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ScienceDirect
Physica D 237 (2008) 2210–2217
PHYSICA D
www.elsevier.com/locate/physd

Kinematic variational principle for motion of vortex rings

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Available online 12 February 2008

Abstract

We show how the ideas of topology and variational principle, opened up by Euler, facilitate the calculation of motion of vortex rings. Kelvin–Benjamin's principle, as generalised to three dimensions, states that a steady distribution of vorticity, relative to a moving frame, is the state that maximizes the total kinetic energy, under the constraint of constant hydrodynamic impulse, on an iso-vortical sheet. By adapting this principle, combined with an asymptotic solution of the Euler equations, we make an extension of Fraenkel–Saffman's formula for the translation velocity of an axisymmetric vortex ring to third order in a small parameter, the ratio of the core radius to the ring radius. Saffman's formula for a viscous vortex ring is also extended to third order.

PACS: 47.10.A-; 47.10.ad; 47.15.ki; 47.32.C-

Keywords: Vortex ring; Translation velocity; Variational principle; Iso-vortical sheet

1. Introduction

Euler opened up the field of topology when he presented the solution to the Königsberg bridge problem in 1735 [1]. As "geometry of position" in the title signifies, Euler envisaged a new type of geometric problem in which distance is not relevant. In 1750, he discovered the polyhedral theorem on the Euler characteristic, a summation of alternately signed numbers of vertices, edges and faces of a polyhedron [2]. This theorem stands as the cornerstone of topology. Almost at the same time, the Euler equations for fluid flows were born.

Euler's 1757 paper [3] certainly overcame the limitation to irrotational velocity field, posed by Bernoulli, and accommodated vorticity. However a century passed before Helmholtz discovered the key to the heart of vortex motion that the vortex lines are frozen into the fluid [4]. Helmholtz' theorem implied that link and knot types of vortex lines remain unchanged throughout the flow evolution. This implication, along with the invariance of circulation, sparked, in Scotland, the construction of atom models by knotted vortex tubes. Inspired by the vortex atom theory, Tait attempted classification of knot and link types [5]. It took another century for the helicity to be discovered [6–9]. This topological invariant is tied with linkage and knottedness of vortex filaments [9]. More precisely, the helicity embodies the Călugăreanu invariant [10], a summation of the writhe and the twist, of a twisted flux tube [11].

The study of the motion of vortex rings started simultaneously with the birth of the field of vortex dynamics [4]. Extending Helmholtz' analysis, Kelvin obtained the formula for velocity of an axisymmetric vortex ring, steadily translating in an inviscid incompressible fluid of infinite extent, for a distribution of vorticity, in the core, proportional to the distance from the axis of symmetry. The assumption is made that the ring is very thin:

$$\varepsilon = \sigma/R_0 \ll 1, \tag{1}$$

where σ is the core radius and R_0 is the ring radius. The formula allowing for an arbitrary distribution of vorticity was found by Fraenkel [12] and Saffman [13] (see also Ref. [14]) as

$$U_0 = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8R_0}{\sigma}\right) + A - \frac{1}{2}\right\}, \tag{2}$$

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doi:10.1016/j.physd.2008.02.003

where Γ is the circulation and

$$A = \lim_{r\to\infty}\left\{\frac{4\pi^2}{\Gamma^2}\int_0^r r' v_0(r')^2 \mathrm{d}r' - \log\left(\frac{r}{\sigma}\right)\right\}, \tag{3}$$

with $v_0(r)$ being the local velocity of circulatory motion of the fluid around the toroidal center circle, as a function only of the local distance r from the circle. In the absence of viscosity, $v_0(r)$ and therefore the local vorticity field may be arbitrary functions of r.

Viscosity acts to diffuse vorticity, and the motion ceases to be steady. For a vortex ring with its toroidal vorticity $\zeta(r,t)$ 'δ-function' concentrated on the circle of radius R_0, at a virtual instant,

$$\zeta(r,0) = \Gamma\delta(\rho - R_0)\delta(z - Z) \quad \text{at } t = 0, \tag{4}$$

with $r^2 = (\rho - R_0)^2 + (z - Z)^2$, it suffices to substitute, into (3), the Oseen diffusing vortex

$$\zeta_0 = \frac{\Gamma}{4\pi\nu t}\mathrm{e}^{-r^2/4\nu t}, \qquad v_0 = \frac{\Gamma}{2\pi r}\left(1 - \mathrm{e}^{-r^2/4\nu t}\right), \tag{5}$$

where ν is the kinematic viscosity and t is the time measured from the instant at which the core is infinitely thin. With this form, (2) supplemented by (3) becomes

$$U_0 = \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{8R_0}{2\sqrt{\nu t}}\right) - \frac{1}{2}(1 - \gamma + \log 2)\right\}, \tag{6}$$

where $\gamma = 0.57721566\cdots$ is Euler's constant. Comparison with the result of numerical simulation of the axisymmetric Navier–Stokes equations [15] illustrates that validity of Saffman's formula (6) is limited to very small times ($\nu t/R_0^2 \ll 1$) [16].

Vortex rings observed in nature are not necessarily thin. Kelvin's formula is an asymptotic solution to $O(\varepsilon)$ for vorticity linear in the distance from the symmetric axis. Dyson [17] accomplished its extension to $O(\varepsilon^3)$ [18]. For this distribution, evidence is available that Dyson's formula fits rather well with the speed of Hill's spherical vortex, the fat limit of Fraenkel–Norbury's family of vortex rings [19]. This unexpected agreement stimulates us to pursue a higher-order correction to (2).

The method of matched asymptotic expansions has been previously developed for a systematic treatment of motion of slender vortex tubes [14,20], and was extended to second order in ε [21]. Derivation of the correction to Fraenkel–Saffman's formula (2) requests us to enter into the third order. A flood of nonlinear terms of a higher order in the Navier–Stokes equations makes our mathematical handling out of control. It was shown that the radius of the circle of vorticity centroid grows linearly in time due to the action of vorticity [22], but reduction of the expression for the speed of a vortex ring remains yet to be attained. The method of Lamb–Saffman–Rott–Cantwell [23,13,24] provides an efficient means.

