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Helicity and Celestial Magnetism

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This informal article discusses the central role of magnetic and kinetic helicity in relation to the evolution of magnetic fields in geophysical and astrophysical contexts. It is argued that the very existence of magnetic fields of the intensity and scale observed is attributable in large part to the chirality of the background turbulence or random-wave field of flow, the simplest measure of this chirality being non-vanishing helicity. Such flows are responsible for the generation of large-scale magnetic fields which themselves exhibit magnetic helicity. In the geophysical context, the turbulence has a 'magnetostrophic' character in which the force balance is primarily that between buoyancy forces, Coriolis forces and Lorentz forces associated with the dynamo-generated magnetic field; the dominant nonlinearity here arises from the convective transport of buoyant elements erupting from the 'mushy zone' at the inner core boundary. At the opposite extreme, in a highly conducting low-density plasma, the near-invariance of magnetic field topology (and of associated helicity) presents the challenging problem of 'magnetic relaxation under topological constraints', of central importance both in astrophysical contexts and in controlled-fusion plasma dynamics. These problems are reviewed and open issues, particularly concerning saturation mechanisms, are reconsidered.

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1. Introduction

Magnetic fields are widely observed in the visible universe on every scale, whether planetary, stellar, galactic or inter-galactic. In every such context, turbulence, or some kind of random wave motion, in the ambient conducting fluid medium plays a key role in the generation and maintenance of the magnetic field by dynamo action. In the case of the Earth, the situation 'closest to home', the fluid medium is the liquid-metal outer core, in which convective currents are driven by thermal and/or compositional convection, the fluid dynamics being strongly influenced, indeed controlled, by a combination of Coriolis forces due to the Earth's rotation, and Lorentz forces associated with the magnetic field that is generated. In the case of the Sun, it is turbulence in the convective zone, again strongly influenced by Coriolis and Lorentz forces, that generates the internal magnetic field, which erupts through the solar photosphere to provide the spectacular displays of solar activity revealed by NASA satellites. In the solar corona, the plasma density is very low, and the dynamics is totally controlled by the magnetic field, which continuously seeks in this region a minimum-energy 'force-free' state in response to evolution forced from below. Thus the Sun and stars like the Sun provide a magnificent 'astrophysical laboratory' for the observation of two complementary magnetohydrodynamic phenomena: dynamo theory on the one hand, and magnetic relaxation under the constraint of conserved field topology on the other.

In the dynamo context, it is widely recognized that the key property of turbulence that is most conducive to dynamo action is its helicity, i.e. the integrated or mean scalar product of velocity $\mathbf{u}(\mathbf{x}, t)$ and vorticity $\boldsymbol{\omega}(\mathbf{x}, t) = \nabla \times \mathbf{u}$. If this mean quantity is of constant sign over a sufficiently large extent of fluid, then dynamo action is inevitable, a magnetic field growing exponentially from an arbitrarily weak seed field until the Lorentz force is strong enough to react back upon the flow in such a way as to establish some kind of statistical equilibrium. The situation is usually considered in terms of turbulence that is both statistically homogeneous in space and stationary in time, an idealisation that requires input of energy at some scale l_0. Conventional turbulence involves a cascade of energy to smaller scales, a cascade that terminates at the Kolmogorov scale $l_v = \mathrm{R}^{-3/4} l_0$ at which viscous stresses come into play and dissipate kinetic energy to heat; here, $\mathrm{R} = \rho_0 u_0 l_0/\mu$ is the Reynolds number of the turbulence, where ρ_0 is the density, u_0 the rms velocity and μ the (dynamic) viscosity of the fluid.

