\right)^3 \psi_m + 3\left(\frac{1}{R}\frac{\partial}{\partial R}\right)^2 \psi_m. \tag{B 12}$$

Rewriting (3.7) using the global cylindrical coordinates,

$$\rho - R = r\cos\theta, \quad z - Z = r\sin\theta, \tag{B 13}$$

derivatives of ψ_m in R are calculated with ease to yield

$$\frac{1}{R}\frac{\partial\psi_m}{\partial R} = -\frac{1}{2\pi}\left(\frac{\cos\theta}{r} + \frac{1}{2R}\left[\log\left(\frac{8R}{r}\right) + \frac{\cos 2\theta}{2}\right] - \frac{r}{8R^2}\left\{\left[\log\left(\frac{8R}{r}\right) - \frac{7}{4}\right]\cos\theta + \frac{\cos 3\theta}{4}\right\}\right) + \cdots, \tag{B 14}$$

$$\left(\frac{1}{R}\frac{\partial}{\partial R}\right)^2 \psi_m = -\frac{1}{2\pi R}\left[\frac{\cos 2\theta}{r^2} + \frac{1}{4Rr}(\cos\theta + \cos 3\theta)\right] + \cdots, \tag{B 15}$$

$$\left(\frac{1}{R}\frac{\partial}{\partial R}\right)^3 \psi_m = -\frac{1}{\pi R^2}\frac{\cos 3\theta}{r^3} + \cdots, \tag{B 16}$$

$$\frac{\partial\psi_m}{\partial z} = \frac{R}{2\pi}\left(\frac{\sin\theta}{r} + \frac{\sin 2\theta}{4R}\right) + \cdots. \tag{B 17}$$

Substitution of (B 5)–(B 17), together with (3.7) and (3.8), into (3.6) gives rise to (3.10).

Appendix C. Proof of absence of the axisymmetric part of vorticity at first order and correction to the ring velocity at second order

We prove first that $\dot{R}^{(1)} = 0$ and secondly that $\zeta_0^{(1)} = \psi_0^{(1)} = 0$. Lastly, $\dot{Z}^{(1)} = 0$ is shown. Unlike in Appendix A, we make full use of (2.17), retaining $\epsilon R^{(1)}$, and, in keeping with this, include axisymmetric terms in the expansions for $\psi^{(1)}$ and $\zeta^{(1)}$. Rather than (4.14) and (4.40), we assume the following expansions:

$$\psi^{(1)} = \psi_0^{(1)} + \psi_{11}^{(1)}\cos\theta, \tag{C 1}$$

$$\zeta^{(1)} = \zeta_0^{(1)} + \zeta_{11}^{(1)}\cos\theta, \tag{C 2}$$

$$\psi^{(2)} = \psi_0^{(2)} + \psi_{11}^{(2)}\cos\theta + \psi_{12}^{(2)}\sin\theta + \psi_{21}^{(2)}\cos 2\theta, \tag{C 3}$$

$$\zeta^{(2)} = \zeta_0^{(2)} + \zeta_{11}^{(2)}\cos\theta + \zeta_{12}^{(2)}\sin\theta + \zeta_{21}^{(2)}\cos 2\theta. \tag{C 4}$$

The vorticity equation (A 4) at $O(\epsilon^2)$ is augmented, on its left-hand side, by

$$-\left[\left(\dot{Z}^{(1)}\sin\theta + \dot{R}^{(1)}\cos\theta\right)\frac{\partial\zeta^{(0)}}{\partial r} + \left(\dot{Z}^{(1)}\cos\theta - \dot{R}^{(1)}\sin\theta\right)\frac{1}{r}\frac{\partial\zeta^{(0)}}{\partial\theta}\right] - \frac{R^{(1)}}{(R^{(0)})^2}\left([\zeta^{(1)},\psi^{(0)}] + [\zeta^{(0)},\psi^{(1)}]\right) + \frac{2R^{(1)}}{(R^{(0)})^3}\frac{\partial\psi^{(0)}}{\partial r}\zeta^{(0)}\sin\theta. \tag{C 5}$$