We show how topological ideas help to bring in a further simplification. It is well known that a stationary configuration of vorticity, embedded in an inviscid incompressible fluid, is realizable as an extremal of energy on an iso-vortical sheet [25–27]. An iso-vortical sheet comprises volume-preserving diffeomorphisms, or smooth maps of fluid particles, with vorticity frozen into the fluid. For a moving state, this conditional variational principle should be augmented by a constraint. Benjamin [28] put forward a variational principle that an axisymmetric vortex ring moving steadily in an inviscid incompressible fluid is realizable as the maximum state of the kinetic energy H on an iso-vortical sheet, subject to the constraint of constant hydrodynamic impulse

$$\boldsymbol{P} = \frac{1}{2}\iiint \boldsymbol{x}\times\boldsymbol{\omega}\,\mathrm{d}V. \tag{7}$$

When translated into three dimensions, Kelvin–Benjamin's principle reads

$$\delta H - \boldsymbol{U}\cdot\delta\boldsymbol{P} = 0, \tag{8}$$

where the velocity $\boldsymbol{U}$ of the region plays the role of the Lagrangian multipliers.

An iso-vortical sheet is infinite dimensional. A family of solutions of the Euler equations include several parameters. By posing some relations on these parameters, we can maintain the solutions on a single iso-vortical sheet, and, when restricted to this family, the dimension of an iso-vortical sheet is reduced to finite. Thus the traveling speed of a vortex ring may be calculable through (8). This is indeed the case for the first-order velocity formula as listed in the book [29]. The principle (8) has a wider applicability as exemplified by a vortex ring governed by the Gross–Pitaevskii equation [30]. In this paper, we adapt this variational principle to deduce the $O(\varepsilon^3)$ correction to the traveling speed. At large Reynolds numbers, the viscosity plays a secondary role only of selecting vorticity profile, and the inviscid formula is applicable to give the correction term to Saffman's formula (6).

We begin with the general variational formulation in three dimensions (Section 2). After a statement of asymptotic expansions of the flow field, the kinetic energy and the impulse (Section 3), we recall the outer and inner solutions [22] in Sections 4 and 5 respectively. Thereafter, we calculate, in Section 6, the energy and the impulse to $O(\varepsilon^2)$ and present, in Section 7, a recipe for implementing (8) to produce the $O(\varepsilon^3)$ correction to Fraenkel–Saffman's formula (2) and Saffman's formula (6) for the traveling speed of vortex rings. It is highly probable that a vortex ring obeying the Euler equations is a maximum-energy state [28,31]. The upper bound of energy, if available, guarantees the existence of this extremal, and is furnished by a topological invariant [32]. Appendix gives a concise description for viewing this invariant as a variant of the helicity [33].

2. Variational principle

Roberts [34] proved the above principle for an axisymmetric vortex ring steadily translating in an inviscid fluid. Below, we extend this principle to three dimensions to gain an insight into the variational structure.

Under the assumption that the fluid is incompressible, we can introduce the vector potential $\boldsymbol{A}$ for the velocity field $\boldsymbol{u}$ ($\boldsymbol{u} = \nabla \times \boldsymbol{A}$). We assume that the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is localised in some finite region in such a way that the velocity decreases sufficiently rapidly. These assumptions admit a representation of the total kinetic energy H of the fluid, filling an unbounded space, as

$$H = \frac{1}{2}\iiint \boldsymbol{u}^2 \mathrm{d}V = \frac{1}{2}\iiint \boldsymbol{\omega}\cdot\boldsymbol{A}\mathrm{d}V, \tag{9}$$

where the density of fluid is set to be unity.

We confine ourselves to steady motion, with constant speed $\boldsymbol{U}$, of a region with vorticity and assume that the flow is stationary in a frame moving with $\boldsymbol{U}$. It is expedient to partition the velocity $\boldsymbol{u}$ as $\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{U}$. By the assumption that the relative velocity $\bar{\boldsymbol{u}}$ is steady, it obeys

$$\nabla \times (\bar{\boldsymbol{u}} \times \boldsymbol{\omega}) = 0. \tag{10}$$

Consequently, there exists a globally defined spatial function $h(\boldsymbol{x})$ such that

$$\bar{\boldsymbol{u}} \times \boldsymbol{\omega} = \nabla h. \tag{11}$$

Suppose that fluid particles undergo an infinitesimal displacement $\delta\xi$ while preserving the volume of an arbitrary fluid element:

$$\boldsymbol{x} \to \tilde{\boldsymbol{x}} = \boldsymbol{x} + \delta\xi(\boldsymbol{x})\,; \qquad \nabla\cdot\delta\xi = 0. \tag{12}$$

We impose the condition that the flux of vorticity through an arbitrary material surface be unchanged throughout the process of the displacement. Its local representation is [26,27,32]

$$\delta\boldsymbol{\omega} = \nabla \times (\delta\xi \times \boldsymbol{\omega})\,. \tag{13}$$

In keeping with the above, we decompose the vector potential $\boldsymbol{A}(\boldsymbol{x})$ as $\boldsymbol{A} = \bar{\boldsymbol{A}} - \boldsymbol{x}\times\boldsymbol{U}/2$. Using the definition $\bar{\boldsymbol{u}} = \nabla\times\bar{\boldsymbol{A}}$, we can deduce, from (11) and (13),

$$\bar{\boldsymbol{A}}\cdot\delta\boldsymbol{\omega} = -\nabla\cdot\left\{h\delta\xi + \bar{\boldsymbol{A}}\times(\delta\xi\times\boldsymbol{\omega})\right\}. \tag{14}$$

