Motion and Expansion of a Viscous Vortex Ring: Elliptical Slowing Down and Diffusive Expansion

Yasuhide Fukumoto and H.K. Moffatt

1 Introduction

The motion of a vortex ring is a venerable problem, and, since the attempts of Helmholtz and Kelvin in the last century, extensive study has been made on various dynamical aspects, such as formation, traveling speed, waves, instability, interactions and so on. Concerning the steady solution for inviscid dynamics, analytical technique has been matured enough to make a highly nonlinear regime tractable. In contrast, the effect of viscosity on the nonlinear dynamics is poorly understood even for an isolated vortex ring.

In this article, we present a large-Reynolds-number asymptotic theory of the Navier–Stokes equations for the motion of an axisymmetric vortex ring of small cross-section. Our intention is to make the nonlinear effect amenable to analysis by constructing a framework for calculating higher-order asymptotics. The nonlinearity is featured by deformation of the core cross-section. We build a general formula for the translation speed incorporating the slowing-down effect caused by the elliptical deformation of the core. Moreover we show that viscosity has the action of expanding the ring radius, simultaneously with swelling the core; starting from an infinitely thin circular loop of radius R_0, the radii $R_s(t)$ of the loop of stagnation points relative to a comoving frame, $R_p(t)$ of the loop of peak vorticity, $R_c(t)$ of the centroid of vorticity all grow linearly in time t as $R_s \approx R_0 + 2.5902739\nu t/R_0$, $R_p \approx R_0 + 4.5902739\nu t/R_0$, and $R_c \approx R_0 + 3\nu t/R_0$. It is pointed out that the asymptotic values of R_p and R_c exhibit a discrepancy, at a finite Reynolds number, from the numerical result of Wang, Chu & Chang (1994).

To begin with, we briefly survey known results. Dyson (1893) (see also Fraenkel 1972) extended Kelvin's formula for the speed U of a thin axisymmetric vortex ring, steadily translating in an inviscid incompressible fluid of infinite extent, to third (virtually fourth) order in a small parameter $\varepsilon = \sigma/R_0$, the ratio of core radius σ to the ring radius R_0, as

$$U = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8}{\varepsilon}\right) - \frac{1}{4} - \frac{3\varepsilon^2}{8}\left[\log\left(\frac{8}{\varepsilon}\right) - \frac{5}{4}\right] + O(\varepsilon^4 \log\varepsilon)\right\}, \qquad (1.1)$$

where Γ is the circulation carried by the ring. The vorticity is assumed to be in proportion to distance from the axis of symmetry. We consider Kelvin's formula (the first two terms) as the first-order and the $O(\varepsilon^2)$-terms as the third. The local self-induced flow consists not only of a uniform flow but also of a straining field. The latter manifests itself at $O(\varepsilon^2)$ and deforms the core into an ellipse, elongated in the propagating direction:

$$r = \sigma\left\{1 - \frac{3\varepsilon^2}{8}\left[\log\left(\frac{8}{\varepsilon}\right) - \frac{17}{12}\right]\cos 2\theta + \cdots\right\}, \tag{1.2}$$

where (r, θ) are local moving cylindrical coordinates about the core center which will be introduced in §2. The inclusion of the third-order term in the propagating velocity gives a remarkable improvement in approximation; (1.1) compares well even with the exact value for the 'fat' limit of Hill's spherical vortex (Fraenkel 1972). In this limit, the parameter ε is as large as $\sqrt{2}$ under a suitable normalisation. This surprising agreement encourages us to explore a higher-order approximation in more general circumstances.

Viscosity acts to diffuse vorticity, and the motion ceases to be steady. Its influence on the traveling speed, at large Reynolds number, was first addressed by Tung & Ting (1967), using the matched asymptotic expansions, for the case where the the vorticity is, at a virtual instant $t = 0$, a 'δ-function' concentrated on a circle of radius R_0. By a different method, Saffman (1970) succeeded in deriving an explicit formula, valid up to first order in $\epsilon \equiv (\nu/\Gamma)^{1/2}$, as

$$U = \frac{\Gamma}{4\pi R_0}\left[\log\left(\frac{8R_0}{2\sqrt{\nu t}}\right) - \frac{1}{2}(1 - \gamma + \log 2) + \cdots\right], \tag{1.3}$$

where ν is the viscosity, t is the time, and $\gamma = 0.57721566\cdots$ is Euler's constant (see also Callegari & Ting 1978). Wang, Chu & Chang (1994) employed a similar method to Tung & Ting (1967), but with a different choice $\sqrt{t}$ as small parameter, and gained a correction to (1.3) originating from the viscous diffusive effect. This correction vanishes in the limit of $\nu \rightarrow 0$. Unfortunately, the existing asymptotic theories all assume a circular symmetric core with a Gaussian distribution of vorticity. It implies that our knowledge of the nonlinear effect is restricted to $O(\epsilon)$. For comprehensive lists of theories of vortex rings, the article of Shariff & Leonard (1992) should be referred to.

Motivated by intriguing pattern variation of the dissipation field visualised from numerical data of simulations of fully developed turbulence, Moffatt, Kida & Ohkitani (1994) developed a large-Reynolds-number asymptotic theory for a steady stretched vortex tube subjected to uniform non-axisymmetric irrotational strain. They demonstrated that the higher-order asymptotics satisfactorily account for the fine structure of the dissipation field previously obtained by numerical computation (Kida & Ohkitani 1992). The corresponding planar problem, though unsteady, is dealt with in a similar manner, and an extension of the result of Ting & Tung (1965) to a higher order was achieved by Jiménez, Moffatt & Vasco (1996). The structure of the solutions have much in common; at leading order, a columnar vortex with circular cores, an exact solution of the Navier–Stokes equations, is obtained. A quadrupole component enters at $O(\nu/\Gamma)$, which is realised as the deformation of the core cross-section into an ellipse. The distinguishing feature is that the major axis of the ellipse is aligned at 45° to the principal axis of the external stain. This result leads us to expectat that the strained cross-section of a vortex ring, observed in nature, is established as an equilibrium between self-induced strain and viscous diffusion. Along the line of this scenario, we elucidate the structure of this strained core and its influence on the traveling speed of an axisymmetric vortex ring.

A powerful technique for our purpose is the method of matched asymptotic expansions. It has been previously developed to derive the velocity of a slender curved vortex tube (see, for example, Ting & Klein 1991). However this method is limited to $O(\epsilon^2)$ (Moore & Saffman 1972; Fukumoto & Miyazaki 1991). In the viscous case also, the self-induced strain, with the resulting elliptical deformation of the core, makes its appearance at $O(\epsilon^2)$, and its influences on the translation speed come up at $O(\epsilon^3)$. We are thus requested to extend asymptotic expansions to a higher order.

In §2, we state the general problem. The existing asymptotic formula for the potential flow associated with a circular vortex loop is not sufficient to carry through our program. In order to work out the correct inner limit of the outer solution, we devise, in §3, a technique to produce a systematic asymptotic expression of the Biot–Savart integral accommodating an arbitrary vorticity distribution. In §4, the inner expansions are scrutinised to $O(\epsilon)$ and are extended to $O(\epsilon^2)$. Based on these, we demonstrate in §5.1 that the radii of the loops of the stagnation points, maximum vorticity and vorticity centroid all grow linearly in time owing to the action of viscosity. Thereafter, we establish in §5.2 a general formula for the translating velocity of a vortex ring. In §6, an equation governing the temporal evolution of the axisymmetric vorticity at $O(\epsilon^2)$ is derived, and an integral representation of the exact solution is given, by which the formula of the preceding section can be closed.

A few ambiguous steps lying in previous theories stand as obstacles to proceeding to higher orders. These highlight the significance of the dipoles distributed along the core centerline and oriented in the propagating direction. It turns out that their strength needs to be prescribed at an initial instant, which solves the problem of undetermined constants at $O(\epsilon)$. As a by-product, a clear interpretation is provided of the general mechanism of the self-induced motion of a curved vortex tube. Because of the limitation of space, we must omit the technical details. A comprehensive account of our theory will be available in the paper of Fukumoto & Moffatt (2000).