We are in the main concerned with the growth of a 'mean magnetic field' on scales large compared with the energy-containing scale l_0 of the turbulence; to analyse such growth is the subject of the 'mean-field electrodynamics' pioneered by Steenbeck et al. (1966). The scale on which a magnetic field $\mathbf{B}$ can grow depends on the resistivity η of the medium through the magnetic Reynolds number $\mathrm{R_m} = u_0 l_0/\eta$. Somewhat surprisingly, dynamo action is possible even when $\mathrm{R_m} \ll 1$, provided the scale L of the fluid domain available for the growth of the field is sufficiently large. In this situation, the field grows on all scales of order $\mathrm{R_m}^{-2} l_0$ and greater (Moffatt 1970), so the scale requirement is that $L \gtrsim \mathrm{R_m}^{-2} l_0$. Note that a magnetic Reynolds number $\tilde{\mathrm{R}}_\mathrm{m} = u_0 L/\eta$ based on the large scale L is then necessarily *large*, indeed $\tilde{\mathrm{R}}_\mathrm{m} = \mathrm{O}(\mathrm{R_m}^{-1})$ or greater. At least in planetary contexts, the low-$\mathrm{R_m}$/high-$\tilde{\mathrm{R}}_\mathrm{m}$ scenario provides a convenient and plausible starting point for detailed analysis.

In astrophysical as opposed to planetary contexts, $\mathrm{R_m}$ is generally large, not small, and this poses severe difficulties, as yet not fully resolved, for mean-field electrodynamics. The problem is this: if we gradually increase $\mathrm{R_m}$ from small values up to values of order unity, the scale L on which magnetic fluctuations grow most rapidly decreases towards the scale l_0 of the turbulence itself, and the averaging technique on which mean-field electrodynamics is based, which strictly requires that $L \gg l_0$, becomes much harder to justify. Nevertheless, the mean-field approach has had modest success, more than might have been anticipated from the outset, in explaining at least qualitatively much of the observed behaviour of the solar magnetic field (for which $\mathrm{R_m}$ is certainly large), most obviously its time-periodic character as revealed by the 11-year sunspot cycle. The theoretical challenge is therefore in part to understand the reasons for this undeserved success.

As indicated above, the helicity of the background turbulence plays a key role in the promotion of dynamo action. This helicity arises when the source of energy derives from convection in a gravity field $\mathbf{g}(\mathbf{x})$ influenced by global rotation $\boldsymbol{\Omega}$. Such input of energy, involving the pseudo-scalar quantity $\mathbf{g}\cdot\boldsymbol{\Omega}$, exhibits the property of 'chirality', i.e. lack of reflexional symmetry, a property that is naturally inherited by the turbulence. Helicity is the simplest measure of this chirality, and it is a property that is naturally passed on to the large-scale magnetic field $\mathbf{B}$ that is generated. The resulting 'magnetic helicity' $\mathcal{H}_m$ is defined by

$$\mathcal{H}_m = \int_V \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}V. \tag{1.1}$$

where $\mathbf{B} = \nabla\times\mathbf{A}$, and V is some suitably defined sub-region of the fluid domain. We can be more specific about this below.

2. Fundamental properties of helicity

If, for the moment, we consider the idealisation of a perfectly conducting fluid medium, then, as recognised by Alfvén (1942), the lines of force of $\mathbf{B}$ (or '$\mathbf{B}$-lines') are 'frozen' in the fluid, i.e. are transported with the flow, the flux of $\mathbf{B}$ through any Lagrangian closed curve C being conserved. If we choose C to be a closed $\mathbf{B}$-line, then, by a double application of this result, the flux of $\mathbf{B}$ across any surface spanning this Lagrangian curve is conserved. Development of this simple consideration leads to the following result: let V be any Lagrangian volume of fluid on whose surface S the normal component of $\mathbf{B}$ vanishes. In particular, V could be any closed flux tube of $\mathbf{B}$, if such exists. Then, with this choice of V, $\mathcal{H}_m$ as defined by (1.1) is both gauge-invariant and constant[1]. If V is a closed unknotted flux tube carrying magnetic flux Φ_1 and linking magnetic flux Φ_2, then $\mathcal{H}_m$ may be easily evaluated as $\pm\Phi_1\Phi_2$, the sign $+$ or $-$ being chosen according as the linkage is right-handed or left-handed (a symptom of chirality). Magnetic helicity thus has a topological interpretation in terms of linkage of magnetic flux tubes[2] (Moffatt 1969).