The variation δH of the kinetic energy, subjected to the variation of fluid-particle positions (12), is calculated as

$$\begin{aligned}\delta H &= \iiint \boldsymbol{A}\cdot\delta\boldsymbol{\omega}\,\mathrm{d}V = \boldsymbol{U}\cdot\left(\frac{1}{2}\iiint \boldsymbol{x}\times\delta\boldsymbol{\omega}\,\mathrm{d}V\right)\\ &\quad - \iint\left\{h\delta\xi + \bar{\boldsymbol{A}}\times(\delta\xi\times\boldsymbol{\omega})\right\}\cdot\boldsymbol{n}\,\mathrm{d}A.\end{aligned} \tag{15}$$

The surface integral is taken over the closed surface receding to infinity that bounds the whole region. The second term of the surface integral vanishes under the assumption that the vorticity $|\boldsymbol{\omega}|$ decays sufficiently rapidly with distance $|\boldsymbol{x}|$, say exponentially in $|\boldsymbol{x}|$. Under the same assumption, h approaches a constant h_∞ at large distances $|\boldsymbol{x}|$, and the first term of the surface integral vanishes, with the aid of the Gauss theorem, owing to (12). Consequently, we are left only with the volume integral in (15). The variation of the hydrodynamic impulse (7) is

$$\delta\boldsymbol{P} = \frac{1}{2}\iiint \boldsymbol{x}\times\delta\boldsymbol{\omega}\,\mathrm{d}V. \tag{16}$$

With this form, (15) is reckoned upon as the variational principle (8) for the translation speed $\boldsymbol{U}$ of the vortex region.

A steadily moving vortex ring would be the maximal of the energy [28,31]. For a compact distribution of vorticity of an axisymmetric vortex ring, an upper bound on the energy is supplied by the topological invariant (A.8), with fluid density $\rho_\mathrm{f} = 1$, which guarantees the existence of the solution for vortex rings. By Poincaré's inequality, the energy (9) is bounded above as $\int \boldsymbol{u}^2\mathrm{d}V \le C\int\boldsymbol{\omega}^2\mathrm{d}V$ for some constant C. Introduce cylindrical coordinates (ρ,ϕ,z) with the z-axis coincident with the axis of symmetry and ρ being the distance from the symmetric axis, Supposing that the vorticity $\boldsymbol{\omega} = \zeta\boldsymbol{e}_\phi$, with $\boldsymbol{e}_\phi$ being the unit vector in the ϕ-direction, is confined to a compact region $\mathcal{A}$ in the meridional plane, the enstrophy is shown to be bounded above in the following way [32]:

$$\begin{aligned}\int_{\mathcal{A}}\zeta^2\rho\mathrm{d}\rho\mathrm{d}z &= \int_{\mathcal{A}}\left(\frac{\zeta}{\rho}\right)^2\rho^2\rho\mathrm{d}\rho\mathrm{d}z\\ &\le \left\{\int_{\mathcal{A}}\left(\frac{\zeta}{\rho}\right)^4\rho\mathrm{d}\rho\mathrm{d}z\int_{\mathcal{A}}\rho^4\rho\mathrm{d}\rho\mathrm{d}z\right\}^{1/2} = \text{const.}\end{aligned} \tag{17}$$

The similar would hold true for a continuous, but localised, distribution of vorticity.

3. Asymptotic expansions of energy and impulse

We confine ourselves to steady motion of axisymmetric vortex rings with vorticity $\boldsymbol{\omega}$ in the toroidal directions. The vector potential $\boldsymbol{A} = -\psi(\rho,z,t)/\rho\boldsymbol{e}_\phi$ possesses azimuthal components only. The scalar field ψ is named the Stokes streamfunction.

We build the solution of the Euler equations in the form of a power series in the small parameter ε, inside and around the core. To this end, it is advantageous to employ the local moving cylindrical coordinates (r,θ), on the meridional plane, with the origin maintained in the core. We nondimensionalize the variables in terms of the circulation Γ, the typical ring radius R_0 and the core radius σ. Attached with an asterisk, the nondimensional variables look like

$$\begin{aligned}&r^* = r/\varepsilon R_0, \qquad t^* = t/\frac{R_0^2}{\Gamma}, \qquad \psi^* = \frac{\psi}{\Gamma R_0},\\ &\zeta^* = \zeta/\frac{\Gamma}{R_0^2\varepsilon^2}, \qquad \boldsymbol{u}^* = \boldsymbol{u}/\frac{\Gamma}{R_0\varepsilon},\\ &(\dot{R}^*,\dot{Z}^*) = (\dot{R},\dot{Z})/\frac{\Gamma}{R_0}.\end{aligned} \tag{18}$$

A glance at the equations written in this moving frame tells the dependence, on θ, of the solution in a power series in ε to be [22]

$$\begin{aligned}\psi &= \psi^{(0)}(r) + \varepsilon\psi_{11}^{(1)}(r)\cos\theta\\ &\quad + \varepsilon^2\left[\psi_0^{(2)}(r) + \psi_{21}^{(2)}(r)\cos 2\theta\right] + O(\varepsilon^3),\end{aligned} \tag{19}$$

$$\begin{aligned}\zeta &= \zeta^{(0)}(r) + \varepsilon\zeta_{11}^{(1)}(r)\cos\theta\\ &\quad + \varepsilon^2\left[\zeta_0^{(2)}(r) + \zeta_{21}^{(2)}(r)\cos 2\theta\right] + O(\varepsilon^3).\end{aligned} \tag{20}$$

We solve the Euler equations in a frame moving with the vortex ring. The origin of this moving frame should have some bearing with the core center in the meridional plane, but, for a core of finite thickness, the definition of the center is subtle. We expand the radial position R of the center in powers of ε as

$$R = 1 + \varepsilon^2 R^{(2)} + O(\varepsilon^3). \tag{21}$$

Keeping the first term to be unity by adjusting the origin of the moving frame would be helpful to the analyses that follow.