2 Formulation

Consider an axisymmetric vortex ring of circulation Γ moving in an infinite expanse of viscous fluid with kinematic viscosity ν. We suppose that the circulation Reynolds number Re_Γ is very large:

$$Re_\Gamma = \Gamma/\nu \gg 1\,. \tag{2.1}$$

Two length scales are available, namely, measures of the core radius σ and the ring radius R_0. Suppose that their ratio σ/R_0 is very small. We focus attention on the translational motion of a 'quasi-steady' core. This means that we exclude stable or unstable wavy motion and fast core-area waves. Then, according to (1.1), the time-scale under question is of order $R_0/(\Gamma/R_0) = R_0^2/\Gamma$. The core spreads over this time to be of order $\sigma \sim (\nu t)^{1/2} \sim (\nu/\Gamma)^{1/2} R_0$. Our assumption of slenderness requires that the relevant small parameter $\epsilon (\ll 1)$ is

$$\epsilon = \sqrt{\nu/\Gamma}\,. \tag{2.2}$$

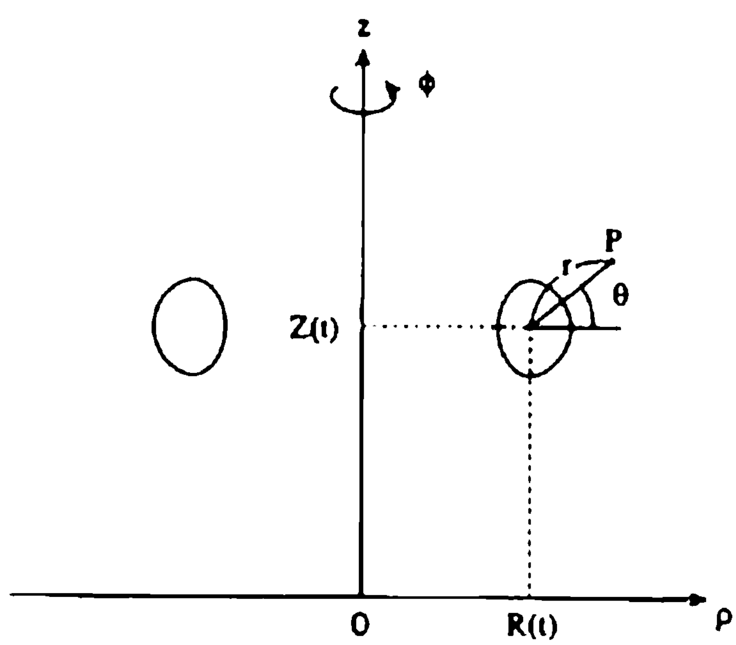

Figure 1

Choose cylindrical coordinates (ρ, ϕ, z) with the z-axis along the axis of symmetry and ϕ along the vortex lines as shown in Figure 1. We consider an axisymmetric distribution of vorticity $\omega = \zeta(\rho, z)e_\phi$ localised about the circle

$(\rho, z) = (R(t), Z(t))$, where e_ϕ is the unit vector in the azimuthal direction. The Stokes streamfunction ψ is given by

$$\psi(\rho, z) = -\frac{\rho}{4\pi}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\frac{\zeta(\rho', z')\rho'\cos\phi' d\rho' d\phi' dz'}{\sqrt{\rho^2 - 2\rho\rho'\cos\phi' + \rho'^2 + (z-z')^2}}. \tag{2.3}$$

The theorems of Kelvin and Helmholtz imply that determination of the ring motion necessitates a knowledge of the flow velocity in the vicinity of the core.

As is well known, the irrotational flow-velocity calculated from (2.3) for an infinitely thin core increases without limit primarily in inverse proportion to the distance from the core. In addition, it entails a logarithmic infinity originating from the curvature effect. These singularities may be resolved by matching the outer flow to an inner vortical flow which decays rapidly as the core center is approached. Thus we are led to inner and outer expansions (Ting & Tung 1965; Tung & Ting 1967). The inner region consists of the core itself and the surrounding toroidal region with thickness of the order of the core radius σ. There we develop an inner asymptotic expansion which matches at each level to the outer solution (2.3).

To this end, it is advantageous to introduce, in the axial plane, local polar coordinates (r, θ) moving with the core center[1] $(R(t), Z(t))$ with $\theta = 0$ in the ρ-direction (Figure 1):

$$\rho = R(t) + r\cos\theta, \qquad z = Z(t) + r\sin\theta. \tag{2.4}$$

Let us make the inner variables dimensionless. The radial coordinate is normalised by the core radius $\epsilon R_0 (= \sigma)$ and the local velocity $\boldsymbol{v} = (u, v)$, relative to the moving frame, by the maximum velocity $\Gamma/(\epsilon R_0)$. In view of (1.1), the normalisation parameter for the ring speed $(\dot{R}(t), \dot{Z}(t))$, the slow dynamics, should be Γ/R_0. The suitable dimensionless inner variables are thus defined as

$$\begin{gathered} r^* = r/\epsilon R_0, \quad t^* = t/\frac{R_0}{\Gamma}, \quad \psi^* = \frac{\psi}{\Gamma R_0}, \quad \zeta^* = \zeta/\frac{\Gamma}{R_0^2\epsilon^2}, \\ v^* = v/\frac{\Gamma}{R_0\epsilon}, \quad (\dot{R}^*, \dot{Z}^*) = (\dot{R}, \dot{Z})/\frac{\Gamma}{R_0}. \end{gathered} \tag{2.5}$$

The difference in normalisation between the last two of (2.5) should be kept in mind.

The equations handled in the inner region are the coupled system of the vorticity equation and the subsidiary relation between ζ and ψ. Dropping the asterisks, they take the following form:

$$\frac{\partial\zeta}{\partial t} + \frac{1}{\epsilon^2}\left(u\frac{\partial\zeta}{\partial r} + \frac{v}{r}\frac{\partial\zeta}{\partial\theta}\right) - \frac{1}{\epsilon\rho^2}\left(\frac{\partial\psi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\cos\theta\right)\zeta$$

[1] The definition of the 'core center' will be discussed at some length in §4.2

$$= \hat{\nu}\left[\Delta\zeta + \frac{\epsilon}{\rho}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\zeta - \frac{\epsilon^2}{\rho^2}\zeta\right], \tag{2.6}$$

$$\zeta = \frac{1}{\rho}\Delta\psi - \frac{\epsilon}{\rho^2}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\psi, \tag{2.7}$$

where $\hat{\nu} = 1$, $\rho = R + \epsilon r\cos\theta$, and Δ is the two-dimensional Laplacian,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{2.8}$$

and u and v are the r- and θ-components of the relative velocity $\boldsymbol{v}$:

$$u = \frac{1}{r\rho}\frac{\partial\psi}{\partial\theta} - \epsilon(\dot{Z}\sin\theta + \dot{R}\cos\theta), \tag{2.9a}$$

$$v = -\frac{1}{\rho}\frac{\partial\psi}{\partial r} - \epsilon(\dot{Z}\cos\theta - \dot{R}\sin\theta). \tag{2.9b}$$

We now postulate the following series expansions of the solution:

$$\zeta = \zeta^{(0)} + \epsilon\zeta^{(1)} + \epsilon^2\zeta^{(2)} + \epsilon^3\zeta^{(3)} + \cdots, \tag{2.10a}$$

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \epsilon^3\psi^{(3)} + \cdots, \tag{2.10b}$$

$$R = R^{(0)} + \epsilon R^{(1)} + \epsilon^2 R^{(2)} + \cdots, \tag{2.10c}$$

$$Z = Z^{(0)} + \epsilon Z^{(1)} + \epsilon^2 Z^{(2)} + \cdots, \tag{2.10d}$$

where $\zeta^{(i)}$ and $\psi^{(i)}$ $(i = 0, 1, 2, 3, \cdots)$ are functions of r, θ and sometimes t. There arises $\log\epsilon$ as well, but we may conveniently take it to be of order unity, since multiples of $\log\epsilon$ happen to be ruled out at least to the above order. Inserting these expansions into (2.6) and (2.7), supplemented by (2.8)–(2.9b), and collecting terms with like powers of ϵ, we obtain the equations to be solved in the inner region.

The permissible solution must satisfy the condition:

$$u \text{ and } v \text{ are finite at } r = 0. \tag{2.11}$$

We emphasise that this condition is better than the restrictive one that $u = v = 0$ at $r = 0$. The requirement that it smoothly match the asymptotic form, valid in the vicinity of the core, of the outer solution will determine the values of $\dot{R}^{(i)}$ and $\dot{Z}^{(i)}$ $(i = 0, 1, 2, \cdots)$. This procedure was already performed by Tung & Ting (1967) and Callegari & Ting (1978) and others, up to first order. Our aim is to explore the second and third orders. Before that, we reconsider the earlier results.

3 Outer solution

For a circular vortex loop of unit strength placed at $(\rho, z) = (R, Z)$, $\zeta = \delta(\rho - R)\delta(z - Z)$ and the Stokes streamfunction (2.3) simplifies to

$$\psi_m(\rho, z; R) = -\frac{\rho}{4\pi}\int_0^{2\pi} \frac{R\cos\phi' d\phi'}{\sqrt{\rho^2 - 2\rho R\cos\phi' + R^2 + (z - Z)^2}}. \tag{3.1}$$

Use of the complete elliptic integrals K and E of the first and second kinds converts (3.1) into Maxwell's well-known formula. We call ψ_m the 'monopole field'. With the aid of the asymptotic behaviour of K and E for modulus close to unity, the asymptotic form of ψ_m for $r \ll R$ is obtainable at once (Dyson 1893; Tung & Ting 1967):

$$\begin{aligned}\psi_m &= -\frac{\Gamma R}{2\pi}\Bigg\{\log\left(\frac{8R}{r}\right) + \frac{r}{2R}\left[\log\left(\frac{8R}{r}\right) - 1\right]\cos\theta \\ &\quad + \frac{r^2}{2^4 R^2}\left(\left[2\log\left(\frac{8R}{r}\right) + 1\right] + \left[-\log\left(\frac{8R}{r}\right) + 2\right]\cos 2\theta\right) \\ &\quad + \frac{r^3}{2^6 R^3}\left(\left[-3\log\left(\frac{8R}{r}\right) + 1\right]\cos\theta + \left[\log\left(\frac{8R}{r}\right) - \frac{7}{3}\right]\cos 3\theta\right)\Bigg\} \\ &\quad + \cdots. \end{aligned} \tag{3.2}$$

It turns out however that, when going to higher orders, (3.1) is not enough to qualify as the outer solution. Investigation of the detailed structure of (2.3) is unavoidable.

