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Dynamo generation of a magnetic field by decaying Lehnert waves in a highly conducting plasma

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Random waves in a uniformly rotating plasma in the presence of a locally uniform seed magnetic field and subject to weak kinematic viscosity ν and resistivity η are considered. These 'Lehnert' waves may have either positive or negative helicity, and it is supposed that waves of a single sign of helicity are preferentially excited by a symmetry-breaking mechanism. A mean electromotive force proportional to $\nu-\eta$ is derived, demonstrating the conflicting effects of the two diffusive processes. Attention is then focussed on the situation $\eta = 0$, relevant to conditions in the universe before and during galaxy formation. An α-effect, axisymmetric about the rotation vector, is derived, decaying on a time-scale proportional to ν^{-1}; this amplifies a large-scale seed magnetic field to a level independent of ν, this field being subsequently steady and having the character of a 'fossil field'. Subsequent evolution of this fossil field is briefly discussed.

Keywords: α-effect, mean field theory, magnetohydrodynamics, fast dynamo, Magneto-Coriolis waves

1. Introduction

It is well known that a field of random inertial waves in a fluid of non-zero resistivity η is capable of exciting a large-scale magnetic field through an α-effect mechanism (Moffatt 1970b, Soward 1975, Walder *et al.* 1980). For dynamo action to occur, the wave field must exhibit chirality (lack of reflexional symmetry), and this requires some mechanism (e.g. gravity $\boldsymbol{g}$ in conjunction with mean rotation $\boldsymbol{\Omega}$, or boundary forcing) that breaks the 'up-down' symmetry of the field. An electromotive force $\mathcal{E}_i = \alpha_{ij} \langle B \rangle_j$ is then generated, where $\langle \boldsymbol{B} \rangle$ is the mean magnetic field, i.e. an α-effect in the terminology of Steenbeck *et al.* (1966), and this leads to exponential growth of magnetic energy until the growing Lorentz force reacts back upon the wave field, leading to a saturated state (Moffatt 1972).

The simplest measure of chirality is the mean helicity $\mathcal{H} = \langle \boldsymbol{u} \cdot \boldsymbol{\omega} \rangle$ of the flow field, and the simplest measure of the α-effect is its trace $\alpha = \alpha_{ii}$. It is frequently found that $\alpha\mathcal{H} < 0$ and that $\alpha \to 0$ as $\eta \to 0$; the dynamo is then 'slow' in the terminology of Vainshtein and Zeldovich (1972). If, on the other hand, $\alpha \to$ const. $\neq 0$ as $\eta \to 0$ (leading also to a non-zero growth rate in this limit), then the dynamo is described as 'fast'; this possibility has been intensively studied (Moffatt and Proctor 1985, Galloway and Proctor 1992, Hollerbach *et al.* 1995; a comprehensive treatment is provided by Childress and Gilbert 1995). In this case, it is generally found that fast-dynamo eigenmodes have a pathological structure, non-differentiable wherever they are non-zero; the applicability of fast-dynamo theory to astrophysical systems is then questionable.

In this paper, we describe a new fast-dynamo mechanism for which the growing magnetic field remains smooth during the whole dynamo process. This results from a random superposition of inertial waves, weakly perturbed by the magnetic field, and decaying through the action

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of viscosity. Non-zero kinematic viscosity ν and magnetic diffusivity η are responsible for a phase shift between the velocity field $\boldsymbol{u}'$ and the perturbation $\boldsymbol{h}'$ of the magnetic field, giving rise to a mean electromotive force (emf) proportional to $\nu-\eta$. This shows first the competing effects of the two diffusivities in generating this emf. Secondly, and more significantly, this emf survives in the limiting situation $\eta = 0$, leading to an α-effect proportional to ν and having the same sign as the helicity $\mathcal{H}$. The price to be paid for this is that the wave field decays through viscous diffusion, so the dynamo is effective only over a finite time. This however has the advantage that the growing field saturates as the wave-field decays, with no need to invoke the Lorentz-force back-reaction. Asymptotically, the fluid comes to rest, and a 'fossil magnetic field', on a large scale L say, remains, steady for a time of order L^2/η when finite-η effects are taken into consideration.