Note that, with V defined as above, the result $\mathcal{H}_m$ = const. holds equally for compressible or incompressible flow. All that matters is that the $\mathbf{B}$-lines are frozen in the fluid, the property that guarantees that all knots and links of $\mathbf{B}$-lines (or better, of $\mathbf{B}$-tubes) are conserved. If the magnetic field is integrable, in the sense that all $\mathbf{B}$-lines lie on closed surfaces, then there is an invariant helicity integral over the volume inside each such surface, thus an infinite family of helicity invariants.

(a) Knotted flux tubes

If magnetic flux is confined to a single knotted flux tube K carrying flux Φ, then the helicity is given by

$$\mathcal{H}_m = \Phi^2(Wr + Tw), \tag{2.1}$$

where Wr and Tw represent writhe and twist respectively (Berger & Field 1984, Moffatt & Ricca 1992). The writhe depends only on the geometry of the axis C of the flux tube:

$$Wr = \frac{1}{4\pi}\oint_C\oint_C \frac{(\mathbf{x}-\mathbf{x}')\cdot\mathrm{d}\mathbf{x}\times\mathrm{d}\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{2.2}$$

Twist on the other hand involves the internal structure of the flux tube. We suppose here for simplicity that the $\mathbf{B}$-lines are closed curves that pass just once the long way round the tube. For a 'uniformly twisted tube', it is sufficient to consider a ribbon bounded by two $\mathbf{B}$-lines (one

[1] A result of this kind was first obtained by Woltjer (1958), although by a questionable argument, in that he adopted a gauge such that $\partial\mathbf{A}/\partial t = \mathbf{u}\times\mathbf{B}$ and required that $\partial\mathbf{A}/\partial t = \mathbf{0}$ on the fixed surface S of a 'closed system'; but, although $\mathbf{n}\cdot\mathbf{u} = \mathbf{n}\cdot\mathbf{B} = 0$ on such a surface S, $\partial\mathbf{A}/\partial t = \mathbf{u}\times\mathbf{B} \neq \mathbf{0}$ on S in general.

[2] It has been recently proved by Enciso & Peralta-Salas (2016) that any integral invariant of volume-preserving transformations within a particular class that they describe as 'regular' is equivalent to the helicity.

being the axis C), and a unit normal spanwise vector on the ribbon $\mathbf{N}(s)$, where s is an arc-length parameter on C. The twist is then defined (Fuller 1971) by

$$Tw = \frac{1}{2\pi}\oint_C (\mathbf{N}' \times \mathbf{N}) \cdot \mathbf{t}\, \mathrm{d}s, \tag{2.3}$$

where $\mathbf{N}' = \mathrm{d}\mathbf{N}/\mathrm{d}s$ and $\mathbf{t}$ is the unit tangent vector on C. This twist is the sum of contributions from the integrated torsion of the curve C and from the number of turns of $\mathbf{N}$ relative to the Frenet triad on this curve in one passage round it. Special care is needed if C is deformed by the transporting flow through an 'inflexional configuration', when these contributions take equal and opposite unit jumps (Moffatt & Ricca 1992). Both Wr and Tw vary continuously under continuous deformation of the tube, but the sum is invariant (Călugăreanu 1959) in accord with the result $\mathcal{H}_m = \text{const}$.

(b) Helicity: an invariant of the Euler equations

The fact that $\mathbf{B}$-lines are frozen in the fluid is sufficient to imply the existence of the family of magnetic helicity invariants. It follows therefore that a similar family of invariants must exist in any situation in which a vector field is transported with conservation of flux by a fluid flow. Since vortex lines are so transported by the flow of an ideal (inviscid) fluid, there must therefore in this situation exist a corresponding helicity invariant analogous to $\mathcal{H}_m$; this is evidently the (kinetic) helicity

$$\mathcal{H} = \int_V \mathbf{u} \cdot \boldsymbol{\omega}\, \mathrm{d}V, \tag{2.4}$$