For axisymmetric flows, the kinetic energy (9) and the hydrodynamic impulse (7) reduces, respectively, to

$$H = -\pi \iint \psi\zeta \,\mathrm{d}\rho \mathrm{d}z, \qquad \boldsymbol{P} = \pi \iint \zeta \rho^2 \mathrm{d}\rho \mathrm{d}z \boldsymbol{e}_z. \tag{22}$$

Correspondingly to (18), these are normalized as

$$H^* = H/\Gamma^2 R_0, \qquad P_z^* = P_z/\Gamma R_0^2, \tag{23}$$

where P_z is the z component of $\boldsymbol{P}$. Upon substitution from (19)–(21), we obtain a representation of (23) to $O(\varepsilon^2)$, as

$$\begin{aligned} H &= -2\pi^2 \int_0^\infty \left\{ r\zeta^{(0)}\psi^{(0)} + \varepsilon^2 r \left[\frac{1}{2}\zeta_{11}^{(1)}\psi_{11}^{(1)} \right.\right. \\ &\quad \left.\left. + \zeta^{(0)}\psi_0^{(2)} + \zeta_0^{(2)}\psi^{(0)} \right] \right\} \mathrm{d}r + O(\varepsilon^3), \end{aligned} \tag{24}$$

$$\begin{aligned} P_z &= \pi + \varepsilon^2 \pi \left[2R^{(2)} + \pi \int_0^\infty \zeta^{(0)} r^3 \mathrm{d}r \right. \\ &\quad \left. + 2\pi \int_0^\infty \zeta_{11}^{(1)} r^2 \mathrm{d}r \right] + O(\varepsilon^3). \end{aligned} \tag{25}$$

It is noteworthy that the kinetic energy H and the impulse P_z are gained, to $O(\varepsilon^2)$, without knowledge of the quadrupole components $\psi_{21}^{(1)}$ and $\zeta_{21}^{(1)}$ of $O(\varepsilon^2)$. Except for cores of uniform ζ/ρ, calculation of $\psi_{21}^{(2)}$ and $\zeta_{21}^{(1)}$ requires numerical integration and stands as an obstacle, though this is not the case with the monopole component $\psi_0^{(2)}$ and $\zeta_0^{(2)}(r)$ of $O(\varepsilon^2)$ and $O(\varepsilon)$ field. Relying on the variational principle, the kinetic energy H and the impulse P_z, to $O(\varepsilon^2)$, are sufficient to deduce the formula for $\boldsymbol{U}$ valid to $O(\varepsilon^3)$. Advent of the variational principle dispenses not only with the quadrupole field of $O(\varepsilon^2)$ but also with the $O(\varepsilon^3)$ field. In the following sections, we enumerate the necessary expressions of flow field.

4. Outer solution

The energy (24) desires the flow field only in the region supported by vorticity, namely the inner solution. In spite of this, the outer solution is necessary to supply the boundary condition on the inner field.

The outer solution is nothing but the Biot–Savart law and is written, for the Stokes streamfunction, as

$$\psi(\rho, z) = -\frac{\rho}{4\pi}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^\infty \mathrm{d}\rho'\mathrm{d}\phi'\mathrm{d}z'\zeta(\rho',z')\rho'\cos\phi' \Big/ \left\{\rho^2 - 2\rho\rho'\cos\phi' + \rho'^2 + (z-z')^2\right\}^{1/2}. \tag{26}$$

Dyson's shift-operator technique is adapted to manipulate the inner limit of (26) for an arbitrary distribution of vorticity in the form of (20) [22]. The asymptotic development of the Biot–Savart law valid to $O(\varepsilon^2)$, in a region $\varepsilon \ll r/R \ll 1$ surrounding the core, is

$$\begin{aligned} \psi &= -\frac{\Gamma}{2\pi}\left[\log\left(\frac{8}{\varepsilon r}\right) - 2\right] \\ &\quad + \varepsilon\left\{-\frac{\Gamma}{4\pi}\left[\log\left(\frac{8}{\varepsilon r}\right) - 1\right] r\cos\theta + d^{(1)}\frac{\cos\theta}{r}\right\} \\ &\quad + \varepsilon^2\left\{-\frac{\Gamma}{2^5\pi}\left(\left[2\log\left(\frac{8}{\varepsilon r}\right) + 1\right] r^2 \right.\right. \\ &\quad \left. - \left[\log\left(\frac{8}{\varepsilon r}\right) - 2\right] r^2\cos 2\theta\right) + \frac{d^{(1)}}{2}\left[\log\left(\frac{8}{\varepsilon r}\right) \right. \\ &\quad \left.\left. + \frac{\cos 2\theta}{2}\right] - \frac{\Gamma R^{(2)}}{2\pi}\log\left(\frac{8}{\varepsilon r}\right) + q^{(2)}\frac{\cos 2\theta}{r^2}\right\} \\ &\quad + O(\varepsilon^3), \end{aligned} \tag{27}$$

where $\Gamma = 2\pi\int_0^\infty r\zeta^{(0)}\mathrm{d}r = 1$, when nondimensionalized, and $d^{(1)} = d_1/\Gamma\sigma^2$ is the strength of the dipole of $O(\varepsilon)$ whose dimensional form will be provided later by (39). The expression of $q^{(2)}$, the strength of the quadrupole at $O(\varepsilon^2)$, is left out, as this is unnecessary.

5. Inner solution

The radial coordinate r^* in (18), normalized by the core radius σ, is peculiar to the inner expansion. The inner solution is found by solving the Euler or Navier–Stokes equations made dimensionless with use of the inner variables (18), subject to the matching condition (27), in powers of the small parameter ε. In the following we write down the resulting expressions of the vorticity and the Stokes streamfunction. The detail is found in Ref. [22].