For this purpose, it is expedient to adapt Dyson's 'shift operator' technique to an arbitrary distribution of vorticity, and to cast (2.3) in the following form:

$$\psi = -\frac{\rho}{4\pi}\iint_{-\infty}^{\infty} dx' d\hat{z}' \zeta(x', \hat{z}') e^{x'\frac{\partial}{\partial R} - \hat{z}'\frac{\partial}{\partial \hat{z}}} \int_0^{2\pi} \frac{R\cos\phi' d\phi'}{\sqrt{\rho^2 - 2\rho R\cos\phi' + R^2 + \hat{z}^2}}, \tag{3.3}$$

where $(x, \hat{z}) = (\rho - R, z - Z)$ are local Cartesian coordinates attached to the moving frame, and ζ is rewritten in terms of them. Hereafter, we use z for $\hat{z}$. Supposing rapid decrease of vorticity with distance from the local origin $r = 0$, the exponential function of the operators is formally expanded in Taylor series as

$$\begin{aligned}\psi(\rho, z) &= \iint_{-\infty}^{\infty} dx' dz' \zeta(x', z')\Bigg\{1 + \left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right) + \frac{1}{2!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^2 \\ &\quad + \frac{1}{3!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^3 + \frac{1}{4!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^4 + \frac{1}{5!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^5 \\ &\quad + \frac{1}{6!}\left(x'\frac{\partial}{\partial R} - z'\frac{\partial}{\partial z}\right)^6 + \cdots\Bigg\}\psi_m(\rho, z; R). \end{aligned} \tag{3.4}$$

We shall find in §4 that, up to $O(\epsilon^3)$, the vorticity distribution has the following dependence on the local polar coordinate θ:

$$\begin{aligned}\zeta(x,z) &= \zeta_0 + \epsilon\,\zeta_{11}^{(1)}\cos\theta + \epsilon^2(\zeta_0^{(2)} + \zeta_{21}^{(2)}\cos 2\theta)\\ &\quad +\epsilon^3(\zeta_{11}^{(3)}\cos\theta + \zeta_{12}^{(3)}\sin\theta + \zeta_{31}^{(3)}\cos 3\theta) + \cdots, \end{aligned}\tag{3.5}$$

where $\zeta_{ij}^{(k)}$ are functions of r and t, and k stands for the order of perturbation, i labels the Fourier mode with $j = 1$ and 2 corresponding to $\cos i\theta$ and $\sin i\theta$ respectively.

With this form, (3.2) and (3.5), along with (2.10c), are substituted into (3.4) and the resulting expression is made dimensionless by use of the normalization (2.5). Using, in advance, $R^{(1)} = 0$ and (6.5), we eventually arrive at the asymptotic development of the Biot–Savart law, valid to $O(\epsilon^3)$, in a region $r \ll R$ surrounding the core:

$$\begin{aligned}\psi = &-\frac{R^{(0)}\Gamma}{2\pi}\log\left(\frac{8R^{(0)}}{\epsilon r}\right) + \epsilon\left\{-\frac{\Gamma}{4\pi}\left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right)-1\right]r\cos\theta + d^{(1)}\frac{\cos\theta}{r}\right\}\\
&+\epsilon^2\left\{-\frac{\Gamma}{2^5\pi R^{(0)}}\left(\left[2\log\left(\frac{8R^{(0)}}{\epsilon r}\right)+1\right]r^2 - \left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right)-2\right]r^2\cos 2\theta\right)\right.\\
&\quad\left.+\frac{d^{(1)}}{2R^{(0)}}\left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right)+\frac{\cos 2\theta}{2}\right] - \frac{\Gamma R^{(2)}}{2\pi}\log\left(\frac{8R^{(0)}}{\epsilon r}\right) + q^{(2)}\frac{\cos 2\theta}{r^2}\right\}\\
&+\epsilon^3\left\{\frac{\Gamma}{2^7\pi(R^{(0)})^2}\left(\left[3\log\left(\frac{8R^{(0)}}{\epsilon r}\right)-1\right]r^3\cos\theta - \left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right)-\frac{7}{3}\right]r^3\cos 3\theta\right)\right.\\
&\quad-\frac{d^{(1)}}{8(R^{(0)})^2}\left(\left[\log\left(\frac{8R^{(0)}}{\epsilon r}\right)-\frac{7}{4}\right]r\cos\theta + \frac{r\cos 3\theta}{4}\right) - \frac{\Gamma R^{(2)}}{4\pi R^{(0)}}r\cos\theta\\
&\quad-\frac{1}{2\pi}\left(\frac{1}{4}\left[2\pi\int_0^\infty r^3\zeta_0^{(2)}dr\right] + R^{(0)}\left[\pi\int_0^\infty r^2\zeta_{11}^{(3)}dr\right] + \frac{1}{4}\left[\pi\int_0^\infty r^3\zeta_{21}^{(2)}dr\right]\right)\frac{\cos\theta}{r}\\
&\quad+\frac{q^{(2)}}{4R^{(0)}r}(\cos\theta+\cos 3\theta) - \frac{1}{\pi R^{(0)}}\left(\frac{1}{3\cdot 2^8}\left[2\pi\int_0^\infty r^7\zeta^{(0)}dr\right]\right.\\
&\quad-\frac{R^{(0)}}{8\cdot 4!}\left[\pi\int_0^\infty r^6\zeta_{11}^{(1)}dr\right] + \frac{(R^{(0)})^2}{4!}\left[\pi\int_0^\infty r^5\zeta_{21}^{(2)}dr\right]\\
&\quad\left.\left.+\frac{(R^{(0)})^3}{6}\left[\pi\int_0^\infty r^4\zeta_{31}^{(3)}dr\right]\right)\frac{\cos 3\theta}{r^3} - \frac{R^{(0)}}{2\pi}\left[\pi\int_0^\infty r^2\zeta_{12}^{(3)}dr\right]\frac{\sin\theta}{r}\right\}\\
&+\cdots,\end{aligned}\tag{3.6}$$

where

$$\Gamma = 2\pi\int_0^\infty r\zeta^{(0)}dr\,,\tag{3.7a}$$

($\Gamma = 1$ when dimensionless), and $d^{(1)}$ and $q^{(2)}$ are the strength of thedipole at $O(\epsilon)$ and quadrupole at $O(\epsilon^2)$:

$$d^{(1)} = -\frac{1}{2\pi}\left\{\frac{1}{4}\left[2\pi\int_0^\infty r^3\zeta^{(0)}dr\right] + R^{(0)}\left[\pi\int_0^\infty r^2\zeta_{11}^{(1)}dr\right]\right\},\tag{3.7b}$$

$$\begin{aligned} q^{(2)} &= -\frac{1}{2\pi R^{(0)}}\Big\{-\frac{1}{2^6}\Big[2\pi\int_0^\infty r^5\zeta^{(0)}dr\Big] + \frac{R^{(0)}}{8}\Big[\pi\int_0^\infty r^4\zeta_{11}^{(1)}dr\Big] \\ &\quad + \frac{(R^{(0)})^2}{2}\Big[\pi\int_0^\infty r^3\zeta_{21}^{(2)}dr\Big]\Big\}. \end{aligned} \tag{3.7c}$$

The terms multiplied by Γ stem from $\Gamma\psi_m$, and only these have been previously employed as the outer solution. We now recognize that, at higher orders, the monopole field needs to be corrected by the induction velocity associated with the di-, quadru-, hexa-poles ... distributed along the centerline $r = 0$ of the core. In the light of (3.7b) and (3.7c), the detailed profile of vorticity in the core is necessary to evaluate these multi-pole induction terms. Parts of (3.6) supply the matching conditions on the inner solution. The distributions of $\zeta_{11}^{(1)}$, $\zeta_0^{(2)}$, $\zeta_{21}^{(2)}$, $\zeta_{11}^{(3)}$, $\zeta_{12}^{(3)}$ and $\zeta_{31}^{(3)}$ are as yet unknown, but will be determined successively by the inner expansions and the matching procedure.

4 Inner expansions up to second order

In this section, we recall the inner expansions at leading and first orders, developed by Tung & Ting (1967), Widnall, Bliss & Zalay (1971) and Callegari & Ting (1978), and extend them to second order.