The theory may have application to the dynamo generation of a magnetic field from an extremely weak seed field in the ionised gas (plasma) of the early universe, both before and during the process of galaxy formation (Zweibel and Heiles 1997). In such a plasma, the condition $\eta \ll \nu$ is satisfied because the plasma density ρ is extremely small and $\nu = \mu/\rho$, where, as already recognised by Maxwell (1867), the dynamic viscosity μ is almost independent of density in the limit $\rho \to 0$. Estimates of $\eta_{\text{galaxy}} \sim 10^3\,\text{m}^2/\text{s}$ for the Milky Way (Spitzer 1962) and $\eta_{\text{cluster}} \sim 3 \times 10^{-3}\,\text{m}^2/\text{s}$ for galaxy clusters (Schekochihin and Cowley 2006) lead to extremely long characteristic diffusive times L^2/η for the magnetic field on the entire length scale L of the system, $t_{\text{galaxy}} \approx 10^{30}\,\text{yr}$, and $t_{\text{cluster}} \approx 10^{34} - 10^{38}\,\text{yr}$, many orders of magnitude greater than the age of the universe $\sim 10^{10}\,\text{yr}$. The kinematic viscosity in these systems is estimated as $\nu_{\text{galaxy}} \sim 10^{17}\,\text{m}^2/\text{s}$ and $\nu_{\text{cluster}} \sim 10^{23} - 10^{26}\,\text{m}^2/\text{s}$ giving viscous diffusion times of order $10^{16}\,\text{yr}$ and $10^8 - 10^9\,\text{yr}$ respectively. Thus, in the case of the intergalactic medium, the viscous diffusion time is one or two orders of magnitude smaller than the age of the universe, whereas the resistive diffusion time is so long that it is reasonable to treat the intergalactic plasma as perfectly conducting ($\eta = 0$) over the 'pre-galactic' period leading up to and including galaxy formation. It is this scenario that we consider in the following sections.

The paper is arranged as follows. In section 2 we state the basic equations for mean and fluctuating fields, and we point out that kinetic energy can be converted to magnetic energy (a dynamo process) even when the total energy is decaying. In section 3 we review the theory of Lehnert waves, identifying particularly those waves that reduce to inertial waves as the mean magnetic field tends to zero. In section 4, we derive the resulting emf proportional to $\nu-\eta$, and in section5 we establish the resulting finite-time dynamo effect. Finally, in section 6, we consider the establishment of the 'fossil field', and reconcile this with conservation of magnetic helicity during what is essentially a fast-dynamo process.

2. Mean and fluctuating equations

We consider the evolution of a magnetic field in a fluid rotating with uniform angular velocity $\boldsymbol{\Omega}$. For simplicity, we suppose that the fluid is incompressible with constant uniform density ρ, kinematic viscosity ν and magnetic resistivity η. Let $\boldsymbol{u}(\boldsymbol{x}, t)$ be the velocity field and $\boldsymbol{B}(\boldsymbol{x}, t) \equiv (\mu_0\rho)^{1/2}\boldsymbol{h}(\boldsymbol{x}, t)$ the magnetic field in the fluid; then the standard governing evolution equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\boldsymbol{u} = -\boldsymbol{\nabla} P - 2\boldsymbol{\Omega} \times \boldsymbol{u} + (\boldsymbol{h} \cdot \boldsymbol{\nabla})\,\boldsymbol{h} + \nu\nabla^2\boldsymbol{u}, \tag{1a}$$

$$\frac{\partial \boldsymbol{h}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\boldsymbol{h} = (\boldsymbol{h} \cdot \boldsymbol{\nabla})\,\boldsymbol{u} + \eta\nabla^2\boldsymbol{h}, \tag{1b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{h} = 0, \tag{1c}$$

where $P = p/\rho + \frac{1}{2}h^2 - \frac{1}{2}(\boldsymbol{\Omega}\times\boldsymbol{x})^2$ is the modified (total) pressure field. We shall consider the situation in which $\boldsymbol{u}$ and $\boldsymbol{h}$ are turbulent fields, spatially homogeneous on a scale ℓ. Let $\langle\cdot\rangle$ denote a space average over any scale $a \gg \ell$, and let

$$E(t) \equiv \tfrac{1}{2}\left\langle \boldsymbol{u}^2\right\rangle \quad \text{and} \quad M(t) \equiv \tfrac{1}{2}\left\langle \boldsymbol{h}^2\right\rangle \tag{2}$$

denote the mean kinetic and magnetic energies respectively. Then from equations (1a-1c) it is easy to derive the energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(E(t)+M(t)) = -\nu\left\langle \boldsymbol{\omega}^2\right\rangle - \eta\left\langle \boldsymbol{j}^2\right\rangle, \tag{3}$$

where $\boldsymbol{\omega} \equiv \boldsymbol{\nabla}\times\boldsymbol{u}$ and $\boldsymbol{j} \equiv \boldsymbol{\nabla}\times\boldsymbol{h}$ are the vorticity and current fields. In the absence of forcing, the total energy therefore decays as a result of viscous and resistive dissipation; but during this process a temporary dynamo effect converting kinetic energy to magnetic energy cannot be excluded; we aim to investigate this possibility in the situation when $\eta \ll \nu$, as relevant in the extremely low density plasma of the early universe before and during the onset of galaxy formation.