In the absence of viscosity, the vorticity profile $\zeta^{(0)}(r)$ may be left unspecified. The local radial velocity is $u^{(0)} = 0$, and the local azimuthal velocity $v^{(0)}$, $\psi^{(0)}$ and $\zeta^{(0)}(r)$ are linked with each other via

$$v^{(0)} = -\frac{\partial\psi^{(0)}}{\partial r}, \qquad \zeta^{(0)} = \Delta\psi^{(0)} = -\frac{1}{r}\frac{\partial}{\partial r}\left(rv^{(0)}\right). \tag{28}$$

Integrating the first of (28), we obtain the leading-order streamfunction, complying with $O(\varepsilon^0)$ of (27), as

$$\begin{aligned} \psi^{(0)} &= -\int_0^r v^{(0)}(r')\mathrm{d}r' + \lim_{r\to\infty}\left\{\int_0^r v^{(0)}(r')\mathrm{d}r' \right. \\ &\quad \left. - \frac{1}{2\pi}\left[\log\left(\frac{8}{r}\right) - 2\right]\right\}. \end{aligned} \tag{29}$$

Viscosity plays the role of selecting the functional form of $\psi^{(0)}$. We introduce a dimensionless parameter switching on the action of viscosity ν.

$$\hat{\nu} = 0 \quad \text{for the inviscid case,} \quad = 1 \text{ for the viscous case.} \tag{30}$$

For the viscous case, $\varepsilon = \sqrt{\nu/\Gamma}$ takes the place of the small parameter. The axisymmetric (or θ-averaged) part of the vorticity equation at $O(\varepsilon^2)$ leads to the heat conduction equation for $\zeta^{(0)}$. For the initial δ-function core (4), the Oseen vortex (5) is picked out.

The first-order solution comprises a dipole field. The streamfunction corresponding to the uniform flow $-\dot{Z}^{(0)}\boldsymbol{e}_z$ in the z direction is given by $-\dot{Z}^{(0)}\rho^2/2$. Here a dot stands for differentiation with respect to time. Denote the dipole coefficient of the streamfunction for the flow, relative to the moving frame, to be $\tilde{\psi}_{11}^{(1)} = \psi_{11}^{(1)} + r\dot{Z}^{(0)}$. The coefficient function $\tilde{\psi}_{11}^{(1)}$ admits an explicit expression, in the form of a repeated integral, as

$$\tilde{\psi}_{11}^{(1)} = \Psi_{11}^{(1)} + c_{11}^{(1)} v^{(0)}, \tag{31}$$

where $c_{11}^{(1)}$ is a constant (which may depend on t), and

$$\Psi_{11}^{(1)} = -v^{(0)} \left\{ \frac{r^2}{2} + \int_0^r \frac{\mathrm{d}r'}{r'[v^{(0)}(r')]^2} \int_0^{r'} r'' \left[v^{(0)}(r'') \right]^2 \mathrm{d}r'' \right\}. \tag{32}$$

The vorticity is calculable through

$$\zeta_{11}^{(1)} = a\tilde{\psi}_{11}^{(1)} + r\zeta^{(0)}, \tag{33}$$

where

$$a(r,t) = -\frac{1}{v^{(0)}} \frac{\partial \zeta^{(0)}}{\partial r}. \tag{34}$$

The Fourier coefficient $\tilde{\psi}_0^{(2)}(r)$ of the monopole component of $O(\varepsilon^2)$, relative to the moving coordinate frame, is defined by $\tilde{\psi}_0^{(2)} = \psi_0^{(2)} + \dot{Z}^{(0)} r^2/4$. The vorticity equation is integrated for this component, resulting in

$$\frac{\partial \tilde{\psi}_0^{(2)}}{\partial r} = \frac{1}{r} \int_0^r r'\zeta_0^{(2)} \mathrm{d}r' + \frac{r}{2} \frac{\partial \tilde{\psi}_{11}^{(1)}}{\partial r} + \left(\frac{r^2}{2} - R^{(2)} \right) v^{(0)}. \tag{35}$$

The $O(\varepsilon^2)$ monopole component $\zeta_0^{(2)}$ of vorticity obeys

$$\frac{\partial \zeta_0^{(2)}}{\partial t} - \hat{\nu} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \zeta_0^{(2)}}{\partial r} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left\{ -\frac{r}{2v^{(0)}} \left[\frac{\partial \zeta_{11}^{(1)}}{\partial t} - \hat{\nu} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} \right) \zeta_{11}^{(1)} \right] \tilde{\psi}_{11}^{(1)} + \frac{\dot{R}^{(2)} r^2}{2} \zeta^{(0)} \right\}. \tag{36}$$

This equation is extracted from the axisymmetric part of the vorticity equations at $O(\varepsilon^4)$. The constraint that no net vorticity is created, $2\pi \int_0^\infty r\zeta_0^{(2)} \mathrm{d}r = 0$, is compatible with (36).

6. Asymptotics of energy and impulse

There are no terms of $O(\varepsilon)$ in the energy (24) and impulse (25). By substitution from (29), we obtain the leading-order term $H^{(0)}$ of (24), which is expressed, in terms of dimensional variables, as

$$H_0/\Gamma^2 = \frac{1}{2} R_0 \left\{ \log\left(\frac{8R_0}{\sigma} \right) + A - 2 \right\}, \tag{37}$$

where $H_0 = \Gamma^2 R_0 H^{(0)}$ and A is defined by (3).