4.1 Zeroth order

At $O(\epsilon^0)$, the Navier–Stokes equations reduce to the Jacobian form of the steady Euler equations:

$$[\zeta^{(0)}, \psi^{(0)}] \equiv \frac{1}{r}\frac{\partial(\zeta^{(0)}, \psi^{(0)})}{\partial(r, \theta)} = 0, \tag{4.1}$$

resulting in $\zeta^{(0)} = \mathcal{F}(\psi^{(0)})$, for some function $\mathcal{F}$.

Suppose that the flow $\psi^{(0)}$ has a single stagnation point at $r = 0$, the streamlines being all closed around that point. Then it is probable that the solution of (4.1), coupled with the ζ-ψ relation

$$\zeta^{(0)} = \frac{1}{R^{(0)}}\Delta\psi^{(0)}, \tag{4.2}$$

must be 'radial'; the streamlines are necessarily all circles (Moffatt *et al.* 1994). This statement may stand as a corollary of the theorem proved by Caffarelli & Friedman (1980) and Fraenkel (1999). In any event, we may certainly *assume* that $\psi^{(0)} = \psi^{(0)}(r)$.

The functional form of $\psi^{(0)}$ and $\zeta^{(0)}$ remains undetermined at this level of approximation, but is determined through the axisymmetric (or θ-averaged) part of the vorticity equation at $O(\epsilon^2)$:

$$\frac{\partial\zeta^{(0)}}{\partial t} = \left(\zeta^{(0)} + \frac{r}{2}\frac{\partial\zeta^{(0)}}{\partial r}\right)\frac{\dot{R}^{(0)}}{R^{(0)}} + \hat{\nu}\left(\frac{\partial^2\zeta^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta^{(0)}}{\partial r}\right), \tag{4.3}$$

(Tung & Ting 1967). It follows that viscosity plays the role of selecting the distribution. For instance, we restrict our attention to a specific initial distribution of a 'δ-function' vorticity concentrated on the circle of radius $R^{(0)}$:

$$\zeta^{(0)} = \delta(\rho - R^{(0)})\delta(z - Z^{(0)}) \quad \text{at} \quad t = 0. \tag{4.4}$$

When $R^{(0)}$ is constant, to be shown in the next subsection, we obtain the Oseen diffusing vortex:

$$\zeta^{(0)} = \frac{1}{4\pi\hat{\nu}t}e^{-r^2/4\hat{\nu}t}. \tag{4.5}$$

In view of (2.9a), (2.9b) and (4.2), the leading-order variables are related to each other through

$$u^{(0)} = \frac{1}{R^{(0)}r}\frac{\partial\psi^{(0)}}{\partial\theta}, \quad v^{(0)} = -\frac{1}{R^{(0)}}\frac{\partial\psi^{(0)}}{\partial r}, \quad \zeta^{(0)} = -\frac{1}{r}\frac{\partial}{\partial r}\left(rv^{(0)}\right). \tag{4.6}$$

These are integrated to provide $u^{(0)} = 0$ and, in the case of the Oseen vortex,

$$v^{(0)} = -\frac{1}{2\pi r}\left(1 - e^{-r^2/4\hat{\nu}t}\right), \quad \psi^{(0)} = \frac{R^{(0)}}{2\pi}\int_0^r \frac{1}{r'}\left(1 - e^{-r'^2/4\hat{\nu}t}\right)dr'. \tag{4.7}$$

This solution automatically fulfills the matching condition, the leading-order part of (3.6).

4.2 First order

Combining the vorticity equation with ζ-ψ relation at $O(\epsilon)$, we see that the first-order perturbation $\psi^{(1)}$ satisfies

$$(\Delta - a)\,\psi^{(1)} = -\cos\theta v^{(0)} + R^{(0)}ra(\dot{Z}^{(0)}\cos\theta - \dot{R}^{(0)}\sin\theta) + 2r\zeta^{(0)}\cos\theta, \tag{4.8}$$

where

$$a(r,t) = -\frac{1}{v^{(0)}}\frac{\partial\zeta^{(0)}}{\partial r}. \tag{4.9}$$

Here we have anticipated that $\zeta_0^{(1)} = 0$, which follows from an analysis of the vorticity equation at $O(\epsilon^3)$.

The solution satisfying the condition (2.11) at $O(\epsilon^3)$ is explicitly written in the following way. The θ-dependence is

$$\psi^{(1)} = \psi_{11}^{(1)}\cos\theta + \psi_{12}^{(1)}\sin\theta. \tag{4.10}$$

The Fourier coefficients are conveniently decomposed into two parts:

$$\psi_{11}^{(1)} = \tilde{\psi}_{11}^{(1)} - R^{(0)} r \dot{Z}^{(0)} , \quad \psi_{12}^{(1)} = \tilde{\psi}_{12}^{(1)} + R^{(0)} r \dot{R}^{(0)} . \tag{4.11}$$

The equation for $\tilde{\psi}_{11}^{(1)}$ then becomes

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{1}{r^2} + a \right) \right] \tilde{\psi}_{11}^{(1)} = -v^{(0)} + 2r\zeta^{(0)} , \tag{4.12}$$

and $\tilde{\psi}_{12}^{(1)}$ is governed simply by the homogeneous part of (4.12). By inspection, we find that $v^{(0)}$ is a solution of (4.12) (Widnall, *et al.* 1971; Callegari & Ting 1978). The general solution is then immediately obtainable as

$$\tilde{\psi}_{11}^{(1)} = \Psi_{11}^{(1)} + c_{11}^{(1)} v^{(0)} , \quad \tilde{\psi}_{12}^{(1)} = c_{12}^{(1)} v^{(0)} , \tag{4.13}$$

where $\Psi_{11}^{(1)}$ is a particular integral of (4.12):

$$\Psi_{11}^{(1)} = -v^{(0)} \left\{ \frac{r^2}{2} + \int_0^r \frac{dr'}{r'[v^{(0)}(r')]^2} \int_0^{r'} r'' \left[v^{(0)}(r'') \right]^2 dr'' \right\} , \tag{4.14}$$

and $c_{11}^{(1)}$ and $c_{12}^{(1)}$ are constants (which may depend on t).

To have an idea of the meaning of these constants, observe that, with spatial translation of the origin of the moving frame by $\epsilon\alpha$ in the ρ-direction and $\epsilon\beta$ in the z-direction, the streamfunction, when redefined as a function of the relative coordinates, is altered to

$$\begin{aligned} &\psi(\rho - (R_0 + \epsilon\alpha), z - (Z_0 + \epsilon\beta)) \\ &= \psi^{(0)} + \epsilon \left(\psi^{(1)} + \alpha R_0 v^{(0)} \cos\theta + \beta R_0 v^{(0)} \sin\theta \right) + O(\epsilon^2) . \end{aligned} \tag{4.15}$$

Comparison of (4.15) with (4.11) and (4.13) suggests that $c_{11}^{(1)}$ is tied in with shift of the moving frame radially outward by $\epsilon c_{11}^{(1)}/R_0$ and $c_{12}^{(1)}$ with shift in the axial direction by $\epsilon c_{12}^{(1)}/R_0$. Alternatively, $c_{11}^{(1)}$ and $c_{12}^{(1)}$ may be reckoned upon as the parameters placing the circular core in a given moving frame, to an accuracy of $O(\epsilon)$, in terms of the inner spatial scale. Without loss of generality, we may put

$$c_{12}^{(1)} = 0 . \tag{4.16}$$

It follows from (4.13), along with (4.7), that a proper formulation of the initial-value problem is completed with the specification of $d^{(1)}(0)$, and the matching condition includes the specification of the strength of dipole as well:

$$\psi^{(1)} \sim \left\{ -\frac{1}{4\pi} \left[\log\left(\frac{8R_0}{\epsilon r} \right) - 1 \right] r + \frac{d^{(1)}(t)}{r} \right\} \cos\theta \quad \text{as } r \to \infty , \tag{4.17}$$

This condition then gives rise to

$$\dot{R}^{(0)} = 0 , \tag{4.18}$$

and

$$\dot{Z}^{(0)} = \frac{1}{4\pi R_0}\left[\log\left(\frac{8R_0}{\epsilon}\right) - \frac{1}{2} + A\right], \tag{4.19a}$$

where

$$A = \lim_{r\to\infty}\left\{4\pi^2\int_0^r r'[v^{(0)}(r')]^2 dr' - \log r\right\}. \tag{4.19b}$$

In (4.19a) and henceforth, we use, with some abuse of notation, R_0 in plac of $R^{(0)}$. In the case of the Oseen vortex (4.5), the translation velocity (4.19a and (4.19b), is identical with Saffman's formula (1.3). The parameter $c_{11}^{(1)}$ i related to $d^{(1)}$. Once the streamfunction is known, the distribution of vorticit is calculable through

$$\zeta^{(1)} = \frac{1}{R_0}\left(a\tilde{\psi}_{11}^{(1)} + r\zeta^{(0)}\right)\cos\theta + \frac{a}{R_0}\tilde{\psi}_{12}^{(1)}\sin\theta. \tag{4.20}$$

As yet, we have no way of finding the temporal evolution of $d^{(1)}(t)$ for $t > 0$ The explanation is that it may be arbitrary. We can verify that whatever the evolution of $d^{(1)}(t)$ or $c_{11}^{(1)}(t)$ for $t > 0$ may be, this arbitrariness is consistently absorbed at $O(\epsilon^3)$, producing the same radial velocity $\dot{R}^{(2)} + \dot{c}_{11}^{(1)}/R_0$ of the ring. This implies a redundant representation of the asymptotic solution of the Navier–Stokes equations which itself is unique. The speed of the ring is expressed, in an infinite variety of ways, as the sum of the speed of the moving coordinates and that of the ring in this frame.