where now the volume V must be bounded by a Lagrangian 'vorticity surface' on which $\boldsymbol{\omega} \cdot \mathbf{n} = 0$. Invariance of $\mathcal{H}$ for each such V is now a consequence of the invariance of knots and links of vortex tubes, a property famously recognised by Thomson (1867) in his ill-fated theory of 'vortex atoms'. The helicity invariant was discovered by Moreau (1961), and independently by Moffatt (1969) who exploited the above analogy. Helicity is invariant under precisely those conditions that ensure that vortex lines are frozen in the fluid, namely (i) the fluid is inviscid, (ii) the flow is either incompressible or barotropic (i.e. pressure p and density ρ are functionally related: $p = p(\rho)$), and (iii) any body forces acting on the fluid are irrotational.

We note further that, whereas the magnetic helicity invariant $\mathcal{H}_m$ is a consequence of a *linear* equation, namely the induction equation with an arbitrarily prescribed velocity field $\mathbf{u}(\mathbf{x}, t)$,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \tag{2.5}$$

the kinetic helicity invariant is associated with the *nonlinear* Euler equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}), \tag{2.6}$$

where, in an unbounded fluid, the velocity field $\mathbf{u}$ is given in terms of $\boldsymbol{\omega}$ by the Biot-Savart law

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{4\pi}\iiint \frac{\boldsymbol{\omega}(\mathbf{x}', t) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\mathrm{d}^3\mathbf{x}'. \tag{2.7}$$

This nonlinearity, so troublesome in other respects, is irrelevant as far as invariance of helicity is concerned.

(c) Cross helicity

The ideal MHD equations, including the curl of the Lorentz force $\mathbf{j} \times \mathbf{B}$ on the right-hand side of (2.6) (with current density $\mathbf{j} = \nabla \times \mathbf{B}$), have a further invariant, namely the cross helicity $\mathcal{H}_c$

defined by

$$\mathcal{H}_c = \int_V \mathbf{u}\cdot\mathbf{B}\,\mathrm{d}V, \tag{2.8}$$

where again V must be chosen to be the volume inside a magnetic surface on which $\mathbf{B}\cdot\mathbf{n}=0$. If $\mathbf{B}$ is confined to a single unknotted flux tube of vanishingly small cross-section centred on a closed curve C and carrying flux Φ , then this reduces to $\mathcal{H}_c = \Phi K$, where K is the flux of vorticity through C. Although the Lorentz force $\mathbf{j}\times\mathbf{B}$ is in general rotational, so that the flux of vorticity through an *arbitrary* Lagrangian closed curve is not conserved, it *is* so conserved when that curve is a $\mathbf{B}$-line, because obviously in this special case,

$$\oint_{\mathbf{B}-\mathrm{line}} \mathbf{j}\times\mathbf{B}\cdot\mathrm{d}\mathbf{x} = 0. \tag{2.9}$$

We note that all helicity invariants are pseudo-scalars that change sign under change from a right-handed to a left-handed frame of reference. They are non-zero only in chiral fields, i.e. fields that lack mirror symmetry. In this respect, although quadratic in the field variables, they are quite distinct in character from energy-type invariants, e.g. kinetic energy in the special case of non-conducting, inviscid, incompressible flow.

3. The elements of mean-field electrodynamics

(a) Magnetic energy equation

Consider now the induction equation in a conducting fluid with magnetic resistivity $\eta > 0$:

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}. \tag{3.1}$$

The central problem of dynamo theory is to understand how a magnetic field $\mathbf{B}$ can be generated and maintained by a velocity field $\mathbf{u}$ against the persistent erosion associated with this positive resistivity. If we consider an idealised situation in which $\mathbf{B}$ is a localised field in a fluid of infinite extent, then from (3.1) we may easily construct an energy equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \tfrac{1}{2}\mathbf{B}^2\,\mathrm{d}V = -\int \mathbf{u}\cdot(\mathbf{j}\times\mathbf{B})\,\mathrm{d}V - \eta\int \mathbf{j}^2\,\mathrm{d}V, \tag{3.2}$$

in which the erosive effect is clearly evident in the final term. The Lorentz force $\mathbf{j}\times\mathbf{B}$ tends to drive a flow $\mathbf{u}$ in the direction of $\mathbf{j}\times\mathbf{B}$, a key element of the process of magnetic relaxation (see §8 below). By contrast, in order to have dynamo action, we need a velocity field $\mathbf{u}$ which, at least on average, is *anti-parallel* to $\mathbf{j}\times\mathbf{B}$, thus working *against* the Lorentz force, and of sufficient magnitude to compensate for Joule dissipation. It is not difficult to construct a hypothetical force field which will drive such a velocity field; much more difficult to establish that such velocity fields can arise from realistic dynamical models, whatever the particular context may be.

(b) Mean electromotive force

In a turbulent medium, we may for simplicity assume that the turbulent velocity $\mathbf{u}$ is statistically steady and homogeneous, with zero mean ($\langle\mathbf{u}\rangle = 0$). The essence of the mean-field electrodynamics of Steenbeck et al. (1966) is then to assume a separation of scales $L \gg l_0$, where l_0 is the energy-containing scale of the turbulence, and L is the scale of potential growth of the field on which we focus. It is then natural to decompose $\mathbf{B}$ in the form

$$\mathbf{B}(\mathbf{x},t) = \langle\mathbf{B}(\mathbf{x},t)\rangle + \mathbf{b}(\mathbf{x},t), \quad \langle\mathbf{b}(\mathbf{x},t)\rangle = 0, \tag{3.3}$$

where now the angular brackets represent an average over some intermediate scale a satisfying $l_0 \ll a \ll L$. The average of (3.1) then gives

$$\frac{\partial\langle\mathbf{B}\rangle}{\partial t} = \nabla\times\boldsymbol{\mathcal{E}} + \eta\nabla^2\langle\mathbf{B}\rangle\,, \tag{3.4}$$

where $\boldsymbol{\mathcal{E}}$ is the mean electromotive force, defined by

$$\boldsymbol{\mathcal{E}} = \langle \mathbf{u} \times \mathbf{b} \rangle . \tag{3.5}$$

Subtracting (3.4) from (3.1) gives the 'fluctuation' equation

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \langle \mathbf{B} \rangle) + \nabla \times (\mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle) + \eta \nabla^2 \mathbf{b}. \tag{3.6}$$

For given $\mathbf{u}$ (i.e. from a purely kinematic point of view), and neglecting for the moment the possibility of unforced instabilities of (3.6), this equation establishes a linear relationship between $\mathbf{b}$ and $\langle \mathbf{B} \rangle$, and so also between $\boldsymbol{\mathcal{E}}$ and $\langle \mathbf{B} \rangle$, and, since these are both mean fields varying on the large scale L, the latter linear relationship must take the form

$$\mathcal{E}_i = \alpha_{ij} \langle \mathbf{B} \rangle_j + \beta_{ijk} \frac{\partial \langle \mathbf{B} \rangle_j}{\partial x_k} + \dots , \tag{3.7}$$

where the coefficient pseudo-tensors[3] $\alpha_{ij}, \beta_{ijk}, \dots$ are determined, at least in principle, solely by the statistical properties of the turbulence and the parameter η. Substitution back in (3.4) provides a linear equation for $\langle \mathbf{B} \rangle$ with constant coefficients, a dramatic simplification compared with the original induction equation (3.1). The series (3.7) is in effect a series in powers of $\epsilon = l_0/L$, and may be expected to converge rapidly when $\epsilon \ll 1$. The first term, involving the pseudo-tensor α_{ij} is then dominant.