Likewise, after some algebra, we obtain $H_2 = \Gamma^2 R_0 \varepsilon^2 H^{(2)}$, the dimensional form of the second-order term $H^{(2)}$ of (24), as follows:

$$\begin{aligned} \frac{H_2}{\Gamma^2} = &-\frac{\pi d_1}{2\Gamma R_0} \left\{ \log\left(\frac{8R_0}{\sigma} \right) - \frac{1}{2} + \frac{4\pi R_0}{\Gamma} U_0 \right\} + \frac{\pi^2 B}{R_0} \\ &- \frac{\pi^2}{\Gamma^2 R_0} \left[\frac{1}{2} \int_0^\infty r^4 \zeta_0 v_0 \mathrm{d}r + \int_0^\infty r a (\tilde{\psi}_{11}^{(1)})^2 \mathrm{d}r \right] \\ &- \frac{4\pi^2 R_0}{\Gamma^2} \int_0^\infty v_0(r) \left[\int_0^r r'\zeta_0^{(2)}(r') \right] \mathrm{d}r \\ &+ \frac{R_2}{2} \left\{ \log\left(\frac{8R_0}{\sigma} \right) + A - 1 \right\}, \end{aligned} \tag{38}$$

where $v_0 = \Gamma v^{(0)}/\sigma$, $\zeta_0 = \Gamma\zeta^{(0)}/\sigma^2$, $U_0 = \Gamma\dot{Z}^{(0)}/R_0$ and $R_2 = R_0\varepsilon^2 R^{(2)}$ are dimensional variables, and

$$d_1 = -\frac{1}{4} \left(\int_0^\infty r^3 \zeta_0 \mathrm{d}r + 2R_0 \int_0^\infty r^2 \zeta_{11}^{(1)} \mathrm{d}r \right), \tag{39}$$

$$B = \lim_{r\to\infty} \left\{ \frac{1}{\Gamma^2} \int_0^r r' v_0 \tilde{\psi}_{11}^{(1)} \mathrm{d}r' + \frac{r^2}{16\pi^2} \left[\log\left(\frac{r}{\sigma} \right) + A \right] + \frac{d_1}{2\pi\Gamma} \log\left(\frac{r}{\sigma} \right) \right\}. \tag{40}$$

It is to be understood that, in the above, the dimensional variables v_0 and ζ_0 are used in place of $v^{(0)}$ and $\zeta^{(0)}$, respectively.

In the inviscid case, (36) is integrated to produce

$$\int_0^r r'\zeta_0^{(2)}(r')\mathrm{d}r = -\frac{r}{4v^{(0)}} a\tilde{\psi}_{11}^{(1)} + \frac{R^{(2)}}{2} r^2 \zeta^{(0)}, \tag{41}$$

and cancellation among several terms is effected in (38).

The second-order term $P^{(2)}$ in (25), the z component of the hydrodynamic impulse, has a link with the strength d_1 of a dipole defined by (39), in such a way that

$$P_2 = \pi\left(2\Gamma R_0 R_2 - 4\pi d_1 \right), \tag{42}$$

where $P_2 = \Gamma R_0^2 \varepsilon^2 P^{(2)}$ [22].

7. Third-order correction to speed

We are now ready to implement the variational calculation (8) to produce the translation speed of an axisymmetric vortex ring. We set, as a natural profile of local velocity field featuring a vortex ring,

$$v_0(r) = -\frac{\Gamma}{2\pi r} f\left(\frac{r}{\sigma} \right), \qquad \zeta_0 = \frac{\Gamma}{2\pi r} \frac{\mathrm{d}}{\mathrm{d}r} f\left(\frac{r}{\sigma} \right), \tag{43}$$

where f is an arbitrary function, except for the requirement that

$$f(\xi) = O(\xi^2) \quad \text{as } \xi \to 0, \qquad f(\xi) \to 1 \quad \text{as } \xi \to \infty. \tag{44}$$

The parameter σ introduces the scale for the core thickness, and (43) includes both the constant vorticity, within the core, and the Gaussian distribution (5).

Suppose that the fluid particles occupying a toroidal region of radius r around the center circle of radius R are mapped to another toroidal region of radius $\hat{r}$ around the center circle of radius $\hat{R}$. To maintain these flow fields on an iso-vortical sheet, it is necessary for the circulation Γ to remain unchanged. Preservation of volume enforces

$$2\pi^2 r^2 R = 2\pi^2 \hat{r}^2 \hat{R}, \qquad 2\pi^2 \sigma^2 R = 2\pi^2 \hat{\sigma}^2 \hat{R}, \tag{45}$$

from which follows $r/\sigma = \hat{r}/\hat{\sigma}$. Consequently, the local circulation around the circle of radius r

$$\Gamma(r) = 2\pi \int_0^r \zeta_0(r')r'\mathrm{d}r' = \Gamma f(r/\sigma) \tag{46}$$

is invariant: $\Gamma(r) = \Gamma(\hat{r})$. Under an infinitesimal perturbation of $R \to \hat{R} = R+\delta R, \sigma \to \hat{\sigma} = \sigma+\delta\sigma$, with $R = R_0+R_2$, (45) demands that, at each order, $\sigma^2 R_0 =$ const. and $\sigma^2 R_2 =$ const., and therefore that

$$2\delta\sigma/\sigma = -\delta R_0/R_0 = -\delta R_2/R_2. \tag{47}$$

We can show that, under this perturbation, $\hat{A} = A + O\left((\delta R)^2\right)$, or $\delta A = 0$. It follows from this and the first of (47) that the variation of (37), under an iso-vortical perturbation, is

$$\delta H_0 = \frac{\Gamma^2}{2}\left[\log\left(\frac{8R_0}{\sigma}\right) + A - \frac{1}{2}\right]\delta R_0. \tag{48}$$

The variation of the leading term of impulse $P_0 = \Gamma\pi R_0^2$ is $\delta P_0 = 2\pi\Gamma R_0 \delta R_0$, and application of $\delta H_0 = U_0 \delta P_0$ restores Fraenkel–Saffman's formula (2). This result supplements the list in Ref. [29].

