Relaxation to steady vortical flows – and knots in the quark-gluon plasma

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Abstract: The method of magnetic relaxation for the determination of solutions of the Euler equations representing steadily propagating vortical structures is reviewed, and compared with alternative artificial relaxation procedures that conserve the topology of the vorticity field. Attention is focussed first on axisymmetric vortex ring configurations, for which such relaxation techniques provide simple proofs of the existence of vortex rings of prescribed 'signature', the imprint of conserved topology. Relaxation of knotted and linked configurations are briefly considered, from which the concept of an energy spectrum of knots and links emerges in a natural way. Attention is drawn to recent work of Buniy & Kephart (2005), which seeks to identify such energy spectra with the spectrum of excitations (glueballs) in the quark-gluon plasma, in a spirit reminiscent of Kelvin's 'vortex theory of atoms' but here at the elementary particle level of quantum chromodynamics.

1 Introduction

I am grateful to the organisers of this Mini-Symposium on *Classical and Quantum Vortex Rings* for inviting me to give this introductory lecture. I shall talk mainly about the classical situation, but will also discuss briefly a novel manifestation of knotted quantum vortices in the quark-gluon plasma, the primitive state of matter in the Universe some microseconds after the Big Bang.

It is widely recognised that the vortex ring is a universal phenomenon. I mention here just three examples: (i) The vortices shed by the sweep of the oars in a rowing boat: these are in fact 'half vortex rings', semi-circular in form and intersecting the water surface. Each half ring propagates under its self-induced velocity in a direction opposite to that of the boat, carrying momentum equal and opposite to that imparted to the boat by the sweep of the oar. (ii) The spectacular ring vortices shed by the volcanic eruption of Mount Etna on 20 February 2000, and captured on film by Marco Fulle (see $\langle http://stromboli.net \rangle$); this is really a smoke ring, but there is little doubt that it coincides with a (turbulent) vortex ring and rises due to a combination of buoyancy and self-induced velocity. (iii) The equally remarkable air-core vortices blown in water by dolphins as described and filmed by Don White (see $\langle http://earthtrust.org/delrings.html \rangle$). Dolphins evidently not only blow such rings, but then pursue them, play with them, and try to swim through them. The smoke rings that we humans used to blow when it was politically correct to do so are but pale shadows in comparison!

I wish to pay particular tribute in this lecture to William Thomson, Lord Kelvin, who died on 20 December 1907, just over 100 years ago. Kelvin was of course, with Helmholtz, one of the great pioneers of vortex dynamics. When he was a student at Cambridge in 1842-1844, he rowed as 'bow' for his College, Peterhouse. In this leading position in the boat, he was well placed to observe the vortices shed from the eight oars, and I like to think that this set him brooding on the mechanism of vortex ring propagation that he was to analyse some 20 years later. Kelvin (or Thomson as he then was) conceived his ill-fated theory of 'vortex atoms' in 1867 (Kelvin 1867, 1869), whereby he sought to explain the fundamental atomic structure of matter in terms of knotted vortices imbedded in the 'ether', imagined as an ideal incompressible fluid permeating the whole universe. Ill-fated it was, because over the subsequent 20 years or so, Kelvin was unable to find any nontrivial stable vortical structures as solutions of the steady Euler equations of fluid motion. I wish to demonstrate here that, if only he had been able to think in terms of magnetic flux tubes, rather than vortex tubes, his theory might have had more lasting impact.