(c) Dynamo action for a helical wave field

The relevance of helicity in the production of the mean electromotive force $\boldsymbol{\mathcal{E}}$ may be most simply illustrated by considering a single circularly polarised Fourier component of velocity:

$$\hat{\mathbf{u}} = u_0(\sin(kz - \omega t), \cos(kz - \omega t), 0), \tag{3.8}$$

with associated vorticity

$$\hat{\boldsymbol{\omega}} = \nabla \times \hat{\mathbf{u}} = k\hat{\mathbf{u}}. \tag{3.9}$$

This is a Beltrami field, and its helicity density, $\mathcal{H} = \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\omega}} = k\hat{\mathbf{u}}^{\mathbf{2}} = ku_0^2$, is uniform and maximal in relation to its energy density $\frac{1}{2}u_0^2$. If we consider a locally uniform mean magnetic field $(0, 0, B_0)$, then, from (3.6), the fluctuating field $\mathbf{b}$ is found as

$$\mathbf{b} = \frac{u_0 B_0 k}{\omega^2 + \eta^2 k^4} \Big(\eta k^2 \cos(kz - \omega t) - \omega \sin(kz - \omega t), -\eta k^2 \sin(kz - \omega t) - \omega \cos(kz - \omega t), 0 \Big) , \tag{3.10}$$

and it follows that

$$\boldsymbol{\mathcal{E}} \equiv \langle \mathbf{u} \times \mathbf{b} \rangle = (0, 0, \alpha B_0), \quad \text{where } \alpha = -\frac{\eta k^2}{\omega^2 + \eta^2 k^4} \mathcal{H}. \tag{3.11}$$

This single-component electromotive force is not sufficient to provide dynamo action; however we may immediately note that α (a pseudo-scalar) is proportional to the helicity $\mathcal{H}$, but with opposite sign. Note further that, if $\omega = 0$, then $\alpha = -(1/\eta k^2)\mathcal{H}$, increasing without limit as $\eta \to 0$; this is because the flow (steady when $\omega = 0$) generates a perturbation field $\mathbf{b} = O(\mathrm{R_m})B_0$, much greater than the mean field B_0 when $\mathrm{R_m} = u_0/\eta k \gg 1$, (just as when a toroidal field is generated from a poloidal field by differential rotation).

The situation is very different for a superposition of three modes of the form (3.8); for $\omega = 0$, this is the 'ABC-flow' [4]

$$\mathbf{u} = (C \sin kz + B \cos ky, A \sin kx + C \cos kz, B \sin ky + A \cos kx), \tag{3.12}$$

a flow whose structure and dynamo properties have been intensively studied (see, for example Hénon 1966, Dombre et al. 1986, Gilbert 1992, Jones & Gilbert 2013, Bouya & Dormy 2013). Here,

[3]That these coefficients are pseudo-tensors is evident from the fact that $\boldsymbol{\mathcal{E}}$, like the current density $\mathbf{j} = \nabla \times \mathbf{B}$, is a polar (i.e. pure) vector, while $\mathbf{B}$ (like vorticity) is an axial (i.e. pseudo-)vector.

[4]Here, I follow Childress & Gilbert (1995) in using the cyclic order $\{A, B, C\}$ in correspondence with $\{x, y, z\}$, rather than the unconventional ordering $\{B, C, A\}$ adopted by earlier authors, e.g. Hénon (1966) and Dombre et al. (1986).

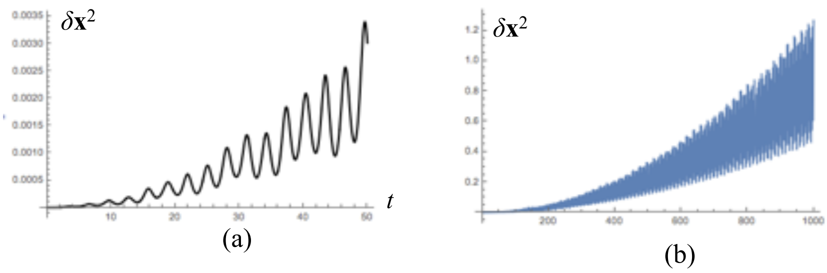

Figure 1. Separation (squared) of fluid particles starting at a typical position in the chaotic region for the flow (3.12), with $A=(2/3)^{1/2}, B=(1/3)^{1/2}, C=1;\ \delta\mathbf{x}(t)=\mathbf{x_1}(t)-\mathbf{x_2}(t)$ with $\mathbf{x}_1(0)=(0.6,0.5,0), \mathbf{x}_2(0)=(0.601,0.5,0)$. (a) Early development of separation up to $t=50$; (b) separation up to time $t=1000$. The growth is exponential in the mean.