A great care should be exercised to proceed to a higher order. Because of the space limitation, we cannot help omitting the detail, and write out the resulting expressions only. The variation of (38) leads, after some manipulations, to

$$\begin{aligned}\delta H_2/\delta R_0 = {} & \frac{2\pi\Gamma d_1}{R_0^2}\left\{\log\left(\frac{8R_0}{\sigma}\right) + \frac{A}{2} - \frac{5}{4}\right\} - \frac{2\pi^2\Gamma^2}{R_0^2}B \\ & + \frac{\pi^2}{R_0^2}\left[\int_0^\infty r^4\zeta_0 v_0 \mathrm{d}r - \int_0^\infty ra(\tilde{\psi}_{11}^{(1)})^2\mathrm{d}r\right] \\ & - 4\pi^2\int_0^\infty v_0(r)\left[\int_0^r r'\zeta_0^{(2)}(r')\right]\mathrm{d}r \\ & + \frac{\Gamma^2 R_2}{2R_0}\left\{\log\left(\frac{8R_0}{\sigma}\right) + A + \frac{1}{2}\right\}.\end{aligned} \tag{49}$$

The hydrodynamic impulse, the second-order term of which is (42), varies as

$$\delta P = [2\pi\Gamma R_0 + 4\pi(\Gamma R_2 + \pi d_1/R_0)]\delta R_0. \tag{50}$$

Enforcement of (8) or $\delta H_0 + \delta H_2 = (U_0 + U_2)\delta P$, eventually gives rise to the desired correction term of $O(\varepsilon^3)$ to the traveling speed:

$$\begin{aligned}U_2 = {} & \frac{d^{(1)}}{2R_0^3}\left\{\log\left(\frac{8R_0}{\sigma}\right) - 2\right\} - \frac{\pi\Gamma}{R_0^3}B \\ & + \frac{\pi}{2\Gamma R_0^3}\left[\int_0^\infty r^4\zeta_0 v_0\mathrm{d}r - \int_0^\infty ra(\tilde{\psi}_{11}^{(1)})^2\mathrm{d}r\right] \\ & - \frac{2\pi}{\Gamma R_0}\int_0^\infty v_0(r)\left[\int_0^r r'\zeta_0^{(2)}(r')\right]\mathrm{d}r \\ & - \frac{\Gamma R_2}{4\pi R_0^2}\left\{\log\left(\frac{8R_0}{\sigma}\right) + A - \frac{3}{2}\right\}.\end{aligned} \tag{51}$$

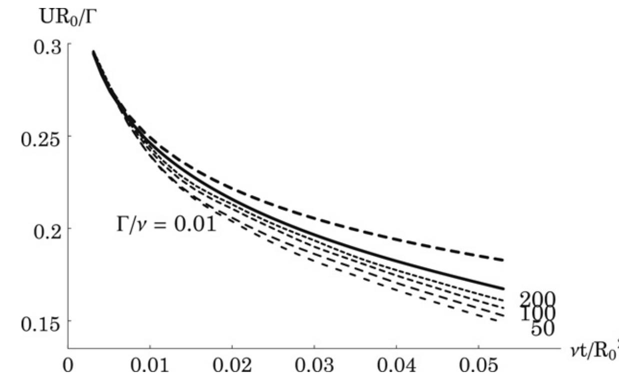

Fig. 1. Variation of speed of a viscous vortex ring with time. The thick line is the higher-order formula (53), while the thick dashed line is the Saffman's formula (6). The dashed lines are the values read off from the graph of numerical simulations [15]. $\Gamma/\nu = 200, 100, 50, 0.01$ from above.

The perturbation R_2 is retained, in order to deal with time variation of the ring radius. For an inviscid vortex ring, the ring radius is constant in time, and we may take $R_2 = 0$ without loss of generality. By exploiting (41) and (51) collapses to

$$\begin{aligned}U_2 = {} & \frac{1}{R_0^3}\left\{\frac{d_1}{2}\left[\log\left(\frac{8R_0}{\sigma}\right) - 2\right] - \pi\Gamma B \right.\\ & \left. + \frac{\pi}{2\Gamma}\int_0^\infty r^4\zeta_0 v_0\mathrm{d}r\right\}.\end{aligned} \tag{52}$$

This is an extension, to $O(\varepsilon^3)$, of Fraenkel–Saffman's formula (2), and is expected to be applicable to a fat core.

The above derivation rests on the assumption of zero viscosity. However, even if viscosity comes into play, the resulting higher-order asymptotics (51) is not invalidated, presumably because, at a large Reynolds number, the action of viscosity is confined to selecting the vorticity profile. In the presence of viscosity ($\nu > 0$), we are forced to solve the inhomogeneous heat-conduction Eq. (36) for the axisymmetric part of the second-order vorticity $\zeta_0^{(2)}$. Taking the initial condition (4), a circular vortex line, of radius R_0, with vanishing thickness, avoids mathematical complication. For this initial condition, (36) is reduced to an ordinary differential equation, with an introduction of similarity variables. The parameters $c_{11}^{(1)}$ in (31) and R_2, both being functions of t, play a common role of specifying the radial position of the ring at $O(\varepsilon^2)$ relative to R_0. This redundancy is removed, for instance, by taking $c_{11}^{(1)} \equiv 0$ and by taking the constant $P_2 = 0$ in (42). This amounts to placing the center $r = 0$ of the local moving frame at the stagnation point relative to this frame [22]. Performing the integration of (36) and then integration in (51), we eventually arrive at an extension of Saffman's formula (6) as

$$U \approx \frac{\Gamma}{4\pi R_0}\left\{\log\left(\frac{4R_0}{\sqrt{\nu t}}\right) - 0.5580 - 3.6716\frac{\nu t}{R_0^2}\right\}. \tag{53}$$

Fig. 1 illustrates the comparison of the asymptotic formula (53) with the direct numerical simulation of the axisymmetric

Navier–Stokes equations [15]. The normalized speed UR_0/Γ of the ring is drawn as a function of normalized time $\nu t/R_0^2$ for its small values. The thick solid line is our formula (53), and the thick broken line is the first-order truncation (6). The thin lines are the results of the numerical simulations. The number attached to each line is the circulation Reynolds number Γ/ν, being no larger than 200. We observe that inclusion of the correction U_2 achieves a significant improvement in approximation. It is remarkable that the large-Reynolds-number asymptotics formula (53), valid to $O(\varepsilon^3)$, compares fairly well with the numerical result of even moderate and small Reynolds numbers.