This modified version of (A 4) is then integrated in θ once, for the non-axisymmetric part, to yield

$$\zeta^{(2)} = \frac{a}{R_0}\psi^{(2)} + ra\left(\dot{Z}^{(1)}\cos\theta - \dot{R}^{(1)}\sin\theta\right) + \left\{\frac{ra}{R_0^2}\psi_0^{(1)} + \frac{r}{R_0}\left(\zeta_0^{(1)} - \frac{a}{R_0}\psi_0^{(1)}\right)\right.$$
$$+\frac{1}{R_0^2 v^{(0)}}\left(a\frac{\partial \psi_0^{(1)}}{\partial r} - R_0\frac{\partial \zeta_0^{(1)}}{\partial r}\right)(\psi_{11}^{(1)} + R_0 r\dot{Z}^{(0)})$$
$$\left.+r\dot{R}^{(1)}\left(\frac{a}{R_0}\dot{Z}^{(0)} - \frac{\zeta^{(0)}}{R_0^2}\right)\right\}\cos\theta + \left\{\frac{b}{4R_0^2}(\psi_{11}^{(1)})^2\right.$$
$$\left.+\frac{r}{2R_0^2}(a + R_0 b\dot{Z}^{(0)})\psi_{11}^{(1)} + \frac{1}{4}\left[b(\dot{Z}^{(0)})^2 + \frac{3}{R_0}a\dot{Z}^{(0)}\right]r^2\right\}\cos 2\theta. \quad \text{(C 6)}$$

The ζ–ψ relation (A 9) at $O(\epsilon^2)$ is augmented, on its right-hand side, by

$$\frac{R^{(1)}}{(R^{(0)})^3}\left[-R^{(0)}\Delta\psi^{(1)} + 2r\cos\theta\Delta\psi^{(0)} + 2\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\psi^{(0)}\right]. \quad \text{(C 7)}$$

The function $\psi_{12}^{(2)}$ has a bearing on the location of the core in the axial direction with an accuracy of $O(\epsilon^2)$. The equation for $\psi_{12}^{(2)}$ is extracted from (C 6) and (A 9) aided by (C 7) as

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{1}{r^2} + a\right)\right]\psi_{12}^{(2)} = -R_0 ra\dot{R}^{(1)}, \quad \text{(C 8)}$$

which is constrained by the matching condition

$$\psi_{12}^{(2)} \propto \frac{1}{r} \quad \text{as } r \to \infty. \quad \text{(C 9)}$$

In effect, the corrections from the $R^{(1)}$-terms have no influence. A general solution of (C 8), regular at $r = 0$, is

$$\psi_{12}^{(2)} = R_0 r\dot{R}^{(1)} + c_{12}^{(2)} v^{(0)}. \quad \text{(C 10)}$$

The matching condition (C 9) then yields

$$\dot{R}^{(1)} = 0, \quad \text{(C 11)}$$

and therefore

$$R^{(1)} = 0 \quad \text{for all } t \geqslant 0. \quad \text{(C 12)}$$

The corresponding vorticity is

$$\zeta_{12}^{(2)} = -\frac{c_{12}^{(2)}}{R_0}\frac{\partial \zeta^{(0)}}{\partial r}. \quad \text{(C 13)}$$

Secondly we treat the axisymmetric part of (A 5). Use of (C 11)–(C 13) helps to simplify this to

$$\frac{\partial \zeta_0^{(1)}}{\partial t} = \hat{\nu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\zeta_0^{(1)}. \quad \text{(C 14)}$$

In the viscous case ($\hat{\nu} \neq 0$), this is solved with the initial condition

$$\zeta_0^{(1)}(r, 0) = 0, \quad \text{(C 15)}$$

giving rise to

$$\zeta_0^{(1)}(r, t) = 0 \quad \text{for all } t \geqslant 0. \quad \text{(C 16)}$$

In the inviscid case ($\hat{\nu} = 0$), (C 14) is compatible with an arbitrary steady spatial distribution of $\zeta_0^{(1)}$, but this may be judiciously neglected.