Following the 'mean-field electrodynamics' approach of Steenbeck *et al.* (1966) we decompose $\boldsymbol{u}$ and $\boldsymbol{h}$ into mean and fluctuating parts

$$\boldsymbol{u} = \langle\boldsymbol{u}\rangle + \boldsymbol{u}', \quad \boldsymbol{h} = \langle\boldsymbol{u}\rangle + \boldsymbol{h}', \tag{4}$$

where $\langle\boldsymbol{u}'\rangle = \langle\boldsymbol{h}'\rangle = 0$. We may further adopt the frame of reference in which $\langle\boldsymbol{u}\rangle = 0$, and we define $\boldsymbol{H} \equiv \langle\boldsymbol{h}\rangle$. We treat $\boldsymbol{H}$ as locally uniform, but varying weakly on scales much greater than the averaging scale a. Averaging equations (1a-1c) then gives

$$\left\langle\left(\boldsymbol{u}'\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}'\right\rangle = -\boldsymbol{\nabla}\langle P\rangle + \left(\boldsymbol{H}\cdot\boldsymbol{\nabla}\right)\boldsymbol{H} + \left\langle\left(\boldsymbol{h}'\cdot\boldsymbol{\nabla}\right)\boldsymbol{h}'\right\rangle, \tag{5a}$$

$$\frac{\partial\boldsymbol{H}}{\partial t} = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}} + \eta\nabla^2\boldsymbol{H}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{H} = 0. \tag{5b}$$

Here $\boldsymbol{\mathcal{E}} \equiv \langle\boldsymbol{u}'\times\boldsymbol{h}'\rangle$ is the mean electromotive force induced by interaction between the small scale fluctuating fields. Subtracting these averaged equations from the full equations (1a)-(1c) gives equations for these fluctuating fields. Here we adopt the 'first-order smoothing approximation' in which squares and products of fluctuating quantities are ignored; the resulting linearised equations are

$$\frac{\partial\boldsymbol{u}'}{\partial t} + \boldsymbol{\nabla}P' - \left(\boldsymbol{H}\cdot\boldsymbol{\nabla}\right)\boldsymbol{h}' + 2\boldsymbol{\Omega}\times\boldsymbol{u}' - \nu\nabla^2\boldsymbol{u}' = 0, \tag{6a}$$

$$\frac{\partial\boldsymbol{h}'}{\partial t} - \left(\boldsymbol{H}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}' - \eta\nabla^2\boldsymbol{h}' = 0, \tag{6b}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{h}' = 0, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u}' = 0. \tag{6c}$$

3. Lehnert waves subject to dissipation

Waves governed by the linearised equations (6a-6c) were first studied in the seminal paper of Lehnert (1954), and deserve the name 'Lehnert waves'[1]. The compact treatment of Moffatt (1978, chapter 10.3) is reproduced here for ease of comparison. The equations admit wave-type solutions of the form

$$(\boldsymbol{u}',\ \boldsymbol{h}',\ P') = \mathrm{Re}\,(\hat{\boldsymbol{u}},\ \hat{\boldsymbol{h}},\ \hat{P})\ \exp\left[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\right], \tag{7}$$

[1] They are sometimes called Magneto-Coriolis (MC) waves, after Finlay (2008)

where $\omega = \omega_r + i\,\omega_i$; these waves decay through the effects of viscosity and resistivity, so we anticipate that $\omega_i < 0$. The divergence conditions (6c) imply that

$$\boldsymbol{k}\cdot\hat{\boldsymbol{u}} = \boldsymbol{k}\cdot\hat{\boldsymbol{h}} = 0, \tag{8}$$

so the waves are transverse.