It is informative to revisit the discrete model in an inviscid flow initially studied by Kelvin (1867) and Dyson (1893). The leading-order flow is 'the Rankine vortex', that is, a straight circular vortex tube of unit radius surrounded by an irrotational flow:

$$\zeta^{(0)} = \begin{cases} \frac{1}{\pi}, \\ 0, \end{cases} \qquad v^{(0)} = \begin{cases} -\frac{r}{2\pi}, & (r \le 1) \\ -\frac{1}{2\pi r}. & (r > 1) \end{cases} \tag{4.21}$$

The choice of $c_{11}^{(1)} = 5/8$ ensures continuity of the relative velocity across the core boundary ($r = 1$) to $O(\epsilon)$ (Widnall *et al.* 1971), and we have the streamfunctions to first order:

$$\psi^{(0)} = \begin{cases} \frac{R_0}{4\pi}\left[r^2 - 1 - 2\log\left(\frac{8R_0}{\epsilon}\right)\right], & (r \le 1) \\ -\frac{R_0}{2\pi}\log\left(\frac{8R_0}{\epsilon r}\right), & (r > 1) \end{cases} \tag{4.22}$$

$$\psi_{11}^{(1)} = \begin{cases} -\frac{1}{4\pi}\left[\log\left(\frac{8R_0}{\epsilon}\right) - 1\right] + \frac{5}{16\pi}(r^3 - r), & (r \le 1) \\ \frac{1}{4\pi}r\log r - \frac{r}{4\pi}\left[\log\left(\frac{8R_0}{\epsilon}\right) - 1\right] - \frac{3}{16\pi r}, & (r > 1) \end{cases} \tag{4.23}$$

where we have used

$$a = -2\delta(r-1)\,. \tag{4.24}$$

The first log-term of (4.23) for $r > 1$ is a particular solution to absorb the inhomogeneous term of (4.12), the curvature effect. Putting aside this term, $\psi_{11}^{(1)}$ signifies, outside the core ($r > 1$), the flow field around a circular cylinder of unit radius, moving in the z-direction with velocity $3/16\pi$, a dipole field, in an imposed uniform flow with positive velocity $[\log(8R_0/\epsilon) - 1]/4\pi$. The summation of these values is equal to Kelvin's velocity. This observation implies that a vortex ring is more than a passive entity convected by the self-induced flow.

Figure 2(a) displays the streamlines $\psi^{(0)} = \text{const.}$, Figure 2(b) $\bar{\psi}_{11}^{(1)}\cos\theta = \text{const.}$, and Figure 2(c) $\psi^{(0)} + \epsilon\bar{\psi}_{11}^{(1)}\cos\theta = \text{const.}$, stream patterns viewed from the moving frame. For clarity, the rather large value $\epsilon = 0.5$ is chosen. We confirm that the origin $r = 0$ coincides with the center of the circular core. As expected, Figure 2(b) exhibits a dipole flow associated with a pair of antiparallel vortices. Its source is twofold. One is the 'apparent' dipole due to displacement of the center as discussed above. The other has a kinematic origin. When a columnar vortex is bent to form a torus, vortex lines are stretched on the outer side and contracted on the inner side. As a consequence, the vorticity is enhanced on the outer side but is diminished on the inner side, implying the creation of a vortex pair at $O(\epsilon)$.

We speculate that the dipole is a key ingredient of a curved vortex tube. A vortex ring may be locally considered as a line of dipoles based at the core centerline embedded in the flow field induced by the circular line vortex. The driving mechanism of the self-propulsion is not only convection due to the self-induced flow but also the thrust provided by the dipoles. The dipole strength depends upon the distribution of vorticity in the core, and this is one of the reasons why we are concerned with the inner field.

4.3 Second order

The second-order equations reveal that the second-order perturbation $\psi^{(2)}$ comprises monopole and quadrupole terms:

$$\psi^{(2)} = \psi_0^{(2)} + \psi_{21}^{(2)}\cos 2\theta\,. \tag{4.25}$$

The latter reflects an elliptical core deformation. The governing equation for the monopole is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\psi_0^{(2)} = R_0\zeta_0^{(2)} + \frac{ra}{2R_0}\bar{\psi}_{11}^{(1)} + \frac{1}{2R_0}\left[rv^{(0)} + r^2\zeta^{(0)} + \frac{\partial\psi_{11}^{(1)}}{\partial r} + \frac{\psi_{11}^{(1)}}{r}\right] + R^{(2)}\zeta^{(0)}\,. \tag{4.26}$$

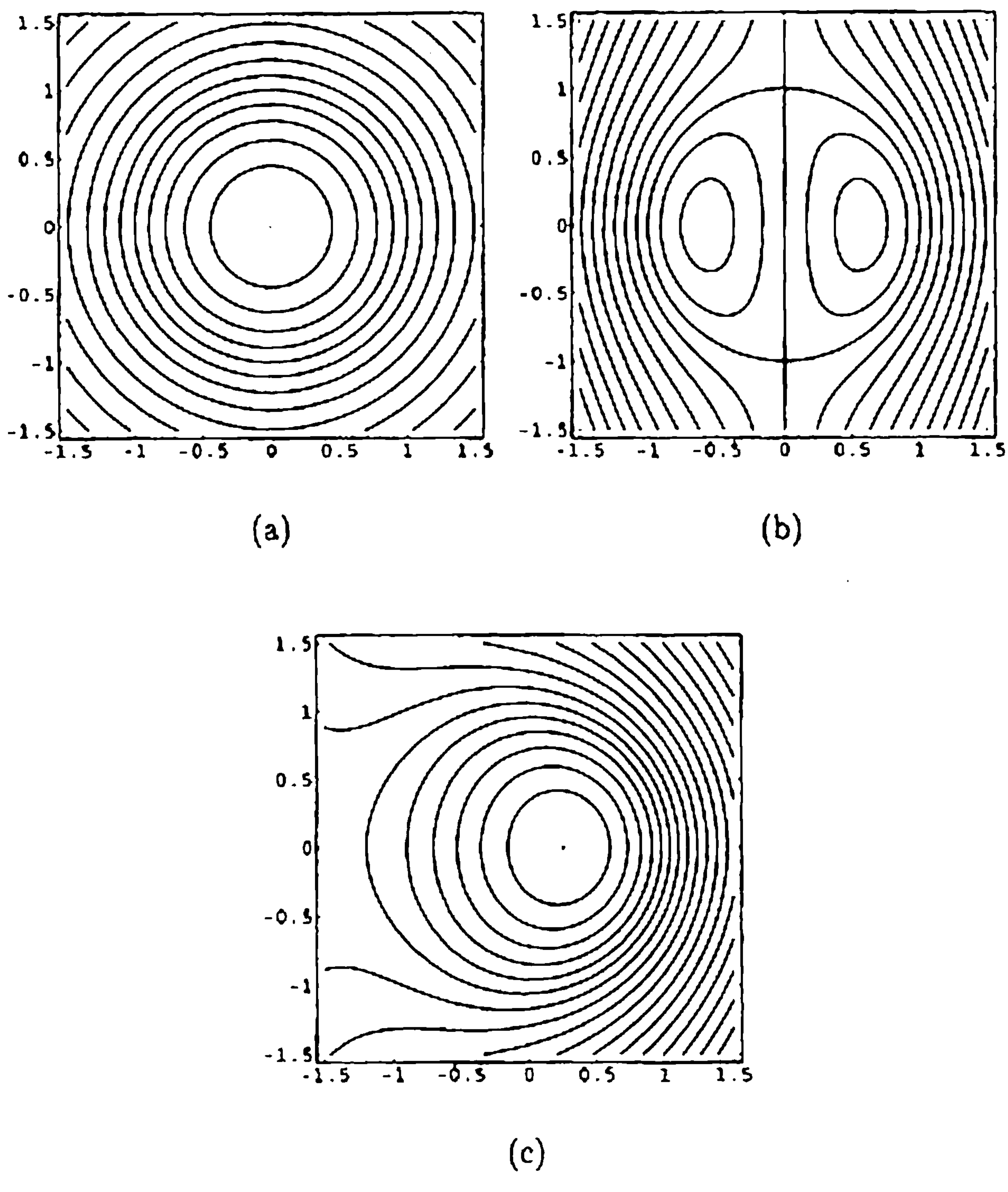

Figure 2. Streamline patterns of the inviscid vortex ring with uniform ζ/ρ, relative to the frame moving at the speed U given by (1.1), up to $O(\epsilon)$ as obtained by (4.22) and (4.23). The small parameter $\epsilon = 0.5$. (a) $\psi^{(0)} = $ const., (b) $\tilde{\psi}_{11}^{(1)} \cos\theta = $ const., (c) $\psi^{(0)} + \epsilon\tilde{\psi}_{11}^{(1)} \cos\theta = $ const.

