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Note on the triad interactions of homogeneous turbulence

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Triad interactions, involving a set of wave-vectors $\{\pm\boldsymbol{k}, \pm\boldsymbol{p}, \pm\boldsymbol{q}\}$, with $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0$, are considered, and the results of triad truncation are compared with the results of exact Euler evolution starting from the same initial conditions. The essential two-dimensionality of the triad interaction is used to separate the problem into two parts: a nonlinear two-dimensional flow problem in the triad plane, and a linear problem of 'passive scalar' type for the evolution of the component of velocity perpendicular to this plane. Several examples of triad evolution are presented in detail, and the marked contrast with Euler evolution is demonstrated. It is known that energy and helicity are conserved under triad truncation; it is shown that the 'in-plane' energy and enstrophy are also conserved. However, it is also shown that, in general, the evolution of the vorticity under triad truncation cannot be represented as transport by any divergence-free velocity field, with the consequence that the detailed topology of the vorticity field is not conserved under this truncation.

Key words: homogeneous turbulence, topological fluid dynamics, turbulence theory

1. Introduction

It is common to consider a field of homogeneous turbulence in terms of the Fourier transform of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, defined by

$$\hat{\boldsymbol{u}}(\boldsymbol{k}, t) = \frac{1}{(2\pi)^3} \int \boldsymbol{u}(\boldsymbol{x}, t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \, \mathrm{d}\boldsymbol{x}, \tag{1.1}$$

where reality of $\boldsymbol{u}(\boldsymbol{x}, t)$ requires that

$$\hat{\boldsymbol{u}}(-\boldsymbol{k}, t) = \hat{\boldsymbol{u}}^*(\boldsymbol{k}, t), \tag{1.2}$$

and the star represents the complex conjugate. For strictly homogeneous turbulence, $\hat{\boldsymbol{u}}(\boldsymbol{k}, t)$ must be understood as a generalized function (or distribution). It is common,

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however, to assume periodicity in the three space directions, so that the integral (1.1) is replaced by a sum. The Fourier transform of the Navier–Stokes equation then takes the form

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) \hat{u}_i(\boldsymbol{k}, t) = -\mathrm{i} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} P_{ij}(\boldsymbol{k}) q_k \hat{u}_k^*(\boldsymbol{p}, t) \hat{u}_j^*(\boldsymbol{q}, t), \tag{1.3}$$

where ν is the kinematic viscosity, and the projection operator $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j/k^2$, deriving from the pressure field, ensures that the incompressibility condition $\boldsymbol{k} \cdot \hat{\boldsymbol{u}}(\boldsymbol{k}, t) = 0$ is satisfied for all t (Kraichnan 1959).

A triad of non-zero wave-vectors $\{\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}\}$ with $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0$ forms a triangle in $\boldsymbol{k}$-space. Such triads evidently provide a coupling of Fourier components that results from the quadratic nonlinearity of the Navier–Stokes equation, and have attracted attention for this reason. Of course, the reality condition (1.2) implies that such 'triads' actually involve a set S of six wave-vectors, specifically

$$S = \{\{\pm\boldsymbol{k}, \pm\boldsymbol{p}, \pm\boldsymbol{q}\}, \boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0\}. \tag{1.4}$$

The importance of such triads was first noted by Kraichnan (1973), who stated that, under a Galerkin truncation of the system (1.3) in which only Fourier modes with wavenumber magnitudes in some range $\{k_1, k_2\}$ are retained, the energy and helicity "are conserved individually by each triad of interacting wave vectors $\{\pm\boldsymbol{k}, \pm\boldsymbol{p}, \pm\boldsymbol{q}\}$". Taken literally, this statement is actually incorrect, because, if we suppose that at some initial instant $t = 0$ we have only a single 'triadic disturbance' corresponding to wave-vectors $\{\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}\}$, and if k_1 is small compared with the minimum of $\{|\boldsymbol{k}|, |\boldsymbol{p}|, |\boldsymbol{q}|\}$, and k_2 large compared with the maximum, then many harmonics of the initial disturbance will be generated before the truncation has any effect, and the excited harmonics must take their energy from the initial triad. It is therefore appropriate to look closely at the argument that Kraichnan gave in justification. He wrote:

> Consider an instantaneous state in which any given set of wave vectors (and their negatives) have arbitrary amplitudes, subject only to the reality condition $\hat{\boldsymbol{u}}(-\boldsymbol{k}) = \hat{\boldsymbol{u}}^*(\boldsymbol{k})$, while the amplitudes at all other wave vectors are zero. Since the energy and helicity expressions are quadratic and diagonal in the wave vector amplitudes, the instantaneous rate of change of the energy and helicity in the instantaneously unexcited wave vectors is zero. Thus the overall conservation implies that the energy and helicity in the excited modes by themselves are conserved. But since the excited modes are chosen arbitrarily and have arbitrary amplitudes,it follows that the conservation is an identity property of the coefficients, in the Navier–Stokes equation, that couple each individual triad of wave vectors. The detailed conservation property can also be verified by direct calculation.