Substitution first in (6b) gives the familiar relationship between $\hat{\boldsymbol{h}}$ and $\hat{\boldsymbol{u}}$, *viz.*

$$\hat{\boldsymbol{h}} = -(\omega + i\,\eta k^2)^{-1}(\boldsymbol{k}\cdot\boldsymbol{H})\hat{\boldsymbol{u}}\,, \tag{9}$$

and then substitution in (6a) and rearrangement of the terms gives

$$-i\sigma\hat{\boldsymbol{u}} + 2\,\boldsymbol{\Omega}\times\hat{\boldsymbol{u}} = -i\,\boldsymbol{k}\hat{P}\,, \tag{10a}$$

where

$$\sigma = (\omega + i\nu k^2) - (\omega + i\eta k^2)^{-1}(\boldsymbol{H}\cdot\boldsymbol{k})^2\,. \tag{10b}$$

To get the dispersion relationship, we first take the cross-product of (10a) with $\boldsymbol{k}$; since $\boldsymbol{k}\cdot\hat{\boldsymbol{u}} = 0$, this gives

$$-i\sigma\boldsymbol{k}\times\hat{\boldsymbol{u}} - 2(\boldsymbol{k}\cdot\boldsymbol{\Omega})\,\hat{\boldsymbol{u}} = 0\,. \tag{11}$$

Taking the cross-product again with $\boldsymbol{k}$ gives

$$i\sigma k^2\hat{\boldsymbol{u}} - 2(\boldsymbol{k}\cdot\boldsymbol{\Omega})\boldsymbol{k}\times\hat{\boldsymbol{u}} = 0\,. \tag{12}$$

From these equations, we have immediately

$$\sigma^2 = 4(\boldsymbol{k}\cdot\boldsymbol{\Omega})^2/k^2, \quad \text{or } \sigma = \mp 2(\boldsymbol{k}\cdot\boldsymbol{\Omega})/k\,, \tag{13}$$

and from (11) we then have the corresponding simple relation between $\hat{\boldsymbol{u}}$ and $\boldsymbol{k}\times\hat{\boldsymbol{u}}$,

$$i\,\boldsymbol{k}\times\hat{\boldsymbol{u}} = \pm k\,\hat{\boldsymbol{u}}\,. \tag{14}$$

Since $\hat{\boldsymbol{\omega}} = i\,\boldsymbol{k}\times\hat{\boldsymbol{u}}$ is the Fourier transform of the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ associated with the waves, it is evident that each constituent wave is of maximal helicity, i.e. for each wave

$$\langle\boldsymbol{u}'\cdot\boldsymbol{\omega}'\rangle = \pm k\;|\hat{\boldsymbol{u}}|^2\,e^{-2|\omega_i|t}\,. \tag{15}$$

These waves are circularly polarised about the wave-vector $\boldsymbol{k}$, the velocity vector $\boldsymbol{u}'$ being constant in magnitude, rotating in direction as $\boldsymbol{k}\cdot\boldsymbol{x}$ increases, at any fixed time; the sense of rotation is left- or right-handed according as the associated helicity $\boldsymbol{u}'\cdot\boldsymbol{\omega}'$ is positive or negative.

In order to have a non-zero net helicity in a superposition of such waves, there must exist some kind of 'up-down' symmetry-breaking mechanism that preferentially excites waves of either positive or negative helicity. This is just as in magnetostrophic turbulence (Moffatt 2008) for which such symmetry-breaking is provided by gravity in conjunction with rotation. We assume that such symmetry breaking exists in the present cosmological context, and this allows us to focus on waves of, say, positive helicity, corresponding to the choice $\sigma = -2(\boldsymbol{k}\cdot\boldsymbol{\Omega})/k$ in (13).

Suppose now that $\boldsymbol{H}$ is a weak 'seed field', so that[2]

$$|\sigma| = 2k^{-1}|\boldsymbol{\Omega}\cdot\boldsymbol{k}| \gg |\boldsymbol{H}\cdot\boldsymbol{k}|\,. \tag{16}$$

Then, provided the diffusion effects associated with ν and η are also weak, the two roots of (10b) (regarded as a quadratic in ω) are given by

$$\omega + i\nu k^2 \approx \sigma + \frac{(\boldsymbol{H}\cdot\boldsymbol{k})^2}{\sigma + i(\eta-\nu)k^2}\,, \tag{17}$$

[2]We neglect the possible role of waves with wave-vector $\boldsymbol{k}$ nearly perpendicular to $\boldsymbol{\Omega}$, for which the inequality (16) is not satisfied.