The functional form of $\zeta_0^{(2)}$, the axisymmetric vorticity component at $O(\epsilon^2)$, will be found by solving the convection–diffusion equation obtained from the solvability condition for the fourth-order equation (§6). The matching condition, a part of (3.6) at $O(\epsilon^2)$, is

$$\psi_0^{(2)} \sim -\frac{1}{2^4\pi R_0}\left[\log\left(\frac{8R_0}{\epsilon r}\right)+\frac{1}{2}\right]r^2+\left(\frac{d^{(1)}}{2R_0}-\frac{R^{(2)}}{2\pi}\right)\log\left(\frac{8R_0}{\epsilon r}\right) \quad \text{as } r\to\infty . \tag{4.27}$$

Under the conditions of (4.27) and finiteness of velocity at $r=0$, (4.26) is readily integrated once to give

$$\frac{\partial\psi_0^{(2)}}{\partial r}=\frac{R_0}{r}\int_0^r r'\zeta_0^{(2)}dr'+\frac{r}{2R_0}\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}+\left(\frac{r^2}{2R_0}-R^{(2)}\right)v^{(0)}-\frac{r}{2}\dot{Z}^{(0)} . \tag{4.28}$$

Next we turn to the quadrupole $\psi_{21}^{(2)}\cos 2\theta$. For convenience, we define $\tilde{\psi}_{21}^{(2)}$ through

$$\psi_{21}^{(2)}=\tilde{\psi}_{21}^{(2)}-\frac{1}{4}r^2\dot{Z}^{(0)} . \tag{4.29}$$

Then the equation for $\tilde{\psi}_{21}^{(2)}$ takes the following form:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}-a\right)\tilde{\psi}_{21}^{(2)}=\frac{b}{4R_0}\left(\tilde{\psi}_{11}^{(1)}\right)^2+\frac{ra}{R_0}\tilde{\psi}_{11}^{(1)} +\frac{1}{2R_0}\left(rv^{(0)}+r^2\zeta^{(0)}+\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}-\frac{\tilde{\psi}_{11}^{(1)}}{r}\right) , \tag{4.30}$$

where

$$b(r,t)=-\frac{1}{v^{(0)}}\frac{\partial a}{\partial r} . \tag{4.31}$$

The boundary conditions demand that

$$\tilde{\psi}_{21}^{(2)}\propto r^2 \quad \text{as } r\to 0 , \tag{4.32a}$$

$$\tilde{\psi}_{21}^{(2)}\sim\frac{r^2}{4}\left\{\dot{Z}^{(0)}+\frac{1}{8\pi R_0}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-2\right]r^2\right\}+\frac{d^{(1)}}{4R_0} \quad \text{as } r\to\infty . \tag{4.32b}$$

The vorticity distribution is calculable from

$$\zeta^{(2)}=\zeta_0^{(2)}+\left[\frac{a}{R_0}\tilde{\psi}_{21}^{(2)}+\frac{b}{4R_0^2}\left(\tilde{\psi}_{11}^{(1)}\right)^2+\frac{ra}{2R_0^2}\tilde{\psi}_{11}^{(1)}\right]\cos 2\theta . \tag{4.33}$$

In the general case, it is unlikely that (4.30) can be further integrated analytically. The numerical computation is postponed to a subsequent paper, but we content ourselves with an explicit solution for Dyson's discrete model:

$$\tilde{\psi}_{21}^{(2)}=\begin{cases}\dfrac{3}{64\pi R_0}r^4+\dfrac{1}{16\pi R_0}\left[3\log\left(\dfrac{8R_0}{\epsilon}\right)-5\right]r^2 , & (r<1)\\[2ex] \dfrac{1}{32\pi R_0}\left[3\log\left(\dfrac{8R_0}{\epsilon}\right)-\log r-\dfrac{5}{2}\right]r^2-\dfrac{3}{64\pi R_0} \\ \qquad +\dfrac{3}{32\pi R_0}\left[\log\left(\dfrac{8R_0}{\epsilon}\right)-\dfrac{3}{2}\right]\dfrac{1}{r^2} . & (r>1)\end{cases} \tag{4.34}$$

The axisymmetric part of vorticity is shown to be suppressed: $\zeta_0^{(2)}=0$.

5 Third-order velocity of a vortex ring

At $O(\epsilon^3)$, a dipole field again shows up as the result of nonlinear interactions among the mono-, di- and quadru-poles of lower orders, and the streamfunction $\psi^{(3)}$ is written as

$$\psi^{(3)} = \psi_{11}^{(3)} \cos\theta + \psi_{12}^{(3)} \sin\theta + \psi_{31}^{(3)} \cos 3\theta . \tag{5.1}$$

The $\cos\theta$ and $\sin\theta$ components, the dipole field, are responsible for the axial and radial velocities respectively.

5.1 Radial expansion

It is not difficult to get $\dot{R}^{(2)}$ from the equation for $\psi_{12}^{(3)}$ in much the same way as getting $\dot{Z}^{(0)}$. Instead, by appeal to a fundamental conservation law, we can skip this procedure. Recall that the hydrodynamic impulse $\boldsymbol{P}$ is conserved, regardless of the inviscid or viscous character of the flow:

$$\boldsymbol{P} = \frac{1}{2}\iiint \boldsymbol{x} \times \boldsymbol{\omega}\, dV = \text{const}. \tag{5.2}$$

In the present axisymmetric problem, only the axial component P_z is nonzero, and upon substitution from $\boldsymbol{\omega} = \zeta \mathbf{e}_\phi$, (2.10a) and (2.10c), it becomes, after normalisation by ΓR_0^2,

$$P_z = \pi R_0^2 + \epsilon^2 P^{(2)} + \cdots , \tag{5.3}$$

where

$$P^{(2)} \equiv \pi \left(2R_0 R^{(2)}(t) - 4\pi d^{(1)}(t)\right) , \tag{5.4}$$

which should be constant throughout the time evolution. Since $R^{(2)} = 0$ at $t = 0$, the radial motion is completely ruled by the evolution of the dipole strength $d^{(1)}(t)$:

$$R^{(2)}(t) = \frac{2\pi}{R_0}\left[d^{(1)}(t) - d^{(1)}(0)\right] . \tag{5.5}$$

We concentrate on the vorticity distribution (4.5) starting from a δ-function core. In this case, $P_z = \pi R_0^2$ identically and hence the $O(\epsilon^2)$ correction term is absent: $P^{(2)}(t) = 0$ for $t \geq 0$. The numerical evaluation of the behaviour of $\Psi_{11}^{(1)}$, for large r, is carried out with ease to yield

$$\Psi_{11}^{(1)} = \frac{r}{4\pi}\left[\log r + \lim_{r\to\infty}\left(4\pi^2 \int_0^r r'[v^{(0)}(r')]^2 dr' - \log r\right) + \frac{1}{2}\right] + \frac{D^{(1)}}{r} + \cdots , \tag{5.6}$$

where

$$D^{(1)} \approx 0.41225489\, t . \tag{5.7}$$

The first-order streamfunction with $c_{11}^{(1)} = 0$ corresponds to a dipole field whose stagnation point is permanently located at $r = 0$ (Klein & Knio 1995). By identifying $D^{(1)}(t)$ with $d^{(1)}(t)$ in (5.5) and restoring dimensional variables, we conclude that, given initially a circular line vortex of radius R_0, the location $\rho = R_s(t)$ of the stagnation point in the core, viewed from the comoving coordinates, drifts outwards linearly in time owing to the action of viscosity as

$$R_s \approx R_0 + 2.5902739 \frac{\nu t}{R_0}. \tag{5.8}$$

The temporal evolution of the radius $R_p(t)$ of the loop of peak vorticity is deducible by choosing $c_{11}^{(1)}(t)$ in (4.13) so that the local origin $r = 0$ is maintained at the maximum of $\zeta^{(0)}+\epsilon\zeta^{(1)}$. Inserting the Gaussian distribution (4.5) and (4.6) into (4.9) and (4.14), we manipulate the behaviour of (4.20) near $r = 0$ as

$$\zeta^{(1)} = \frac{1}{R_0}\left[\left(\frac{c_{11}^{(1)}}{8\pi t^2} + \frac{1}{4\pi t}\right)x + O(r^2 x)\right], \tag{5.9}$$

where $x = r\cos\theta$. The condition of maximum vorticity $\zeta^{(0)} + \epsilon\zeta^{(1)}$ at $r = 0$ brings in

$$c_{11}^{(1)} = -2t. \tag{5.10}$$

In the light of (4.13) and (5.6) along with (5.7), the dipole strength becomes

$$d^{(1)}(t) \approx 4.5902739 \times \frac{t}{2\pi}, \tag{5.11}$$

when the origin of the moving frame is kept sitting at the location of the maximum vorticity. Relying upon the formula (5.5), we gain the expansion law of the circle of peak vorticity, which is expressed, in terms of the dimensional variables as

$$R_p \approx R_0 + 4.5902739 \frac{\nu t}{R_0}. \tag{5.12}$$

We point out that the asymptotic value $4.5902739\cdots$ at large Reynolds number exhibits a marked difference from $1.65\cdots$ at $Re_\Gamma = 10^4$ calculated numerically by Wang *et al.* (1994). In spite of this, at the same value of Re_Γ, their numerical value $0.5779\cdots$ in the formula of the translation velocity is in conformity with Saffman's asymptotic value $0.57796576\cdots$.