This argument shows that the rate of change of energy and helicity in the initially excited modes is zero at the initial instant $t = 0$, but does not show that this rate of change remains zero for all $t > 0$. I surmise that what Kraichnan actually meant was that, if (1.3) are truncated *so that only the contributions from the wave-vectors* $\{\pm\boldsymbol{k}, \pm\boldsymbol{p}, \pm\boldsymbol{q}\}$ *are retained for every* $t \geqslant 0$, then both the energy and helicity of this triad are conserved.

This extreme form of truncation may be described as 'triad truncation'. It has a slightly arbitrary character because, for example, elementary quadratic interaction of modes with wave-vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ leads to harmonics with wave-vectors $\boldsymbol{p} \pm \boldsymbol{q}$;

of these, $\boldsymbol{p}+\boldsymbol{q}=-\boldsymbol{k}\in S$ (S being defined by (1.4)) is retained in triad truncation, whereas $\boldsymbol{p}-\boldsymbol{q}\notin S$ is not. (Note that, even if, in Kraichnan's notation, we take $k_1=\min\{|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{k}|\}$ and $k_2=\max\{|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{k}|\}$, Galerkin truncation is still not sufficient to give the required conservation, because the wave-vector magnitude $|\boldsymbol{p}-\boldsymbol{q}|$ may well lie in the range (k_1, k_2).) Despite this artificiality (which cannot be cured by extending the set to $S^{\pm}=\{\{\pm\boldsymbol{k}, \pm\boldsymbol{p}, \pm\boldsymbol{q}\}, \boldsymbol{k}\pm\boldsymbol{p}\pm\boldsymbol{q}=0\}$), triad truncation has been invoked by subsequent authors (see, in particular, Waleffe (1992)) in analysis of local and non-local interactions in $\boldsymbol{k}$-space, with a view to providing a better understanding of the cascades of energy and helicity.

It therefore seems desirable to reconsider triad interactions in order to better understand their supposed relevance to exact Euler evolution. This is the purpose of the present note.

2. Two-dimensional representation of the triad interaction

The first point to recognize is that the flow associated with a triad interaction is actually two-dimensional (in the sense that it depends on only two Cartesian variables). This is because the triangle defined by the vectors $\{\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}\}$ with $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0$ lies in a plane, which we may take to be the $\{x, y\}$ plane; then, with $\boldsymbol{k}\cdot\boldsymbol{x}=k_x x+k_y y$, $\boldsymbol{p}\cdot\boldsymbol{x}=p_x x+p_y y$ and $\boldsymbol{q}\cdot\boldsymbol{x}=q_x x+q_y y$, it is evident that

$$\boldsymbol{u}(\boldsymbol{x}, t)=\hat{\boldsymbol{u}}(\boldsymbol{k}, t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}+\hat{\boldsymbol{u}}(\boldsymbol{p}, t)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}+\hat{\boldsymbol{u}}(\boldsymbol{q}, t)\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}+\text{c.c.}, \tag{2.1}$$

where c.c. represents the complex conjugate, is a function only of x, y and t, i.e., say,

$$\boldsymbol{u}(\boldsymbol{x}, t)=(u(x, y, t), v(x, y, t), w(x, y, t)). \tag{2.2}$$

The incompressibility condition $\partial u/\partial x+\partial v/\partial y=0$ is satisfied by the introduction of a stream function $\psi(x, y, t)$, such that

$$u=\partial\psi/\partial y, \quad v=-\partial\psi/\partial x. \tag{2.3}$$

The vorticity field $\omega=\nabla\times\boldsymbol{u}$ then has components

$$\omega_x=\partial w/\partial y, \quad \omega_y=-\partial w/\partial x, \quad \omega_z=-\nabla^2\psi, \tag{2.4}$$

and the Navier–Stokes equations may be written in the standard simplified form

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\omega_z=\frac{\partial(\psi, \omega_z)}{\partial(x, y)} \tag{2.5}$$

and

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)w=\frac{\partial(\psi, w)}{\partial(x, y)}. \tag{2.6}$$

The z-component of velocity $w(x, y, t)$ has no effect on the evolution of $\psi(x, y, t)$, and is itself convected and diffused like a passive scalar field. We shall be concerned in what follows with the inviscid situation $\nu=0$.

In an appendix to his paper, Waleffe (1992) actually treated the two-dimensional situation described by (2.5), but did not appear to recognize the applicability of his analysis to the general triad interaction. The following treatment is somewhat simpler than that of Waleffe.