2 Vortons

I use the word *vorton* to mean a localised blob of vorticity ω that propagates without change of structure, with constant velocity, $-\mathbf{U_0}$ say, relative to the fluid at ∞. In this respect, although in no other, it resembles a *soliton*, justifying the use of the term. [The word *vorton* has occasionally been used with a different meaning, and with less justification, to signify an artificial type of point singularity of vorticity. There should be no confusion with the meaning adopted here.] In a frame of reference moving with the vorton, the flow field $\mathbf{u}(\mathbf{x})$ is steady, and $\mathbf{u} \sim \mathbf{U_0}$ at ∞. The vorton carries with it a domain $\mathcal{D}$ of fluid, bounded by a surface $\partial\mathcal{D}$, which is fixed in the frame of the vorton. Outside $\mathcal{D}$, the streamlines all extend to infinity, and since the vorticity is zero at infinity, this means that $\omega \equiv \mathbf{0}$ outside $\mathcal{D}$. From this perspective, a vortex ring is simply an axisymmetric vorton, steady only insofar as viscous diffusion may be neglected. Although such rings will be our primary concern, it is instructive to consider the possibility of non-axisymmetric vortons also. The question of existence of steady solutions of the Euler equations of this type is then of central concern; and if they do exist, how are they to be classified?

Note that streamline topology (unlike vorticity topology) changes under Galilean transformation; thus for example, Hill's spherical vortex has closed streamlines inside a sphere in the frame of reference in which the flow is steady; but relative to a frame of reference moving with a sufficiently large velocity, the instantaneous streamlines are all perturbations of straight lines extending (both ways) to infinity.

A vorton is characterised by four properties that are invariants of Euler flows even under unsteady conditions: in the frame in which the velocity vanishes at infinity, these are the momentum $\mathbf{P}$, the angular momentum $\mathbf{M}$, the kinetic energy E, and the helicity $\mathcal{H}$, defined (for a fluid of unit density) by:

$$\mathbf{P} = \tfrac{1}{2}\int \mathbf{x} \times \omega \, dV \,, \tag{1}$$

$$\mathbf{M} = \tfrac{1}{3}\int \mathbf{x} \times (\mathbf{x} \times \omega) \, dV \,, \tag{2}$$

$$E = \tfrac{1}{2}\int \mathbf{A} \cdot \omega \, dV \,, \tag{3}$$

$$\mathcal{H} = \int \mathbf{u} \cdot \omega \, dV \,, \tag{4}$$

where $\mathbf{A}$ is a vector potential for $\mathbf{u}$; the gauge is arbitrary, but usually taken so that $\nabla \cdot \mathbf{A} = 0$. These integrals are deliberately written in terms of ω, and can be taken to be over the finite domain $\mathcal{D}$, since $\omega = 0$ outside $\mathcal{D}$. The helicity is a topological invariant, providing a measure of the degree of linkage of the vortex lines in $\mathcal{D}$ (Moffatt 1969), and is conserved whenever conditions are such the 'vortex lines are frozen in the fluid'. Actually the integral in this case may be taken over any volume inside a Lagrangian surface on which $\omega \cdot \mathbf{n} = 0$, a condition that persists under frozen field evolution.

Steady Euler flows satisfy the equations:

$$\mathbf{u} \times \omega = \nabla h \,, \quad \omega = \nabla \times \mathbf{u} \,, \quad \nabla \cdot \mathbf{u} = 0 \,, \quad h = p + \tfrac{1}{2}\mathbf{u}^2 \,, \tag{5}$$

where p is the pressure field, and h the 'total head'. It follows that $\mathbf{u}$-lines (i.e. streamlines) and ω-lines (i.e vortex lines) lie on surfaces $h =$ const. The only way that the $\mathbf{u}$-lines can escape from this 'surface' constraint is if $\nabla h \equiv 0$, and $\mathbf{u} = \alpha\omega$, with $\alpha =$ const., (and of course then the vortex lines coincide with the streamlines). For this type of Beltrami flow, the streamlines need not lie on surfaces, and may be chaotic in sub-domains of $\mathcal{D}$. The 'ABC-flow' of Arnol'd (1966) is of this type, and the chaotic character of its streamlines (albeit in a periodic domain) was explored by Dombre et al (1986).