The axisymmetric part of (A 8) reads

$$\Delta\psi_0^{(1)} = R_0\zeta_0^{(1)} = 0. \tag{C 17}$$

This equation implies that $\psi_0^{(1)}$ is a harmonic function regular on the entire plane, and therefore

$$\psi_0^{(1)} = \text{const.} \tag{C 18}$$

Setting $R^{(1)} = \zeta_0^{(1)} = 0$ and $\psi_0^{(1)} = \text{const.}$, the equation for $\psi_{11}^{(2)}$ reduces to

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{1}{r^2} + a\right)\right]\psi_{11}^{(2)} = aR_0 r\dot{Z}^{(1)}. \tag{C 19}$$

The matching condition is

$$\psi_{11}^{(2)} \propto \frac{1}{r} \quad \text{as } r \to \infty. \tag{C 20}$$

In the same way as above, we find

$$\dot{Z}^{(1)} = 0. \tag{C 21}$$

Appendix D. Alternative derivation of the third-order radial velocity

We present a brief outline of the derivation of the expansion law (5.6) of the core radius by means of the matching procedure. Recalling (4.14) with $\psi_{12}^{(1)} = 0$, (4.21) and (4.40), the $\cos\theta$-part of (A 5) is reducible to

$$\frac{1}{r}\left\{\frac{\partial\zeta^{(0)}}{\partial r}\psi_{12}^{(3)} + v^{(0)}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi_{12}^{(3)}\right)\right]\right\} - R_0\dot{R}^{(2)}\frac{\partial\zeta^{(0)}}{\partial r}$$
$$= R_0\left\{-\frac{\partial\zeta_{11}^{(1)}}{\partial t} + \hat{\nu}\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\zeta_{11}^{(1)} + \frac{1}{R_0}\frac{\partial\zeta^{(0)}}{\partial r}\right]\right\}. \tag{D 1}$$

The boundary conditions are

$$\psi_{12}^{(3)} \propto r \quad \text{as } r \to 0, \tag{D 2a}$$

$$\psi_{12}^{(3)} \propto \frac{1}{r} \quad \text{as } r \to \infty. \tag{D 2b}$$

As described at the end of § 4.2, we have only to integrate (D 1) with respect to r, after multiplication by r^2, and to take the limit of $r \to \infty$. For this, we make use of the identity

$$\int_0^\infty r^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\zeta_{11}^{(1)}\mathrm{d}r = 0. \tag{D 3}$$

The heat-conduction equation (4.2) for $\zeta^{(0)}$ with $\dot{R}^{(0)} = 0$ is utilized to simplify the result in such a way that

$$\hat{\nu}\int_0^\infty r^2\frac{\partial\zeta^{(0)}}{\partial r}\mathrm{d}r = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \zeta^{(0)}r^3\mathrm{d}r. \tag{D 4}$$

With the aid of $2\pi\int_0^\infty r\zeta^{(0)}\mathrm{d}r = 1$, we are led to

$$\dot{R}^{(2)} = \frac{2\pi}{R_0}\dot{d}^{(1)}. \tag{D 5}$$

In this way, we have arrived at (5.6) again, through a familiar route.

Appendix E. Necessary formulae to deduce the third-order translation velocity

The integral formulae appearing in the derivation of (7.16) and (7.17), by integration of (7.13) supplemented by (7.14) and (7.15), are enumerated in what follows. Some of these are derivable by turning to the second-order differential equations (4.16), (4.41) and (4.50), and by partial integration, combined with the corresponding boundary conditions. The following expressions are valid at very large values of r:

$$\int_0^r r\left\{v^{(0)}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi_{11}^{(3)}\right)\right]+\frac{\partial\zeta^{(0)}}{\partial r}\psi_{11}^{(3)}\right\}\mathrm{d}r$$
$$=-\frac{3}{2^6\pi^2R_0^2}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-\frac{7}{12}\right]r^2+\frac{d^{(1)}}{8\pi R_0^2}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-\frac{9}{4}\right]+\frac{R^{(2)}}{4\pi^2R_0}, \tag{E 1}$$

$$\int_0^r r\left\{\frac{1}{r}\left(\int_0^r r'\zeta_0^{(2)}\mathrm{d}r'\right)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi_{11}^{(1)}\right)\right]-\frac{\partial\zeta_0^{(2)}}{\partial r}\psi_{11}^{(1)}\right\}\mathrm{d}r=0, \tag{E 2}$$