Let us suppose that $\psi(x, y, t)$ and $w(x, y, t)$ have the triadic Fourier representations

$$\psi(x, y, t) = A_1(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + A_2(t)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + A_3(t)\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} + \text{c.c.} \tag{2.7}$$

and

$$w(x, y, t) = B_1(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + B_2(t)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + B_3(t)\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} + \text{c.c.} \tag{2.8}$$

Then

$$\psi_x = \mathrm{i}k_x A_1(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + \mathrm{i}p_x A_2(t)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + \mathrm{i}q_x A_3(t)\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} + \text{c.c.} \tag{2.9}$$

and similarly for ψ_y, w_x and w_y. Note also that

$$\omega_z = C_1(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + C_2(t)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} + C_3(t)\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} + \text{c.c.}, \tag{2.10}$$

where, since $\omega_z = -\nabla^2\psi$,

$$C_1 = k^2 A_1, \quad C_2 = p^2 A_2, \quad C_3 = q^2 A_3. \tag{2.11}$$

If we look first at (2.6), there are 72 terms in all in the expansion of the Jacobian $\partial(\psi, w)/\partial(x, y) = \psi_x w_y - \psi_y w_x$. Only 24 of these have wave-vectors in the set S, and, applying triad truncation, we may gather these surviving terms together as follows:

$$\begin{aligned}\psi_x w_y - \psi_y w_x \;=\; & (p_x q_y - p_y q_x)(A_3^* B_2^* - A_2^* B_3^*)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} + (q_x k_y - q_y k_x)(A_1^* B_3^* - A_3^* B_1^*)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \\ & + (k_x p_y - k_y p_x)(A_2^* B_1^* - A_1^* B_2^*)\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} + \text{c.c.}\end{aligned} \tag{2.12}$$

Here we may use the convenient geometrical result that

$$p_x q_y - p_y q_x = q_x k_y - q_y k_x = k_x p_y - k_y p_x = 2\Delta, \tag{2.13}$$

where Δ is the area of the triangle formed by the vectors $\{\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}\}$. Hence, (2.6) (with $\nu = 0$) yields the equations

$$\dot{B}_1 = 2\Delta(A_3^* B_2^* - A_2^* B_3^*), \quad \dot{B}_1^* = 2\Delta(A_3 B_2 - A_2 B_3), \tag{2.14}$$

and four similar equations for $\dot{B}_2$, $\dot{B}_2^*$ and $\dot{B}_3$, $\dot{B}_3^*$, obtained by cyclic permutation of suffices. Writing $\boldsymbol{A} = (A_1, A_2, A_3)$ and $\boldsymbol{B} = (B_1, B_2, B_3)$, these equations may now be written in the compact vector notation

$$\dot{\boldsymbol{B}} = 2\Delta\,(\boldsymbol{B}^* \times \boldsymbol{A}^*), \quad \dot{\boldsymbol{B}}^* = 2\Delta\,(\boldsymbol{B} \times \boldsymbol{A}). \tag{2.15}$$

Turning now to (2.5), we obviously arrive in the same way at the equations

$$\dot{\boldsymbol{C}} = 2\Delta\,(\boldsymbol{C}^* \times \boldsymbol{A}^*), \quad \dot{\boldsymbol{C}}^* = 2\Delta\,(\boldsymbol{C} \times \boldsymbol{A}), \tag{2.16}$$

where $\boldsymbol{C} = (k^2 A_1, p^2 A_2, q^2 A_3)$. These nonlinear equations describe the evolution of the complex vector $\boldsymbol{A}(t)$ and hence of the stream function ψ under triad truncation. They may be written in the equivalent form

$$\left.\begin{aligned} k^2 \dot{A}_1 &= 2\Delta(p^2 - q^2)A_2^* A_3^*, \\ p^2 \dot{A}_2 &= 2\Delta(q^2 - k^2)A_3^* A_1^*, \\ q^2 \dot{A}_3 &= 2\Delta(k^2 - p^2)A_1^* A_2^*, \end{aligned}\right\} \tag{2.17}$$

together with their complex conjugates. Note that if A_1, A_2 and A_3 are real at $t = 0$, then according to these equations, they remain real for all $t > 0$; in this case, the (2.17) have the same structure as the Euler equations for the free rotation of a rigid body, an analogy pointed out by Waleffe (1992).