3 Existence of Vortons

To establish existence results, it is expedient to switch attention to an analogous problem: magnetostatics in a perfectly conducting fluid, described by the equations:

$$\mathbf{j} \times \mathbf{B} = \nabla p\,, \quad \mathbf{j}(\mathbf{x}) = \nabla \times \mathbf{B}\,, \quad \nabla \cdot \mathbf{B} = 0\,. \tag{6}$$

Here, $\mathbf{B}$ is the magnetic field distribution in the fluid, $\mathbf{j}$ the corresponding current distribution, and $\mathbf{j} \times \mathbf{B}$ the associated Lorentz force. There is an exact analogy between the systems (5) and (6), the analogous pairs of variables being:

$$\{\mathbf{u}\,, \mathbf{B}\}\,, \quad \{\omega\,, \mathbf{j}\}\,, \quad \{h\,, p_0 - p\}\,, \tag{7}$$

where p_0 is a constant. Note the change of sign, the Lorentz force being $\mathbf{j} \times \mathbf{B}$, not $\mathbf{B} \times \mathbf{j}$!

In switching attention to the magnetostatic problem (6), it would appear that we have gained nothing: we have merely replaced one problem by another that is apparently identical. However, the magnetic problem lends itself to a physically natural *relaxation* procedure for the determination of *stable (minimum energy) equilibrium states*.

This invokes a *different* magnetic field analogy: under unsteady conditions, the magnetic field evolves according to the induction equation

$$\partial \mathbf{B}/\partial t = \nabla \times (\mathbf{v} \times \mathbf{B})\,, \tag{8}$$

where $\mathbf{v}$ is now the velocity field (and we deliberately use the symbol $\mathbf{v}$ here rather than $\mathbf{u}$.) This equation has the well-known consequence that the **B**-lines are frozen in the fluid (like the vortex lines in unsteady Euler flow). However, $\mathbf{B}$ is not here constrained to be the curl of the convecting field $\mathbf{v}$, and equation (8) is linear if $\mathbf{v}$ is regarded as known. One consequence of (8) is conservation of *m*agnetic helicity

$$\mathcal{H}_M = \int \mathbf{A} \cdot \mathbf{B}\, dV\,, \tag{9}$$

where now $\mathbf{A}$ is a vector potential for $\mathbf{B}$, and the integral is again taken over any volume inside a Lagrangian surface on which $\mathbf{B} \cdot \mathbf{n} = 0$. This result was first obtained by Woltjer (1958), albeit in more restricted form.

The magnetic helicity invariant provides an important lower bound on the magnetic energy $M = \frac{1}{2} \int \mathbf{B}^2\, dV$, namely

$$M \geq 2q\, |\mathcal{H}_M| \tag{10}$$

where q is a constant with the dimensions of $(\text{length})^{-1}$, which depends only on the domain topology, geometry and scale (or the scale of the support of $\mathbf{B}$ if the fluid is unbounded) (Arnol'd 1974)

It is not difficult to construct a process whereby magnetic energy decreases monotonically to a minimum compatible with this lower bound, while conserving the 'frozen-in' character of the field, and therefore conserving its topology however complex this may be. In this process, the field-lines contract (due to Maxwell tension) as far as permitted by this topology. In the asymptotic stable minimum energy state, the fluid is at rest, and the field satisfies the magnetostatic equilibrium equations (6), reflecting a balance between the Lorentz force and the pressure gradient.

Now we can exploit the Euler-magnetic (or $\{\mathbf{u}\,,\mathbf{B}\}$) analogy: to each solution of the magnetostatic problem, there corresponds via this analogy a solution of the steady Euler equations. Hence we have an amazing result: given an arbitrary solenoidal velocity field $\mathbf{U}(\mathbf{x})$, there apparently exists a steady Euler flow $\mathbf{u}(\mathbf{x})$ whose streamlines are obtained by continuous deformation (induced by the 'image' relaxation process) of the streamlines of $\mathbf{U}(\mathbf{x})$. Thus there is a huge family of steady Euler flows (e.g. given any knot K, there exists a steady Euler flow for which each streamline is closed and in the form of the knot K); and this result applies equally if the field $\mathbf{U}$ is uniform at infinity (care being needed in this case in handling the surface integrals). Detaile discussion and justification of these statements may be found in Moffatt(1985, 1986, 1988).