and

$$\omega + \mathrm{i}\eta k^2 \approx -\sigma^{-1}(\boldsymbol{H}\cdot\boldsymbol{k})^2\,, \tag{18}$$

where σ is still given by (13). Clearly (17) represents a damped inertial wave whose frequency is weakly modified by the presence of the magnetic field; in fact from (17) we have $\omega = \omega_{\mathrm{r}} + \mathrm{i}\,\omega_{\mathrm{i}}$, where (provided $|\eta-\nu|k^2 \ll |\sigma|$)

$$\omega_{\mathrm{r}} \approx \sigma + \sigma^{-1}(\boldsymbol{H}\cdot\boldsymbol{k})^2 \approx \sigma, \quad \omega_{\mathrm{i}} \approx -\nu k^2 - (\boldsymbol{H}\cdot\boldsymbol{k})^2(\eta-\nu)k^2\sigma^{-2} \approx -\nu k^2\,. \tag{19}$$

We focus attention on this type of weakly perturbed, damped, inertial wave in what follows.

4. Mean emf induced by damped wave mode

We can now calculate the contribution to the mean emf $\boldsymbol{\mathcal{E}}$ from this wave. Using (9), this is given by

$$\hat{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}) = \langle \boldsymbol{u}'\times\boldsymbol{h}'\rangle = \tfrac{1}{2}\mathrm{Re}\left(\hat{\boldsymbol{u}}^*\times\hat{\boldsymbol{h}}\right)\mathrm{e}^{-2|\omega_{\mathrm{i}}|t} = -\tfrac{1}{2}(\boldsymbol{k}\cdot\boldsymbol{H})\mathrm{Re}\left[\frac{\hat{\boldsymbol{u}}^*\times\hat{\boldsymbol{u}}}{\omega_{\mathrm{r}}+\mathrm{i}\,\omega_{\mathrm{i}}+\mathrm{i}\eta k^2}\right]\mathrm{e}^{-2\nu k^2 t}. \tag{20}$$

Since $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{u}}^*$ are perpendicular to $\boldsymbol{k}$, we have

$$\hat{\boldsymbol{u}}^*\times\hat{\boldsymbol{u}} = k^{-2}[(\hat{\boldsymbol{u}}^*\times\hat{\boldsymbol{u}})\cdot\boldsymbol{k}]\,\boldsymbol{k} = \mathrm{i}\,k^{-2}[\hat{\boldsymbol{u}}^*\cdot(\mathrm{i}\,\boldsymbol{k}\times\hat{\boldsymbol{u}})]\,\boldsymbol{k} = \mathrm{i}\,k^{-2}[\hat{\boldsymbol{u}}^*\cdot\hat{\boldsymbol{\omega}}]\boldsymbol{k}\,, \tag{21}$$

and $\hat{\boldsymbol{u}}^*\cdot\hat{\boldsymbol{\omega}}$ is purely real. Hence (20) becomes

$$\begin{aligned}\hat{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}) &= -\tfrac{1}{2}(\boldsymbol{k}\cdot\boldsymbol{H})\mathrm{Re}\left[\frac{\mathrm{i}k^{-2}\,(\hat{\boldsymbol{u}}^*\cdot\hat{\boldsymbol{\omega}})\,\boldsymbol{k}}{\omega_{\mathrm{r}}+\mathrm{i}\,\omega_{\mathrm{i}}+\mathrm{i}\eta k^2}\right]\mathrm{e}^{-2\nu k^2 t}\\ &= \frac{-(\omega_{\mathrm{i}}+\eta k^2)}{2k^2\left[{\omega_{\mathrm{r}}}^2+(\omega_{\mathrm{i}}+\eta k^2)^2\right]}\,\hat{\mathcal{H}}(\boldsymbol{k})\,(\boldsymbol{k}\cdot\boldsymbol{H})\,\boldsymbol{k}\,\mathrm{e}^{-2\nu k^2 t}\,,\end{aligned} \tag{22}$$

where $\hat{\mathcal{H}}(\boldsymbol{k})$ is the helicity at time $t=0$ of the considered wave. Hence finally, from (19), at leading order in the small parameters $|\boldsymbol{H}\cdot\boldsymbol{k}|/|\sigma|$ and $|(\nu-\eta)k^2|/|\sigma|$,

$$\hat{\boldsymbol{\mathcal{E}}}(\boldsymbol{k}) = \frac{\nu-\eta}{2\sigma^2}\,\hat{\mathcal{H}}(\boldsymbol{k})\,(\boldsymbol{k}\cdot\boldsymbol{H})\,\boldsymbol{k}\,\mathrm{e}^{-2\nu k^2 t}\,. \tag{23}$$