3. Energy and helicity

The mean energy E of the flow (2.2) is given by

$$2E = \langle u^2 + v^2 + w^2 \rangle = \langle |\nabla\psi|^2 + |w|^2 \rangle = k^2|A_1|^2 + p^2|A_2|^2 + q^2|A_3|^2 + \langle |w|^2 \rangle. \tag{3.1}$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}t}|A_1|^2 = A_1\dot{A}_1^* + \dot{A}_1 A_1^* = 2\varDelta(p^2 - q^2)(A_1A_2A_3 + \text{c.c.}) \tag{3.2}$$

and similar equations for $\mathrm{d}|A_2|^2/\mathrm{d}t$ and $\mathrm{d}|A_2|^2/\mathrm{d}t$; hence, since terms cancel in pairs,

$$\frac{\mathrm{d}}{\mathrm{d}t}(k^2|A_1|^2 + p^2|A_2|^2 + q^2|A_3|^2) = 0. \tag{3.3}$$

Further, with $\langle |w|^2 \rangle = |\boldsymbol{B}|^2$, and using (2.15), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle |w|^2 \rangle = \boldsymbol{B}\cdot\dot{\boldsymbol{B}}^* + \dot{\boldsymbol{B}}\cdot\boldsymbol{B}^* = 0. \tag{3.4}$$

Hence the energy is indeed constant, and, more strongly, the contributions to energy from the components in and perpendicular to the 'triad plane' of the wave-vectors $\{\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}\}$ are separately constant.

Similarly, the mean helicity $\mathscr{H}$ of the flow (Moffatt 1969) is given by

$$\mathscr{H} = \langle \boldsymbol{u}\cdot\omega \rangle = \langle \nabla\psi\cdot\nabla w \rangle - \langle w\nabla^2\psi \rangle = 2\langle \nabla\psi\cdot\nabla w \rangle. \tag{3.5}$$

From (2.7), (2.8) and (2.11), we easily find

$$\langle \nabla\psi\cdot\nabla w \rangle = \boldsymbol{C}\cdot\boldsymbol{B}^* + \boldsymbol{C}^*\cdot\boldsymbol{B}, \tag{3.6}$$

so that

$$\frac{1}{4\varDelta}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} = \boldsymbol{C}\cdot(\boldsymbol{B}\times\boldsymbol{A}) + \boldsymbol{B}^*\cdot(\boldsymbol{C}^*\times\boldsymbol{A}^*) + \boldsymbol{C}^*\cdot(\boldsymbol{B}^*\times\boldsymbol{A}^*) + \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = 0. \tag{3.7}$$

Hence the mean helicity is also constant under triad truncation.

By virtue of the analogy between w and ω_z evident in (2.5) and (2.6), the constancy of $\langle w^2 \rangle$ implies also the constancy of $\varOmega = \langle \omega_z^2 \rangle$. This is the contribution to enstrophy from just the z-component of vorticity; from (2.10), we thus have

$$\varOmega = \langle \omega_z^2 \rangle = |\boldsymbol{C}|^2 = k^4|A_1|^2 + p^4|A_2|^2 + q^4|A_3|^2 = \text{const.} \tag{3.8}$$

4. Examples

4.1. *Example 1*

Suppose we adopt an initial condition with

$$\psi = 2\cos px, \quad w = 2\cos px + 2\cos qy \quad \text{at } t = 0. \tag{4.1}$$

Thus $\boldsymbol{p} = (p, 0, 0)$, $\boldsymbol{q} = (0, q, 0)$, $\boldsymbol{k} = (-p, -q, 0)$, $2\varDelta = pq$, and the initial amplitudes (all real) are

$$A_1 = A_3 = 0, \quad A_2 = 1; \quad B_1 = 0, \quad B_2 = B_3 = 1 \quad \text{at } t = 0. \tag{4.2}$$

Here, the As are obviously constant, and from (2.14), the Bs satisfy the equations

$$\dot{B}_1 = -pqB_3, \quad \dot{B}_2 = 0, \quad \dot{B}_3 = pqB_1, \tag{4.3}$$

with solution satisfying the initial conditions (4.2),

$$B_1 = -\sin pqt, \quad B_2 = 1, \quad B_3 = \cos pqt. \tag{4.4}$$

Hence the 'triad solution' for w is

$$w = -2\sin pqt\cos(px + qy) + 2\cos px + 2\cos pqt\cos qy. \tag{4.5}$$

In contrast, the Euler equation for $w(x, y, t)$ (with no truncation) is

$$w_t = \psi_x w_y - \psi_y w_x = -(2p\sin px)w_y, \tag{4.6}$$

and the exact solution of this equation satisfying the initial condition (4.1) is

$$w(x, y, t) = 2\cos px + 2\cos[qy - 2pqt\sin px]. \tag{4.7}$$

This may be compared with the triad solution (4.5); for small t, (4.5) gives

$$w(x, y, t) = 2\cos px + 2\cos qy - 2pqt\cos(px + qy) + O(t^2), \tag{4.8}$$

whereas, for small t, the exact Euler solution (4.7) gives

$$\begin{aligned} w(x, y, t) &= 2\cos px + 2\cos qy + 4pqt\sin px\sin qy + O(t^2) \\ &= 2\cos px + 2\cos qy - 2pqt\cos(px + qy) + 2pqt\cos(px - qy) + O(t^2). \end{aligned} \tag{4.9}$$

Even at order $O(t)$, the triad solution departs from the exact Euler solution, because the contribution with wave-vector $(p, -q, 0)$ has been filtered out in the triad truncation.