the problem is that the 'awkward' term $\nabla\times(\mathbf{u}\times\mathbf{b}-\langle\mathbf{u}\times\mathbf{b}\rangle)$ in (3.6) is non-zero, so calculation of the fluctuating field $\mathbf{b}$ is difficult, the more so as $\mathrm{R_m}$ increases. The difficulty is associated with the existence of chaotic islands within the flow field, within which particle trajectories diverge exponentially in the mean. Such behaviour is illustrated in Fig. 1, which shows the time-evolution of the separation of two particles initially very close together in the chaotic regime; this is for the particular flow studied by Hénon 1966 and Dombre et al. 1986 for which $A=(2/3)^{1/2}, B=(1/3)^{1/2}, C=1$.

However, if $\mathrm{R_m}\ll 1$, the term $\nabla\times(\mathbf{u}\times\mathbf{b}-\langle\mathbf{u}\times\mathbf{b}\rangle)$ in (3.6) is negligible, and an isotropic 'alpha-effect' is generated:

$$\boldsymbol{\mathcal{E}}\equiv\langle\mathbf{u}\times\mathbf{b}\rangle=\alpha\,\langle\mathbf{B}\rangle\quad\text{where now } \alpha=-\frac{1}{3\eta k^2}\mathcal{H}, \tag{3.13}$$

and where $\mathcal{H}$ is the *mean* helicity, the helicity density being now periodic in x, y and z. With this expression for $\boldsymbol{\mathcal{E}}$, the mean field equation (3.4) may be written

$$\frac{\partial\,\langle\mathbf{B}\rangle}{\partial t}=\alpha\,\nabla\times\langle\mathbf{B}\rangle-\eta\,\nabla\times(\nabla\times\langle\mathbf{B}\rangle). \tag{3.14}$$

We now note that any 'force-free' Beltrami field with structure satisfying $\nabla\times\langle\mathbf{B}\rangle=K\,\langle\mathbf{B}\rangle$ will behave exponentially:

$$\langle\mathbf{B}\rangle\sim e^{pt}\quad\text{where } p=\alpha K-\eta K^2. \tag{3.15}$$

This shows exponential dynamo growth provided K is chosen to have the same sign as α, and $\alpha K>\eta K^2$, i.e. provided the scale $|K|^{-1}$ is sufficiently large. Here, it is clearly the non-zero mean helicity of the ABC-flow that is responsible for this large-scale dynamo action. We note that for this type of force-free mode,

$$\langle\mathbf{B}\rangle\cdot\nabla\times\langle\mathbf{B}\rangle\,/\,\langle\mathbf{B}\rangle^2=K, \tag{3.16}$$

and that the growth rate is maximal for $K=\alpha/2\eta$.

If A, B and C are unequal and nonzero, and $\mathrm{R_m}\ll 1$, the resulting α_{ij} is non-isotropic, but still symmetric, and dynamo action still occurs on a sufficiently large length-scale. This is the situation that, when coupled to the effect of differential rotation, is relevant to conditions in the liquid core of the Earth (see §6 below).

The ABC-flow is of course highly idealised. However, the result (3.11) carries over to isotropic (but nevertheless chiral) turbulence, by just replacing the expression in (3.11) for α by the more general form

$$\alpha = -\frac{1}{3\eta}\int \frac{1}{k^2}\hat{\mathcal{H}}(k)\,\mathrm{d}k, \tag{3.17}$$

where $\hat{\mathcal{H}}(k)$ is the 'helicity spectrum' of the turbulence (Moffatt 1970).