There are two immediate qualifications to this result. First, tangential discontinuities of $\mathbf{B}$ may appear asymptotically in the magnetic relaxation process (an important phenomenon in magnetohydrodynamics, believed to be responsible for flares and heating of the solar corona, Parker 1994); this means that the asymptotic state is not strictly topologically equivalent to the initial state. It may however be described as being *topologically accessible* from this initial state (Moffatt 1985). The analogue of the tangential discontinuity in the Euler flow is a vortex sheet, which may be expected to be subj ect to Kelvin-Helmholtz instability.

This leads naturally to the second qualification: whereas the magnetostatic solutions are stable, being minimum energy states under frozen field perturbations, in fully three-dimensional situations the analogous Euler flows are generally unstable with respect to isovortical perturbations for which the vorticity field is frozen in the fluid, as natural for the unsteady Euler equations (Rouchon 1991). Although the analogous steady equilibrium states (magnetostatic and Euler) are in effect identical, their stability properties are quite different. Kelvin's vortex atom theory was doomed because of the intrinsic instability of steady three-dimensional Euler flows! It is moreover this intrinsic instability that may be recognised as the primary cause of turbulence!

4 The axisymmetric situation

The same magnetic relaxation procedure may be carried out when the initial magnetic fields is axisymmetric and uniform ($\mathbf{B}_0$, say) at infinity. Obviously if all field lines extend to infinity, then the field can and will relax to the uniform field $\mathbf{B}_0$ everywhere. Only if there is a domain of closed $\mathbf{B}$-lines does the field relax to a non-trivial state. Let χ be the flux-function (the analogue of the Stokes streamfunction ψ) of the magnetic field. During relaxation the toroidal surfaces $\chi =$ const. move with the fluid, and, by virtue of incompressibility, the volume $V(\chi)$ inside each such surface is conserved. So also is the flux $W(\chi)$ of field in the zonal direction inside each such torus, by virtue of the frozen field behaviour (and Alfvén's theorem). The pair of functions $S = \{V(\chi), W(\chi)\}$ constitutes a *signature* for the field, conserved, as a signature should be, during the relaxation process. The 'amazing' result here is that, for any initial field uniform at infinity and having a domain $\mathcal{D}$ of closed streamlines, and arbitrary signature S within $\mathcal{D}$, there exists a steady Euler flow with the same topology and the same signature S. These are vortex rings, and it is apparent from this argument that a very wide family of such rings exist. The function $W(\psi)$ corresponds to the distribution of swirl within the vortex around the axis of symmetry; of course, this W may be identically zero.

The domain $\mathcal{D}$ may have either spherical or toroidal topology. The former type of vortex ring is sometimes described as 'fat', the latter 'thin'. There is an important distinction here in that tangential discontinuities cannot form if the relaxing field has fat topology, for this would require infinite stretching of field lines on the osculating surfaces, and this is incompatible with monotonic decrease of magnetic energy to a minimum. Tangential discontinuities can however form for the thin topology, due to possible collapse of the field at the ring of saddle points on the inner part of $\partial\mathcal{D}$ (Moffatt 1988). Such collapse has been described for 2-dimensional field relaxation by Linardatos (1993), and the situation is similar here.

Hence it is only for fat rings that the magnetic relaxation process provides a family of rings with everywhere continuous velocity.

5 The wind-up process of Vallis, Carnevale & Young

The magnetic relaxation process suffers from the disadvantage that it is the topology of the velocity field, rather than that of the vorticity field, that can be 'prescribed'. It would be far more useful to devise a relaxation process that conserves vorticity topology, because that is what is conserved by the unsteady Euler equations themselves.