It is immediately evident from this result that viscosity and resistivity have opposite effects in generating this electromotive force, and indeed cancel completely when $\nu=\eta$, i.e. when the magnetic Prandtl number $\mathrm{Pr_m}\equiv\nu/\eta = 1$. Second, and more significantly in the pre-galactic context, the emf survives even in the perfectly conducting limit $\eta=0$. In this limit, it is viscosity that is responsible for the phase shift between $\boldsymbol{u}'$ and $\boldsymbol{h}'$ that is required to give a non-zero mean $\langle\boldsymbol{u}'\times\boldsymbol{h}'\rangle$. This is an interesting and novel situation, because, as indicated in the Introduction, the $\eta\to 0$ limit usually possesses severe difficulties in dynamo theory. Here however, we have potentially a well-defined α-effect, albeit temporary because of the exponential decay factor in (23). We shall explore the consequences in the following section.

First however, note that a spectrum of waves of different $\boldsymbol{k}\in\mathcal{K}=\{\boldsymbol{k}:|\sigma|\gg|\boldsymbol{k}\cdot\boldsymbol{H}|\}$ (see footnote on p.3) will provide additive contributions to the total emf $\boldsymbol{\mathcal{E}}$ which then takes the form

$$\boldsymbol{\mathcal{E}} = (\nu-\eta)\,\sum\!\!\!\!\!\!\int_{\mathcal{K}}\frac{k^2}{8(\boldsymbol{k}\cdot\boldsymbol{\Omega})^2}\hat{\mathcal{H}}(\boldsymbol{k})(\boldsymbol{k}\cdot\boldsymbol{H})\boldsymbol{k}\,\mathrm{e}^{-2\nu k^2 t}\,\mathrm{d}^3\boldsymbol{k}\,, \tag{24}$$

where the symbol $\sum\!\!\!\!\!\!\int$ allows for a discrete and/or continuous spectrum as, generally speaking, the spectrum can involve both, continuous and discrete intervals. Thus

$$\mathcal{E}_i = \alpha_{ij} H_j \quad \text{where} \quad \alpha_{ij} = (\nu - \eta) \sum\!\!\!\!\!\!\int_{\mathcal{K}} \frac{k^2 k_i k_j}{8(\boldsymbol{k} \cdot \boldsymbol{\Omega})^2} \hat{\mathcal{H}}(\boldsymbol{k})\, \mathrm{e}^{-2\nu k^2 t}\, \mathrm{d}^3\boldsymbol{k}\,. \tag{25}$$

This is an α-effect, in general anisotropic[3]. For simplicity, let us suppose that the helicity spectrum $\hat{\mathcal{H}}(\boldsymbol{k})$ is symmetric about the direction of $\boldsymbol{\Omega}$; then α_{ij} is also axisymmetric. With $\boldsymbol{\Omega} = (0,\, 0,\, \Omega)$ in cartesian coordinates $\mathrm{O}xyz$, and with $\hat{\mathcal{H}}(\boldsymbol{k}) = \hat{\mathcal{H}}(k,\, \theta)$ where θ is co-latitude, we then have

$$\alpha_{ij} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \tag{26}$$

where, on the additional assumption that $\hat{\mathcal{H}}(k, \theta)$ is symmetric about the plane $\theta = \pi/2$,

$$\alpha = \frac{\pi(\nu - \eta)}{4\Omega^2} \int \mathrm{d}k \int_0^{(\pi/2)-\epsilon} k^4\, \hat{\mathcal{H}}(k, \theta)\, \mathrm{e}^{-2\nu k^2 t}\, \tan^2\theta \sin\theta\, \mathrm{d}\theta\,, \tag{27}$$

$$\gamma = \frac{\pi(\nu - \eta)}{2\Omega^2} \int \mathrm{d}k \int_0^{(\pi/2)-\epsilon} k^4\, \hat{\mathcal{H}}(k, \theta)\, \mathrm{e}^{-2\nu k^2 t}\, \sin\theta\, \mathrm{d}\theta\,. \tag{28}$$

The small angle ϵ in the upper limit $(\pi/2) - \epsilon$ of (27) and (28) is chosen to correspond to the set $\mathcal{K}$, thus excluding vectors $\boldsymbol{k}$ that do not satisfy (16).