For completeness, we derive the expansion law of the circle of radial centroid $\rho = R_c(t)$ of vorticity defined by

$$R_c \equiv \frac{1}{2}\iiint \frac{(\boldsymbol{x}\times\boldsymbol{\omega})\cdot\boldsymbol{P}}{P^2}\rho\, dV = \frac{\pi}{P_z}\iint \rho^3\zeta r\, dr\, d\theta. \tag{5.13}$$

This is conveniently decomposed as

$$\begin{aligned} R_c &= \frac{\pi}{P_z}\iint (R + \epsilon r\cos\theta)\rho^2\zeta r\, dr\, d\theta \\ &\approx R_0 + \epsilon^2\left\{R^{(2)} + \frac{\pi^2}{P_z}\int_0^\infty (2R_0 r^3\zeta^{(0)} + R_0^2\zeta_{11}^{(1)})\, r^2 dr\right\}. \end{aligned} \tag{5.14}$$

With the help of (3.7b) and (5.5), (5.14) is reduced to

$$R_c(t) \approx R_0 - \frac{2\pi\epsilon^2}{R_0} d^{(1)}(0) + \frac{3\epsilon^2}{4R_0}\left[2\pi \int_0^\infty r^3 \zeta^{(0)} dr\right]. \tag{5.15}$$

For the initial 'δ-function' core, the radial centroid evolve as

$$R_c(t) \approx R_0 + \frac{3\nu t}{R_0}, \tag{5.16}$$

again being in contradiction with Wang *et al.*'s claim that R_c is a constant.

It is to be noted that the expansion laws of R_s, R_p and R_c do not include the parameter Γ. These laws are all attributable to the effect of viscous diffusion of curved vortex lines, which is linear in ν.

5.2 The third-order correction to the translation velocity

The equations to be manipulated to obtain $\dot{Z}^{(2)}$ reduce, after numerous cancellations, to

$$\begin{aligned}
&\frac{1}{r}\left\{ v^{(0)} \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi_{11}^{(3)}\right)\right] + \frac{\partial \zeta^{(0)}}{\partial r}\psi_{11}^{(3)} \right\} \\
&\quad + R^{(2)} \dot{Z}^{(0)} \frac{\partial \zeta^{(0)}}{\partial r} + R_0 \left(\dot{Z}^{(2)} \frac{\partial \zeta^{(0)}}{\partial r} + \dot{Z}^{(0)} \frac{\partial \zeta_0^{(2)}}{\partial r} \right) \\
&- \frac{1}{r}\left\{ \frac{1}{r}\left(\int_0^r r' \zeta_0^{(2)} dr'\right) \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\psi_{11}^{(1)}\right)\right] - \frac{\partial \zeta_0^{(2)}}{\partial r}\psi_{11}^{(1)} \right\} - \frac{2v^{(0)}}{r^2}\left(\int_0^r r'\zeta_0^{(2)} dr'\right) \\
&\quad + \frac{2}{R_0}\left(\frac{1}{R_0}\frac{\partial \psi^{(0)}}{\partial r}\Delta\psi_0^{(2)} + \frac{\partial \psi_0^{(2)}}{\partial r}\zeta^{(0)} + 2R^{(2)} v^{(0)} \zeta^{(0)} \right) \\
&\quad - \frac{v^{(0)}}{R_0 r}\left(\frac{1}{2}\frac{\partial \tilde{\psi}_{21}^{(2)}}{\partial r} + \frac{\tilde{\psi}_{21}^{(2)}}{r} \right) \\
&\quad + \frac{1}{r}\left\{ \frac{1}{2}\left(-\frac{\partial \zeta_{21}^{(2)}}{\partial r}\tilde{\psi}_{11}^{(1)} + \frac{1}{R_0} a\tilde{\psi}_{11}^{(1)} \frac{\partial \tilde{\psi}_{21}^{(2)}}{\partial r} \right) - \zeta_{21}^{(2)} \frac{\partial \tilde{\psi}_{11}^{(1)}}{\partial r} \right. \\
&\quad\quad \left. + \frac{1}{R_0}\left[\frac{\partial}{\partial r}\left(a\tilde{\psi}_{11}^{(1)}\right) + r\frac{\partial \zeta^{(0)}}{\partial r} \right]\tilde{\psi}_{21}^{(2)} \right\} - \frac{1}{4R_0^2}\left[a\tilde{\psi}_{11}^{(1)}\frac{\partial \tilde{\psi}_{11}^{(1)}}{\partial r} + \frac{a}{r}\left(\tilde{\psi}_{11}^{(1)}\right)^2 \right] \\
&\quad + \frac{5v^{(0)}}{4R_0^2}\left(\frac{\partial \tilde{\psi}_{11}^{(1)}}{\partial r} + \frac{\tilde{\psi}_{11}^{(1)}}{r} \right) - \frac{r\zeta^{(0)}}{R_0^2}\frac{\partial \tilde{\psi}_{11}^{(1)}}{\partial r} \\
&\quad + \frac{3rv^{(0)}}{4R_0^2} a\tilde{\psi}_{11}^{(1)} + \frac{5}{4R_0^2} r\left(v^{(0)}\right)^2 + \frac{\dot{Z}^{(0)}}{R_0}\left(r\zeta^{(0)} - 2v^{(0)}\right) = 0.
\end{aligned} \tag{5.17}$$

The matching condition (3.6) reads

$$\psi_{11}^{(3)} \sim \frac{3}{2^7\pi R_0^2}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-\frac{1}{3}\right]r^3 - \frac{d^{(1)}}{8R_0^2}\left[\log\left(\frac{8R_0}{\epsilon r}\right)-\frac{7}{4}\right]r - \frac{R^{(2)}}{4\pi R_0}r + \frac{d_1^{(3)}}{r} \quad \text{as } r \to \infty . \tag{5.18}$$

The coefficient $d_1^{(3)}$ is irrelevant to the determination of $\dot{Z}^{(2)}$. It is worth emphasising that the terms multiplied by $d^{(1)}$ represent the flow induced by a line of dipoles arranged on the core centerline, which has been overlooked in previous studies.

The remaining task is to multiply (5.17) by r^2, integrate with respect to r, and seek the limiting form as $r \to \infty$. Using (6.5) and the non-singular condition at $r = 0$, and taking the limit $r \to \infty$, we eventually arrive at the desired formula:

$$\begin{aligned}\dot{Z}^{(2)} = & -\frac{3d^{(1)}}{8R_0^3}\left[\log\left(\frac{8R_0}{\epsilon}\right)+\frac{4}{3}A-\frac{7}{6}\right] - \frac{P^{(2)}}{8\pi^2 R_0^3}\left[\log\left(\frac{8R_0}{\epsilon}\right)+A-\frac{3}{2}\right] \\ & -\frac{\pi}{4R_0^3}\left(B-\frac{13}{8}\int_0^\infty r^4\zeta^{(0)}v^{(0)}dr\right) \\ & -\frac{2\pi}{R_0}\int_0^\infty\left[\int_0^r r'\zeta_0^{(2)}(r')dr'\right]v^{(0)}(r)dr - \frac{\pi}{2R_0^3}\int_0^\infty\left(2ra+b\tilde{\psi}_{11}^{(1)}\right)rv^{(0)}\tilde{\psi}_{21}^{(2)}dr \\ & +\frac{\pi}{8R_0^3}\int_0^\infty\left[ra\left(\tilde{\psi}_{11}^{(1)}-3r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right)\tilde{\psi}_{11}^{(1)}+b\left(\tilde{\psi}_{11}^{(1)}-r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right)\left(\tilde{\psi}_{11}^{(1)}\right)^2\right]dr ,\end{aligned} \tag{5.19a}$$

where A and $P^{(2)}$ are given by (4.19b) and (5.4) and

$$B = \lim_{r\to\infty}\left[\int_0^r r'v^{(0)}(r')\tilde{\psi}_{11}^{(1)}(r')dr' + \frac{1}{16\pi^2}(\log r + A)r^2 + \frac{d^{(1)}}{2\pi}\log r\right] . \tag{5.19b}$$