The triad evolution is strikingly different from the exact Euler evolution as time t increases, as shown in figure 1. Here, by way of illustration, we have taken $p = 1$, $q = 2$. The periodic evolution given by (4.5) is shown in the left-hand column. The exact Euler evolution given by (4.7) is shown in the right-hand column; this shows the development of increasingly high harmonics of the initial field due to the shearing of the w-contours in the y-direction (of course, viscosity would ultimately arrest this development) by the rectilinear velocity field $(u, v) = (0, 2\sin x)$. It is hard to believe that triad truncation can give any genuine insights relevant to real Euler or Navier–Stokes evolution in such circumstances.

4.2. *Example 2*

With $\boldsymbol{p} = (p, 0, 0)$, $\boldsymbol{q} = (0, p, 0)$, we take initial condition

$$\psi = 2\cos px + 2\cos py, \quad w(x, y, t) = 2\cos px + 4\cos py. \tag{4.10}$$

Thus

$$A_1 = 0, \quad A_2 = A_3 = 1; \quad B_1 = 0, \quad B_2 = 1, \quad B_3 = 2 \quad \text{at } t = 0. \tag{4.11}$$

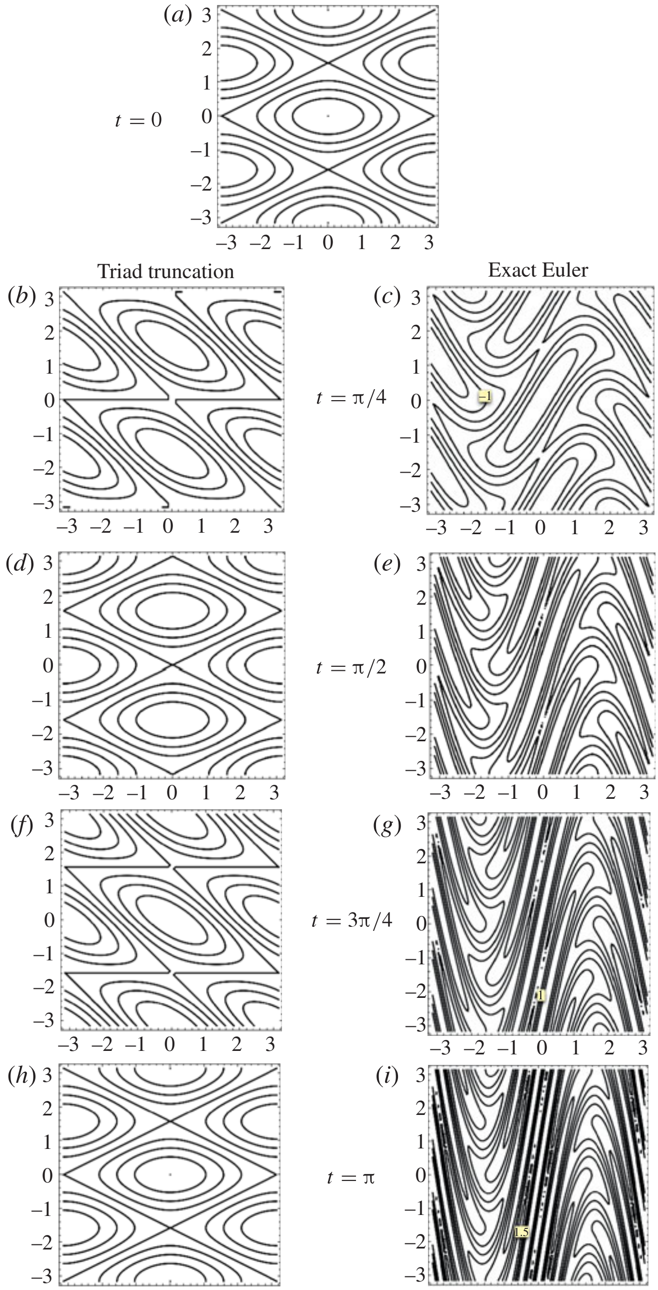

FIGURE 1. Example 1: evolution of the field $w(x, y, t)$ for $p = 1$, $q = 2$; the left-hand column shows the time-periodic evolution under triad truncation; the right-hand column shows the exact Euler evolution. The box covers the area $|x| \leqslant \pi$, $|y| \leqslant \pi$.