5. Dynamo action with $\eta = 0$.

In this section, we suppose that $\eta = 0$, and we explore the consequences of the resulting α-effect. This is isotropic ($\gamma = \alpha$) in the special circumstance that all the constituent wave vectors are concentrated on the cone of angle $\theta = \theta_0$ where $\tan^2\theta_0 = 2$, i.e. $\theta_0 \approx 54.7^{\circ}$. In this situation, $\alpha_{ij} = \alpha(t)\delta_{ij}$, where

$$\alpha(t) = \frac{\pi\nu}{2\Omega^2} \int \mathrm{d}k\, k^4\, \hat{\mathcal{H}}(k)\, \mathrm{e}^{-2\nu k^2 t}\,. \tag{29}$$

If further the waves are concentrated around a dominant wave-number $k = k_0$, then

$$\alpha(t) = \frac{\pi\, \nu k_0^2}{2\,\Omega^2}\, \mathcal{H}_0\, \mathrm{e}^{-2\nu k_0^2\, t}\,, \tag{30}$$

where $\mathcal{H}_0$ (supposed positive) is now the helicity of the velocity field at time $t = 0$. In order to illustrate the possible behaviour in the simplest way, we now adopt this simple formula[4].

With $\boldsymbol{\mathcal{E}} = \alpha(t)\boldsymbol{H}$, the mean-field equation (5b) (with $\eta = 0$) becomes

$$\frac{\partial \boldsymbol{H}}{\partial t} = \alpha(t)\, \boldsymbol{\nabla} \times \boldsymbol{H}\,. \tag{31}$$

This equation admits growing modes with force-free structure $\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{x},\, t) = K\boldsymbol{H}(\boldsymbol{x},\, t)$, where $K \ll k_0$ (to satisfy the scale-separation implicit in the mean-field theory). Then (31)

[3]In the strong field regime, the helicity spectrum $\hat{\mathcal{H}}(\boldsymbol{k})$ depends on the mean magnetic field $\boldsymbol{H}$ because of the action of the Lorentz force; however, in the weak field regime $|\boldsymbol{H} \cdot \boldsymbol{k}|/|\sigma| \ll 1$ considered here, $\hat{\mathcal{H}}(\boldsymbol{k})$ does not depend on $\boldsymbol{H}$ at leading order, so the EMF is linearly related to the mean field.
[4]The assumption of isotropy is not necessary; if $\gamma \neq \alpha$ but $\gamma\alpha > 0$, magnetic energy growth still occurs, as described for the case of constant α_{ij} by Moffatt(1970a).

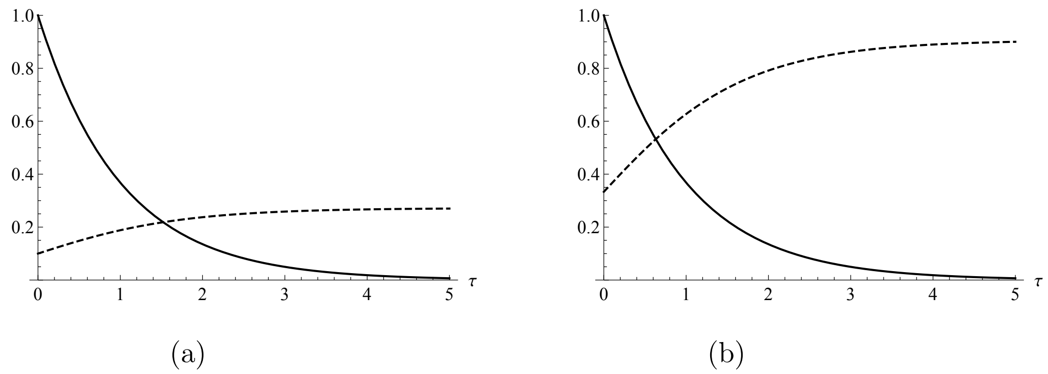

Figure 1. Decay of kinetic energy $E(\tau)/E(0)$ (solid) and simultaneous growth of magnetic energy $M(\tau)/E(0)$ as given by (35); dimensionless time $\tau = 2\nu k_0^2 t$; $\pi K \mathcal{H}_0/2\Omega^2 = 1$; (a) from an initial level $M(0) = 0.1E(0)$; (b) $M(0) = (1/3)E(0)$; in the latter case, about half the initial kinetic energy is converted to magnetic energy by dynamo action.

becomes

$$\frac{\partial \boldsymbol{H}}{\partial t} = K\alpha(t)\,\boldsymbol{H}, \tag{32}$$

with solution

$$\boldsymbol{H}(\boldsymbol{x},\,t) = \boldsymbol{H}(\boldsymbol{x},\,0)\exp\left[\frac{\pi\,K\,\mathcal{H}_0}{4\,\Omega^2}\left(1 - \mathrm{e}^{-2\nu k_0^2\,t}\right)\right]. \tag{33}$$