If, in particular, $\zeta^{(0)}$ evolves as (4.5), then $P^{(2)} = 0$ and (5.19a) reduces to

$$\begin{aligned}\dot{Z}^{(2)} = & -\frac{3d^{(1)}}{8R_0^3}\left\{\log\left(\frac{8R_0}{\epsilon}\right)+\frac{2}{3}\left[\log\left(\frac{1}{4\sqrt{t}}\right)+\gamma-\frac{7}{4}\right]\right\} - \frac{\pi B}{4R_0^3} - \frac{39}{2^7\pi R_0^3}\hat{\nu}t \\ & -\frac{2\pi}{R_0}\int_0^\infty\left[\int_0^r r'\zeta_0^{(2)}(r')dr'\right]v^{(0)}(r)dr - \frac{\pi}{2R_0^3}\int_0^\infty\left(2ra+b\tilde{\psi}_{11}^{(1)}\right)rv^{(0)}\tilde{\psi}_{21}^{(2)}dr \\ & +\frac{\pi}{8R_0^3}\int_0^\infty\left[ra\left(\tilde{\psi}_{11}^{(1)}-3r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right)\tilde{\psi}_{11}^{(1)}+b\left(\tilde{\psi}_{11}^{(1)}-r\frac{\partial\tilde{\psi}_{11}^{(1)}}{\partial r}\right)\left(\tilde{\psi}_{11}^{(1)}\right)^2\right]dr .\end{aligned} \tag{5.20}$$

Our formula is completed with construction of a formal solution for $\zeta_0^{(2)}$, which is the topic of the following section. In a general case, numerical computation of $\tilde{\psi}_{21}^{(2)}$ and $\zeta_0^{(2)}$ and numerical integration in the above formula are required to evaluate $\dot{Z}^{(2)}$. The remaining computation is left for a subsequent paper.

Fortunately the explicit solution is at hand for the Rankine vortex (4.21) at $O(\epsilon^0)$. In this case, $A = 1/4$, $d^{(1)} = -3/(16\pi)$ as read off from (4.23), $R^{(2)} = 0$ and $\zeta_0^{(2)} = 0$. ¿From (5.19b), $B = 11/(3\cdot 2^7\pi^2)$. The definition (5.4) gives $P^{(2)} = 3\pi/4$. Introduction of these values, (4.21), (4.23) and (4.34) into the formula (5.19a) gives rise to the third-order correction in Dyson's formula (1.1).

6 The axisymmetric part of second-order vorticity

The vorticity equation at $O(\epsilon^3)$ is solved for $\zeta_{12}^{(3)}$, giving

$$\zeta_{12}^{(3)} = \frac{a}{R_0}\tilde{\psi}_{12}^{(3)} + \frac{r}{v^{(0)}}\left\{-\frac{\partial\zeta_{11}^{(1)}}{\partial t} + \hat{\nu}\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\zeta_{11}^{(1)} + \frac{1}{R_0}\frac{\partial\zeta^{(0)}}{\partial r}\right]\right\}, \tag{6.1}$$

where we have defined

$$\psi_{12}^{(3)} = \tilde{\psi}_{12}^{(3)} + R_0 r\dot{R}^{(2)}. \tag{6.2}$$

Substituting $\psi^{(1)} = \psi_{11}^{(1)}\cos\theta$, (4.25), (5.1) and the associated vorticity distribution (4.20) and (4.33) into the vorticity equation at $O(\epsilon^4)$, we get, after some manipulation, a somewhat simple convection–diffusion equation for $\zeta_0^{(2)}$. By virtue of (6.1), a further simplification is achieved, leaving

$$\frac{\partial\zeta_0^{(2)}}{\partial t} - \hat{\nu}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\zeta_0^{(2)}}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left\{-\frac{r}{2R_0 v^{(0)}}\left[\frac{\partial\zeta_{11}^{(1)}}{\partial t}\right.\right.$$
$$\left.\left.-\hat{\nu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\zeta_{11}^{(1)}\right]\tilde{\psi}_{11}^{(1)} + \frac{\dot{R}^{(2)}r^2}{2R_0}\zeta^{(0)}\right\}. \tag{6.3}$$

The appropriate initial condition is

$$\zeta_0^{(2)}(r,0) = 0 \quad \text{for } 0 \le r < \infty. \tag{6.4}$$

A glance at (6.3) shows that, whether viscosity is present or not,

$$\int_0^\infty r\zeta_0^{(2)}(r,t)dr = 0 \quad \text{for all } t \ge 0, \tag{6.5}$$

under the initial condition (6.4).

By using a Green's function, the unique solution of (6.3) for $\hat{\nu} = 1$ may be written out. If, in particular, a δ-function core is assumed at the initial instant, then it admits the following solution:

$$\begin{aligned}\zeta_0^{(2)}(r,t) &= \frac{1}{2R_0 t}\left(\int_0^t t'\dot{R}^{(2)}(t')dt'\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\zeta^{(0)}\right)\\ &-\frac{1}{4R_0\hat{\nu}}\int_0^t dt'\frac{\exp\{-r^2/4\hat{\nu}(t-t')\}}{t-t'}\int_0^\infty dr'\exp\left\{-\frac{r'^2}{4\hat{\nu}(t-t')}\right\}I_0\left(\frac{rr'}{2\hat{\nu}(t-t')}\right)\\ &\times\frac{\partial}{\partial r'}\left\{\frac{r'\tilde{\psi}_{11}^{(1)}(r',t')}{v^{(0)}(r',t')}\left[\frac{\partial}{\partial t'}-\hat{\nu}\left(\frac{\partial^2}{\partial r'^2}+\frac{1}{r'}\frac{\partial}{\partial r'}-\frac{1}{r'^2}\right)\right]\zeta_{11}^{(1)}(r',t')\right\},\end{aligned} \tag{6.6}$$

where I_0 is the modified Bessel function of zeroth order of the first kind.

References

Caffarelli, L.A., Friedman, A. (1980) 'Asymptotic estimates for the plasma problem', *Duke Math. J.* **47**, 795–742.

Callegari, A. J., Ting, L. (1978) 'Motion of a curved vortex filament with decaying vortical core and axial velocity', *SIAM J. Appl. Maths* **35**, 148–175.

Dyson, F. W. (1893) 'The potential of an anchor ring. – part II', *Phil. Trans. R. Soc. Lond. A* **184**, 1041–1106.

Fraenkel, L. E. (1972) 'Examples of steady vortex rings of small cross-section in an ideal fluid', *J. Fluid Mech.* **51**, 119–135.

Fraenkel, L. E. (1999) *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*, Chapters 3, 4, Cambridge University Press.

Fukumoto, Y., Miyazaki, T. (1991) 'Three-dimensional distortions of a vortex filament with axial velocity', *J. Fluid Mech.* **222**, 369–416.

Fukumoto, Y., Moffatt, H. K. (2000) 'Motion and expansion of a viscous vortex ring. Part 1. A higher-order asymptotic formula for the velocity', to appear in *J. Fluid Mech.*

Jiménez, J., Moffatt, H. K., Vasco, C. (1996) 'The structure of vortices in freely decaying two-dimensional turbulence', *J. Fluid Mech.* **313**, 209–222.

Kelvin, Lord. (1867) 'The translatory velocity of a circular vortex ring', *Phil. Mag. (4)* **35**, 511–512.

Kida, S., Ohkitani, K. (1992) 'Spatiotemporal intermittency and instability of a forced turbulence', *Phys. Fluids A* **4**, 1018–1027.

Klein, R., Knio, O. M. (1995) 'Asymptotic vorticity structure and numerical simulation of slender vortex filaments', *J. Fluid Mech.* **284**, 275–321.

Moffatt, H. K., Kida, S., Ohkitani, K. (1994) 'Stretched vortices – the sinews of turbulence; large-Reynolds-number asymptotics', *J. Fluid Mech.* **259**, 241–264.

Moore, D. W., Saffman, P. G. (1972) 'The motion of a vortex filament with axial flow', *Phil. Trans. R. Soc. Lond. A* 272, 403–429.

Saffman, P. G. (1970) 'The velocity of viscous vortex rings', *Stud. Appl. Math.* 49, 371–380.

Shariff, K., Leonard, A. (1992) 'Vortex rings', *Ann. Rev. Fluid Mech.* **24**, 235–279.

Ting, L., Klein, R. (1991) *Viscous Vortical Flows*, Lecture Notes in Physics **374**, Springer.

Ting, L., Tung, C. (1965) 'Motion and decay of a vortex in a nonuniform stream', *Phys. Fluids* **8**, 1039–1051.

Tung, C., Ting, L. (1967) 'Motion and decay of a vortex ring', *Phys. Fluids* **10**, 901–910.

Wang, C.-T., Chu, C.-C., Chang, C. C. (1994) 'Initial motion of a viscous vortex ring', *Proc. R. Soc. Lond. A* **446**, 589-599.

Widnall, S. E., Bliss, D. B., Zalay, A. (1971) 'Theoretical and experimental study of the stability of a vortex pair', In *Aircraft Wake Turbulence and its Detection*, (eds. Olsen, Goldberg & Rogers), 305–338, Plenum.