One such procedure has been devised by Vallis et al (1989). This is potentially extremely useful for two-dimensional or axisymmetric situations, but fails for three-dimensional situations, for reasons indicated below. We suppose that a localised vorticity field evolves according to the equation

$$\partial\omega/\partial t = \nabla\times(\mathbf{v}\times\omega), \text{ where } \omega = \nabla\times\mathbf{u},\ \mathbf{v} = \mathbf{u}+\alpha\partial\mathbf{u}/\partial t \text{ and } \nabla\cdot\mathbf{u} = 0\,. \quad (11)$$

This may be described as 'artificial dynamics', but it is a dynamical evolution for which the vorticity field is transported by $\mathbf{v}$, and so has conserved topology. The energy E is not however conserved: elementary manipulations give

$$dE/dt = -\alpha \int [\partial \mathbf{u}/\partial t]^2 \, dV \,, \tag{12}$$

so that E decreases or increases according as α is positive or negative. This result is useful only if a positive lower bound or a finite upper bound can be placed on E (since otherwise, E may relax to zero in the former case, or increase to infinity in the latter, and in neither case can a useful conclusion be drawn). Unfortunately, there is in general no lower bound on E analogous to (10). However, there is a finite *upper* bound in the axisymmetric situation without swirl, as may be proved using a combination of Schwartz and Poincaré inequalities coupled with the fact that $[\omega_\varphi / r \sin\theta]$ is a Lagrangian invariant of the flow $\mathbf{v}$ (Moffatt 1990b); this means that in this case, if we take α negative, so that E *increases* monotonically, it must tend to a limit not greater than this upper bound. This process is perhaps better described as a 'wind-up' rather than a 'relaxation' process, because of the increase of kinetic energy. Anyway, the important thing is that the process tends to a limit in which $\partial \mathbf{u}/\partial \mathbf{t} = 0$ everywhere; in this limit, $\mathbf{v} \equiv \mathbf{u}$, and we have a steady Euler flow, whose signature is that of the (arbitrary) initial field.

It had previously been asserted (Benjamin 1976) that vortex rings are steady solutions of the Euler equations of maximum energy with respect to 'rearrangement' of the vortex lines, or what amounts to the same thing, isovortical perturbations. The above 'wind-up' process has provided some justification for this assertion. Such solutions should also be stable with respect to axisymmetric isovortical perturbations, by arguments due to Arnol'd (1966), but the question of their stability with respect to non-axisymmetric perturbations remains open.

6 Energy of knotted flux tubes

A flux tube of volume V, carrying magnetic flux Φ (the analogue of κ) and knotted in the form of a knot of type K , has magnetic helicity $\mathcal{H}_M = h\Phi^2$, where h is the conserved writhe-plus-twist of the tube (Moffatt & Ricca 1992). On dimensional grounds, the minimum energy of this knotted tube, in its fully relaxed state, is

$$M_{min} = m_K(h) \Phi^2 V^{-1/3} \,, \tag{13}$$

where $m_K(h)$ is a dimensionless function of h, determined solely by the knot type K (Moffatt 1990b). Moreover, this state, being stable, will be characterised by a spectrum of real frequencies ω_n, which, again on dimensional grounds, are given by

$$\omega_n = \Omega_{Kn}(h)\Phi V^{-1} \,. \tag{14}$$

This is just the sort of result that Kelvin needed as a basis for his 'vortex' atom theory! If he had been able to formulate his ideas in terms of magnetic flux tubes in an 'ideal' conducting fluid, instead of vortex tubes, then these ideas could really have taken root! Although all the ingredients of electromagnetism were by then available, another 75 years were to pass before the marriage of electromagnetic theory and fluid dynamics was successfully consummated in the work of Alfvén (1942).

When the internal twist of a flux tube is small, minimum energy is achieved when the tube is contracted to minimum length (for prescribed volume in response to Maxwell tension); in this configuration, the tube is in a 'pulled-tight' state, making contact with itself on contact surfaces of tangential discontinuity.