Since $\nabla^2\psi = -p^2\psi$, the flow in the $\{x, y\}$ plane is steady, so that the As are constant, and the Bs satisfy

$$\dot{B}_1 = p^2(B_2 - B_3), \quad \dot{B}_2 = -p^2 B_1, \quad \dot{B}_3 = p^2 B_1. \tag{4.12}$$

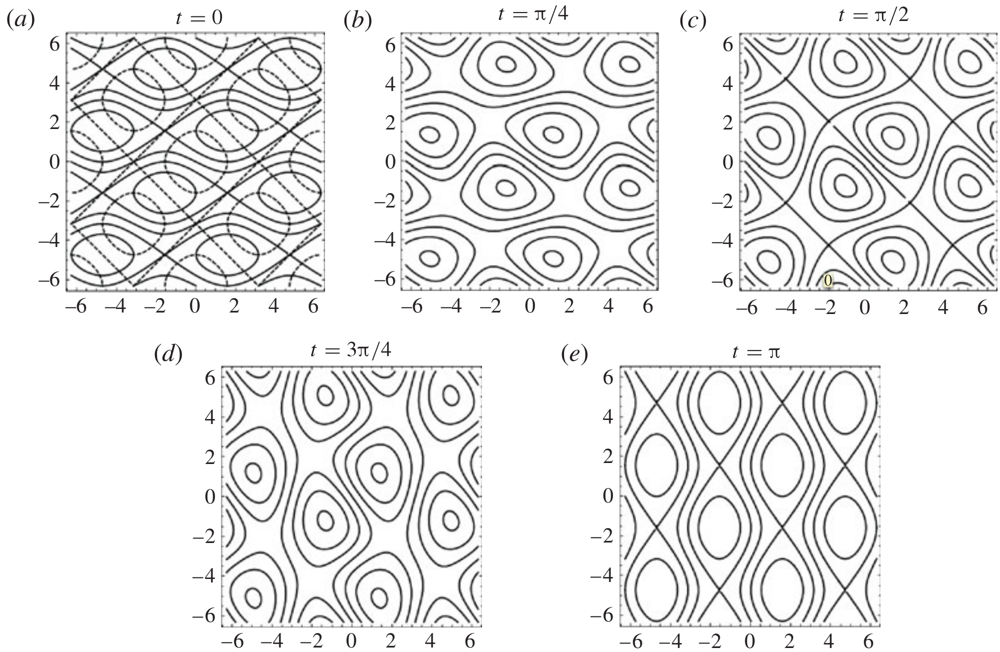

FIGURE 2. Example 2: evolution of the contours $w = \text{const.}$ under triad truncation, given by (4.14), during a first half-period; here $\tau = \sqrt{2}\, t$ and $p = 1$. The box covers the area $|x| \leqslant 2\pi$, $|y| \leqslant 2\pi$. Panel (*a*) shows the contours $\psi = \text{const.}$ (dashed), superposed on the initial w contours (solid).

The solution satisfying the initial conditions (4.11) is

$$2B_1 = -\sqrt{2} \sin \sqrt{2}\, p^2 t, \quad 2B_2 = 3 - \cos \sqrt{2}\, p^2 t, \quad 2B_3 = 3 + \cos \sqrt{2}\, p^2 t, \tag{4.13}$$

and the corresponding triad-truncated solution for w, time periodic with period $\sqrt{2}\, \pi/p^2$, is

$$\begin{aligned} w \; = \; & \sqrt{2} \sin \sqrt{2}\, p^2 t \sin p(x+y) + (3 - \cos \sqrt{2}\, p^2 t) \sin px \\ & + (3 + \cos \sqrt{2}\, p^2 t) \sin py. \end{aligned} \tag{4.14}$$

The change in the contours $w = \text{const.}$ during an initial half-period are shown in figure 2. Figure 2(*a*) also contains the contours ψ =const, which remain steady throughout.

In this case, the exact solution of the Euler equations cannot easily be found; however, physically, it is clear that in reality the contours $\psi = \text{const.}$ are wound up by the flow $(\psi_y, -\psi_x)$ into increasingly tight double spirals within each region of closed streamlines $\psi = \text{const.}$, in the manner much as described by Gilbert (1988); again, the triad truncation can give no hint of this complex behaviour.