Mathematical labor to reach the same formula for the speed of a vortex ring dramatically decreases in order of the method of matched asymptotic expansions [22], Lamb's method and the variational principle. By appealing to the variational principle, we have succeeded in achieving higher-order extension of Fraenkel–Saffman's and Saffman's formulae, which are applicable to fat cores, Hopefully this principle encompasses helical vortex tubes if allowance is made for the rotation of the system (*cf.* [35]).

Appendix. Unified view of topological invariants

The helicity is a topological invariant of an ideal fluid in three dimensions. Two-dimensional ideal flows admit an integral of any function of vorticity as topological invariants. This is extended to axisymmetric flows. However, Noether's theorem associated with the particle relabeling symmetry does not discriminate between two and three dimensions. Inspired by this fact, it can be shown that these are variants of the cross helicity [33]. This appendix gives a brief sketch of this unified view.

We start from the vorticity equations for a barotropic fluid filling a domain $\mathcal{D}$:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) . \tag{A.1}$$

Since we are concerned with the kinematics of ideal barotropic flows, the advection velocity $\boldsymbol{u}$ may be an arbitrary smooth vector field so that the vorticity ω may be unrelated to $\nabla \times \boldsymbol{u}$. We take compressibility into account, and the fluid density ρ_{f} obeys the equation of continuity $D\rho_{\mathrm{f}}/Dt + \rho_{\mathrm{f}}\nabla \cdot \boldsymbol{u} = 0$, Here $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{u} \cdot \nabla$ is the Lagrangian derivative. The law of mass conservation holds true without reference to the detailed form of velocity field $\boldsymbol{u}$, and therefore pertains to the kinematics.

Suppose that $\mathcal{D}$ is simply connected. Impose the following boundary condition on ω:

$$\omega \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\mathcal{D}, \tag{A.2}$$

or in the case the domain $\mathcal{D}$ is unbounded,

$$|\omega| \to 0 \quad \text{sufficiently rapidly as } |\boldsymbol{x}| \to \infty. \tag{A.3}$$

Then for a given solenoidal vector field $\omega(\boldsymbol{x}, t)$, there exists a vector potential $\boldsymbol{v}(\boldsymbol{x}, t)$ defined, over $\mathcal{D}$, by $\omega = \nabla \times \boldsymbol{v}$. The vector potential is determined only up to the gauge transformation. The evolution equation of $\boldsymbol{v}$, obtained by taking the uncurl of (A.1), is named the Euler–Poincaré equations [36], and, when specialized as $\boldsymbol{v} = \boldsymbol{u}$, is made coincident with the Euler equations.

Let us introduce another solenoidal vector field $\boldsymbol{B}(\boldsymbol{x}, t)$ which is frozen into the fluid. The equation of $\boldsymbol{B}$ takes the same form as (A.1), and the boundary condition to be imposed is the same as (A.2) or (A.3). The cross helicity

$$\mathcal{H}[\omega, \boldsymbol{B}] = \int_{\mathcal{D}} \boldsymbol{v} \cdot \boldsymbol{B} \, \mathrm{d}V \tag{A.4}$$

is invariant even if the advection velocity field $\boldsymbol{u}$ is different from $\boldsymbol{v}$ [33,37]. The helicity [6–9] is a special case of (A.4) of taking $\boldsymbol{B} = \boldsymbol{\omega}$ and $\boldsymbol{u} = \boldsymbol{v}$.

For two-dimensional flows on the xy-plane with velocity provided by $\boldsymbol{u}(\boldsymbol{x}, t) = (u_x(x, y, t), u_y(x, y, t), 0)$, there is a family of integral invariants for planar flows in a domain $\mathcal{A}$, namely integrals of arbitrary function of $\omega = \partial u_y/\partial x - \partial u_x/\partial y$. For a compressible barotropic fluid, it is superseded by

$$Q = \int_{\mathcal{A}} \omega f \left(\frac{\omega}{\rho_{\mathrm{f}}} \right) \mathrm{d}A, \tag{A.5}$$

where f is an arbitrary function. This integral is termed the generalised enstrophy [27]. Invariance of (A.5) is a direct consequence of the restriction of (A.1) to two dimensions,

$$\frac{\mathrm{D}}{\mathrm{D}t} f \left(\frac{\omega}{\rho_{\mathrm{f}}} \right) = 0, \tag{A.6}$$

and the conservation law of the vorticity flux or Kelvin's circulation theorem. Introducing $\boldsymbol{F} = \nabla \times f \boldsymbol{e}_z$, (A.6) is converted into

$$\frac{\partial \boldsymbol{F}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{F}) . \tag{A.7}$$

A topological invariant is manufactured by replacing $\boldsymbol{B}$ by $\boldsymbol{F}$ in (A.4) with the volume integral of unit length in z over the domain $\mathcal{A}$. This integral is reduced, after a partial integration, to (A.5), except for a boundary term. The latter vanishes in a typical case that $f(\omega/\rho)$ approaches zero sufficiently rapidly as the boundary $\partial\mathcal{A}$ recedes to infinity,

The axisymmetric counterpart of (A.5) is

$$Q_{\mathrm{A}} = \int_{\mathcal{A}} \zeta f \left(\frac{\zeta}{\rho_{\mathrm{f}} \rho} \right) \mathrm{d}\rho \mathrm{d}z, \tag{A.8}$$

for an arbitrary function f of $\zeta/(\rho_{\mathrm{f}}\rho)$ [32]. The vector $\boldsymbol{F} = \nabla \times f \boldsymbol{e}_\phi/\rho$ fulfills (A.4) and (A.7), taking $\boldsymbol{F}$ in place of $\boldsymbol{B}$, coincides with (A.8) except for a boundary term.

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