This configuration is close to the 'ideal knot' concept of Stasiak et al(1998). One physically illuminating way to construct an ideal knot is as follows: take a unit volume of plasticine in the form of a ' standard' cylinder of circular cross-section whose height and diameter are equal, and roll this out so that it becomes a cylinder, still of circular cross-section, of length L. For large L, the work done in this process is roughly proportional to L, and it is reasonable to *define* the (dimensionless) energy of this rolled-out state as $E = L$. Now for a given knot K, we choose L to be the minimum length for which the tube can be tied in this knot, the ends being then joined together (the bending energy expended in this knotting process can be supposed negligible, and the cross-section can be assumed to remain circular and of constant radius). The result is the ' ideal' (or what I would prefer to call the 'pulled-tight') configuration of the knot K.

This differs from the magnetic flux tube in two minor respects: (i) the minimum energy knotted flux-tube does not have uniform circular cross-section (although this can be imposed as an additional constraint – see Chui & Moffatt 1995); and (ii) it does in general have internal twist, which is not present in the above plasticine construction (although it could be added). Nevertheless, the ideal knot may be recognised as a close relative of the minimum energy knotted flux tube.

The pulled-tight configurations of all prime knots of up to 8 crossings and their associated lengths have been computed by Stasiak *et al* (1998) (see also Flammini & Stasiak 2006). Thus the 'ground-state' energies of these knots may be regarded as known. As pointed out by Moffatt (1990b), higher energy states (corresponding to local energy minima) may be expected to exist also, and we should therefore allow for an 'energy spectrum' for each knot; it is the lowest, or ground state, energy that is of greatest potential interest.

7 Knots in the quark-gluon plasma

These arcane considerations come to life in the extreme context of the 'quark-gluon' plasma, the hypothesised state of matter in the universe just microseconds after the Big Bang. Experiments in the Relativistic Heavy Ion Collider (Brookhaven National Laboratory) involving collision of gold nuclei produce a quark-gluon plasma that, remarkably, behaves like "a liquid of very low viscosity" [as described in *New Scientist*, 14 July 2007].

Gluons are the quantum chromodynamic (QCD) flux tubes that connect quarks allowing the transmission of force between them. It has been conjectured that 'glueballs' (solitonic solutions of the QCD equations) are knotted QCD tubes within such a plasma (Buniy & Kephart 2005); these are manifest as excitations in the plasma with an energy spectrum which is in principle measurable.

Bunyi & Kephart have found a remarkable correlation between the low-energy levels measured in these experiments, and the ground-state energies of knots of low crossing number: the two are linearly related up to knots of crossing number nine (as far as measurements are available). The suggestion that these fundamental excitations in matter in its most primitive form are knotted QCD flux tubes is quite compelling. Kelvin might have said "I told you so"! But of course here, the physical context is very different. Most importantly, the QCD tubes are quantised with universal cross-sectional radius on the quantum scale determined by the QCD force field, so relaxation to a minimum energy state conserves cross-section rather than volume. Nevertheless, the concept of knotted flux tubes in a minimum energy configuration returns to centre-stage!

8 Conclusions and perspectives

I may perhaps be allowed to conclude with three philosophical, not to say, theological, remarks. First, the old 19th century concept of an ether (an ideal incompressible fluid permeating all space), so ardently embraced by Kelvin, is now replaced in this 21st century by the 'quark-gluon plasma', the most primitive manifestation of matter in the very early universe.

Second, as we have seen above, Kelvin's knotted vortices in the ether representing atoms (in his time the most fundamental building blocks of the material world) are now replaced by knotted chromo-magnetic flux tubes, i.e. glueballs, the most fundamental excitations in the quark-gluon plasma.

And third, God, that 'immortal, invisible' influence promoting the creation of matter and life as we know it, is replaced in current cosmology by 'Dark Energy', that equally intangible concept required to render astronomical observations compatible with currently-known fundamental principles of physics.

Plus ça change, plus c'est la même chose?

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