4.3. *Example 3*

Suppose we now take $\boldsymbol{k} = (-1, -2, 0)$, $\boldsymbol{p} = (1, 0, 0)$, $\boldsymbol{q} = (0, 2, 0)$ and initial conditions

$$\psi = 2 \cos x + 4 \cos 2y, \quad w = 2 \cos x + 6 \cos 2y, \tag{4.15}$$

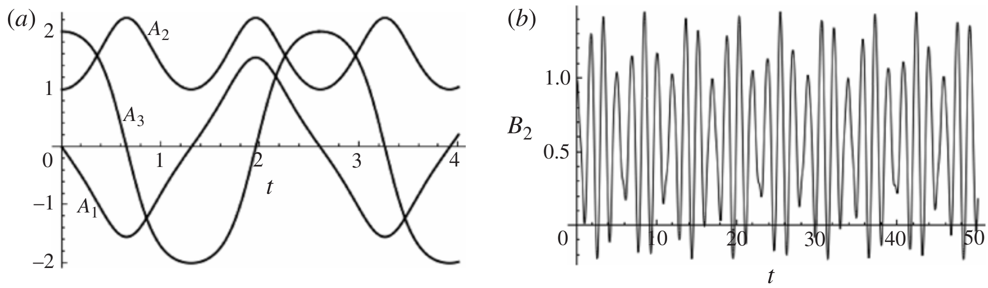

FIGURE 3. Example 3: (a) time-periodic functions $A_1(t)$, $A_2(t)$ and $A_3(t)$ from integration of (4.17); (b) resulting function $B_2(t)$ for $0 < t < 50$ from integration of (4.20), showing no clear sign of periodicity; the behaviour of the functions $B_1(t)$ and $B_3(t)$ is similar.

so that now

$$A_1 = 0, \quad A_2 = 1, \quad A_3 = 2; \quad B_1 = 0, \quad B_2 = 1, \quad B_3 = 3 \quad \text{at } t = 0. \tag{4.16}$$

Again, the A remain real, and (2.17) become

$$\mathrm{d}A_1/\mathrm{d}t = -6A_2A_3/5, \quad \mathrm{d}A_2/\mathrm{d}t = -2A_3A_1, \quad \mathrm{d}A_3/\mathrm{d}t = 2A_1A_2. \tag{4.17}$$

We have the two integrals

$$E = 5A_1^2 + A_2^2 + 4A_3 = 17 \quad \text{and} \quad \Omega = 25A_1^2 + A_2^2 + 16A_3 = 65. \tag{4.18}$$

The solution, periodic on the curve of intersection of these two ellipsoids in the A-space, is shown in figure 3(a); the period is approximately $T = 2.61$.

The B also remain real and satisfy the equations

$$\dot{B}_1 = 2(B_2A_3 - B_3A_2), \quad \dot{B}_2 = 2(B_3A_1 - B_1A_3), \quad \dot{B}_3 = 2(B_1A_2 - B_2A_1). \tag{4.19}$$

These admit the single integral

$$B_1^2 + B_2^2 + B_3^2 = \text{const.} = 10. \tag{4.20}$$

Figure 3(b) shows $B_2(t)$ for $0 < t < 50$; this shows no clear sign of periodicity; $B_1(t)$ and $B_3(t)$ are qualitatively similar. Figure 4 shows contour plots $w(x, y, t) = \text{const.}$ at times $t = 0$, $4T$ and $16T$.

5. Non-conservation of topology under triad truncation

Under exact Euler evolution, the w contours and the ω_z contours are transported by the flow $(\psi_y, -\psi_x)$, and the topology of these fields is therefore conserved. One measure of this topology is the conserved mean helicity.

The fact that, under triad truncation, the mean helicity is still conserved suggests that the fields w and ω_z may be likewise transported, not by the velocity field $(\psi_y, -\psi_x)$ (because the truncated solution does not satisfy the Euler equations), but possibly by some related two-dimensional velocity field, $\tilde{\boldsymbol{v}}$, say, with $\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{v}} = 0$. Suppose that such a field exists; then under the triad truncation, w satisfies the equation

$$\partial w/\partial t = -\tilde{\boldsymbol{v}} \cdot \boldsymbol{\nabla} w = -\boldsymbol{\nabla} \cdot (\tilde{\boldsymbol{v}} w). \tag{5.1}$$

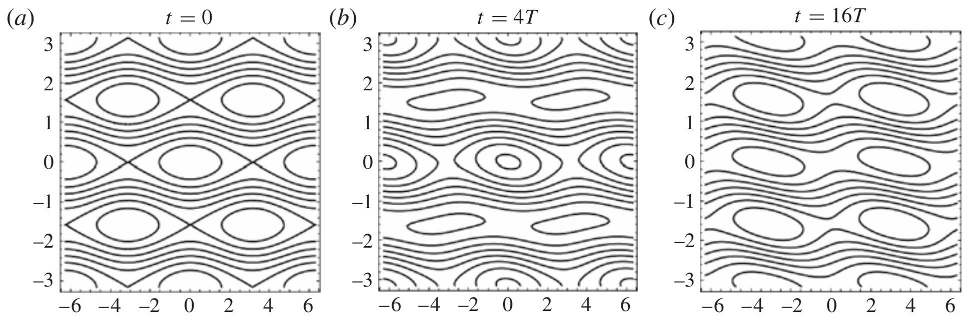

FIGURE 4. Example 3: contour plots of w for the initial conditions (4.15) at times $t = 0$, $4T$ and $16T$, the ψ field being time-periodic with period T.