because of (3.9),

$$\int_0^r\frac{2r^2}{R_0}\left(\frac{1}{R_0}\frac{\partial\psi^{(0)}}{\partial r}\Delta\psi_0^{(2)}+\frac{\partial\psi_0^{(2)}}{\partial r}\zeta^{(0)}+2R^{(2)}v^{(0)}\zeta^{(0)}\right)\mathrm{d}r$$
$$=-\frac{r^2}{8\pi R_0^2}\log\left(\frac{8R_0}{\epsilon r}\right)-\frac{d^{(1)}}{2\pi R_0^2}, \tag{E 3}$$

$$-\int_0^r\frac{rv^{(0)}}{R_0}\left(\frac{1}{2}\frac{\partial\tilde{\psi}_{21}^{(2)}}{\partial r}+\frac{\tilde{\psi}_{21}^{(2)}}{r}\right)\mathrm{d}r$$
$$=\frac{r^2}{8\pi R_0}\dot{Z}^{(0)}+\frac{r^2}{2^6\pi^2R_0^2}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-\frac{9}{4}\right]+\frac{d^{(1)}}{2^4\pi R_0^2}$$
$$+\frac{1}{16R_0^2}\int_0^r r^2v^{(0)}b(\tilde{\psi}_{11}^{(1)})^2\mathrm{d}r+\frac{1}{4R_0^2}\int_0^r r^3v^{(0)}a\tilde{\psi}_{11}^{(1)}\mathrm{d}r$$
$$+\frac{1}{8R_0^2}\int_0^r\left[r^2v^{(0)}\left(\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}-\frac{\tilde{\psi}_{11}^{(1)}}{r}\right)+r^3\left(v^{(0)}\right)^2+r^4\zeta^{(0)}v^{(0)}\right]\mathrm{d}r, \tag{E 4}$$

$$\int_0^r\left\{\frac{r}{2}\left(-\frac{\partial\zeta_{21}^{(2)}}{\partial r}\tilde{\psi}_{11}^{(1)}+\frac{1}{R_0}a\tilde{\psi}_{11}^{(1)}\frac{\partial\tilde{\psi}_{21}^{(2)}}{\partial r}\right)-r\zeta_{21}^{(2)}\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right.$$
$$\left.+\frac{1}{R_0}\left[\frac{\partial}{\partial r}(a\tilde{\psi}_{11}^{(1)})+r\frac{\partial\zeta^{(0)}}{\partial r}\right]r\tilde{\psi}_{21}^{(2)}\right\}\mathrm{d}r=-\frac{1}{2R_0}\int_0^r(2ra+b\tilde{\psi}_{11}^{(1)})rv^{(0)}\tilde{\psi}_{21}^{(2)}\mathrm{d}r$$
$$+\frac{1}{4R_0^2}\int_0^r\left[ra\left(\tilde{\psi}_{11}^{(1)}-r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right)\tilde{\psi}_{11}^{(1)}+\frac{b}{2}\left(\tilde{\psi}_{11}^{(1)}-r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right)(\tilde{\psi}_{11}^{(1)})^2\right]\mathrm{d}r, \tag{E 5}$$

$$\frac{\dot{Z}^{(0)}}{R_0}\int_0^r(r^3\zeta^{(0)}-2r^2v^{(0)})\mathrm{d}r=\frac{\dot{Z}^{(0)}}{2\pi R_0}r^2, \tag{E 6}$$

$$\int_0^r r^3 \left(v^{(0)}\right)^2 \mathrm{d}r = \frac{r^2}{8\pi^2} + \int_0^r r^4 \zeta^{(0)} v^{(0)} \mathrm{d}r, \tag{E 7}$$

$$\int_0^r r^2 \zeta^{(0)} \tilde{\psi}_{11}^{(1)} \mathrm{d}r = \frac{d^{(1)}}{4\pi} - \frac{1}{4} \int_0^r r^4 \zeta^{(0)} v^{(0)} \mathrm{d}r. \tag{E 8}$$

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