The corresponding magnetic energy is given by

$$M(t) = M(0)\exp\left[\frac{\pi\,K\,\mathcal{H}_0}{2\,\Omega^2}\left(1 - \mathrm{e}^{-2\nu k_0^2\,t}\right)\right] \sim M(0)\exp\left[\frac{\pi\,K\,\mathcal{H}_0}{2\,\Omega^2}\right] \quad \text{as } t\to\infty. \tag{34}$$

The behaviour is sketched in figure 1, with (a) $M(0)/E(0) = 0.1$ and (b) $M(0)/E(0) = 1/3$, with $\pi\,K\,\mathcal{H}_0/2\,\Omega^2 = 1$, and with dimensionless time $\tau = 2\nu k_0^2 t$. At early times (specifically for $\tau \ll 1$), the growth rate is exponential:

$$M(t) \sim M(0)\exp\left[\frac{\pi\,K\,\mathcal{H}_0}{\Omega^2}\nu k_0^2\,t\right], \tag{35}$$

as for a conventional α^2-dynamo. However, due to the viscous decay of the wave field, the α-effect is here quenched within a time of order $(\nu k_0^2)^{-1}$, i.e. $\tau = \mathrm{O}(1)$, leading to the saturated state (34). For an initial state of maximal helicity, $\mathcal{H}_0$ is related to the initial mean kinetic energy $E(0) = \frac{1}{2}\langle u_0^2\rangle$ by the equation $\mathcal{H}_0 = 2k_0E(0)$. Since the increase in magnetic energy in (34) is derived entirely from the initial kinetic energy $E(0)$, we must evidently satisfy the inequality $M(\infty) - M(0) < E(0)$, i.e.

$$M(0) < E(0)\left(\exp\left[\frac{\pi\,K\,k_0E(0)}{\Omega^2}\right] - 1\right)^{-1}. \tag{36}$$

If the energy of the initial seed field $M(0)$ is low enough to satisfy this inequality, then field saturation occurs purely as a consequence of the temporary nature of the dynamo process. If the inequality is not satisfied, then nonlinear effects associated with the back-reaction of the Lorentz force must presumably result in earlier field saturation.

6. The fossil field

Provided (36) is satisfied, what is particularly interesting about this mechanism of large-scale field intensification is that, while the kinetic energy of the wave field decays to zero, the magnetic field extracts energy from the same wave field according to (35) and saturates at the level (34) which subsequently remains steady. This is of course because we have set $\eta = 0$, but, as argued in the introduction, this is legitimate in the 'pre-galactic' cosmos, when $\eta \ll \nu \ll u_0\ell$, where u_0 is the initial rms plasma velocity. Note that the asymptotic level (34) is independent of the viscosity ν, but the time taken to establish this asymptotic state is of order $(\nu k_0^2)^{-1}$. We may legitimately describe the asymptotic field as a 'fossil field', resulting as it does from an earlier period of evolution, and providing information about the conditions during that earlier period that led to its growth.

During the intensification process, since $\eta = 0$, the total (mean) magnetic helicity is conserved. This total helicity is

$$\mathcal{H}_M = \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle + \langle \boldsymbol{a} \cdot \boldsymbol{b} \rangle = \mathcal{H}_M^L + \mathcal{H}_M^S, \quad \text{say}, \tag{37}$$

where $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, and $\boldsymbol{b} \left(\equiv (\mu_0\rho)^{-1/2}\boldsymbol{h}'\right) = \boldsymbol{\nabla} \times \boldsymbol{a}$. The large-scale magnetic helicity $\mathcal{H}_M^L$ grows in tandem with the magnetic energy $M(t)$, and since $\mathcal{H}_M$ is constant, an equal and opposite small-scale helicity $\mathcal{H}_M^S \approx -\mathcal{H}_M^L$ must grow at the same time, i.e. net linkage of the large-scale field is compensated by net linkage of the small-scale field.

When the growing magnetic energy is of the same order of magnitude as the exponentially decaying kinetic energy, the process of magnetic relaxation as described by Moffatt (1985) must gradually take over. This however is a relatively slow process, being driven by the weak Lorentz force. Relaxation could in principle completely reverse the earlier intensification process, although on a much longer time-scale. More important perhaps is the fact that the fossil field will be subject to subsequent effects associated with galaxy formation, when issues involving self-gravitation, turbulence and non-zero η can no longer be ignored.

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