Integrating this equation over any area $\mathscr{A}$, say, inside a contour $\partial\mathscr{A}$ on which $w = 0$, we have

$$\int_{\mathscr{A}} (\partial w/\partial t)\, \mathrm{d}x\, \mathrm{d}y = -\oint_{\partial\mathscr{A}} w(\tilde{\boldsymbol{v}} \boldsymbol{\cdot} \boldsymbol{n})\, \mathrm{d}l = 0. \tag{5.2}$$

This, then, is a necessary condition for the existence of the hypothetical 'transporting' field $\tilde{\boldsymbol{v}}$.

Consider the case of example 1 above, for which, at time $t = 0$, $\mathscr{A}$ is the parallelogram bounded by the lines $px \pm qy = \pm\pi/2$ (see figure 1*a*). Now from the triad-truncated solution (4.5),

$$\partial w/\partial t|_{t=0} = -2pq \cos(px + qy), \tag{5.3}$$

which is negative throughout the interior of $\mathscr{A}$; in fact, as may be easily verified,

$$\int_{\mathscr{A}} (\partial w/\partial t)\, \mathrm{d}x\, \mathrm{d}y = -4\pi. \tag{5.4}$$

Thus, for this case, the condition (5.2) is not satisfied, even at time $t = 0$, so a transporting, divergence-free, velocity field $\tilde{\boldsymbol{v}}$ does not exist. The same conclusion holds for examples 2 and 3 also; and indeed there is no reason to believe that the condition (5.2) will be satisfied for arbitrary triads and arbitrary initial conditions (under triad truncation) except possibly in exceptional circumstances. In particular, the area $\mathscr{A}(w)$ inside contours $w = \text{const.}$ (the signature function of w in the terminology of Moffatt (1986)) is *not* conserved under triad truncation. Similarly, the signature function $\mathscr{A}(\omega_z)$ is not conserved, i.e. the evolution of the vorticity field under triad truncation cannot be represented in terms of transport by any divergence-free velocity field. I am grateful to a referee who has pointed out that any non-trivial truncation will similarly destroy the Lagrangian structure (and so presumably the detailed topological properties) of the flow.

6. Conclusions

The aim of this note has been to clarify the meaning and nature of the triad interactions introduced by Kraichnan (1973), and given prominence by him and later authors in the context of inter-scale energy and helicity transfer in homogeneous turbulence. Triad truncation, involving, as it does, retention of amplitudes corresponding to the wave-vector $\boldsymbol{p}+\boldsymbol{q}$ resulting from interaction between two wave-vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, but rejection of those corresponding to $\boldsymbol{p}-\boldsymbol{q}$, is to that extent somewhat arbitrary. By exploiting the essential two-dimensionality of the triad interaction, we have shown that triads exhibit four quadratic invariants, the energy and helicity (as previously known), and, in addition, the energy and enstrophy of the flow in the plane of the triad triangle. Three examples of triad evolution have been presented showing that: (i) this evolution is, as might be expected, strikingly different from Euler evolution from the same initial conditions; (ii) triad evolution is time-periodic for particular symmetries if the initial condition involves just two wave-vectors of equal wavenumber magnitude, but is otherwise non-periodic; and (iii) certain topological features of the velocity field that are conserved under Euler evolution are not in general conserved under triad truncation.

There is no doubt that triad truncation has intrinsic interest, just as do other truncations like standard Galerkin truncation, or truncations based on helical modes, as adopted by Cambon & Jacquin (1989), Waleffe (1992) and more recently Biferale, Musacchio & Toschi (2013). The special interest of triad truncation lies in the fact that evolution under this truncation can be explicitly calculated and compared with evolution under the Euler equations starting from the same initial conditions. It seems likely that the rapid divergence of these solutions is a property of any truncation, and leads one to urge caution in the interpretation of results based on direct numerical simulations of the Navier–Stokes equations, and to urge that particular attention be paid to the robustness of the truncation adopted.

Acknowledgements

My interest in this problem was stimulated by the recent paper of Zhu, Yang & Zhu (2013), who provide a unified account of conjectured energy and helicity cascade processes in a variety of hydrodynamic and magnetohydrodynamic contexts; the 'rugged invariance' of energy and helicity under Galerkin truncation provides a starting point for the theoretical discussion of these authors.

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