4. Large $\mathrm{R_m}$ and the fast dynamo

If we go to the opposite limit $\mathrm{R_m} \gg 1$ that appears to be most relevant in astrophysical contexts, a number of difficulties arise. First, there may exist instabilities of the equation (3.6) even if $\langle \mathbf{B}\rangle = 0$. This is because a fluctuating field $\mathbf{b}$ on the same scale as $\mathbf{u}$ may grow due to the exponential growth of material line elements that is characteristic of any turbulent flow, and the associated stretching of the $\mathbf{b}$-lines of a transported magnetic field. This undoubtedly leads to exponential growth of magnetic energy when $\eta = 0$ ($\mathrm{R_m} = \infty$). However the situation is complicated by the fact that there is an associated persistent decrease of scale of the magnetic field. When $\eta \neq 0$, however small, this leads ultimately to a rate of Joule dissipation that may erode the stretching effect. This is analogous to the rate of viscous dissipation of kinetic energy that remains finite in turbulent flow in the limit $\nu \to 0$. The stretching effect and the erosive effect are in competition, and it is by no means obvious which wins. This basic turbulent dynamo problem was clearly enunciated by Batchelor (1950), who conjectured, on the basis of the analogy between magnetic field and vorticity, that stretching would win if and only if the magnetic Prandtl number $\mathrm{P_m} > 1$. Batchelor's analogy is known to be flawed, in that modes of growth are available to the magnetic field that are not available to vorticity (which is of course constrained to be the curl of the convecting velocity), but it nevertheless holds some appeal. Direct numerical simulations (DNS) of MHD turbulence show exponential growth of magnetic energy from very low initial levels even when $\mathrm{P_m} < 1$ (see for example Brandenburg & Subramanian 2005). This may be because modes of growth induced by the 'stretch-twist-fold' process of Vainshtein & Zel'dovich (1972) (or some similar process) may intensify field without obvious decrease of scale; but a rigorous theoretical demonstration of this growth process, sustained for all time, for any value of $\mathrm{P_m}$ is still lacking.

The second difficulty for a fast dynamo (i.e. one for which the growth rate is independent of η in the limit $\eta \to 0$) arises from the invariance of magnetic helicity $\mathcal{H}_m$ in this limit, as least for fields that are twice differentiable. In an elementary search for dynamo action under the effects of a steady velocity field $\mathbf{u}(\mathbf{x})$, it is customary to seek solutions of the induction equation (3.1) (supplemented by appropriate boundary conditions) of eigenfunction form $\mathbf{B} \sim \hat{\mathbf{B}}(\mathbf{x})e^{pt}$ with $p \neq 0$; but this would imply that $\mathcal{H}_m \sim e^{2pt}$, in obvious conflict with conservation of $\mathcal{H}_m$, unless it happens that $\mathcal{H}_m = 0$, and this for all subdomains bounded by magnetic surfaces. Actually, as shown by Moffatt & Proctor (1985), even in this exceptional case exponential growth is impossible unless the field $\hat{\mathbf{B}}$ is non-differentiable or zero nearly everywhere. In practice, this means that any eigenfunction solution of (3.1) must become increasingly sparse as $\eta \to 0$, being in effect concentrated in filaments and/or sheets within which the transverse gradient of the field is $\mathrm{O}(\mathrm{R_m^{1/2}})$ as $\mathrm{R_m} \to \infty$. This prediction is indeed borne out by numerical computation at large $\mathrm{R_m}$.

The reason for this behaviour may be partially understood in terms of a 'magnetic expulsion' effect. It is well known that a magnetic field that crosses through any region of closed streamlines is 'wound up' by the differential rotation within the region leading to expulsion of the field on a time-scale $\mathrm{O}(\mathrm{R_m^{1/3}})$ (Weiss 1966, Moffatt & Kamkar 1983). The behaviour is most easily visualised for a steady two-dimensional velocity field with space-periodic stream function, e.g. $\psi \sim \sin kx \sin ky$; a locally uniform magnetic field is expelled from each region of closed streamlines, forming layers of thickness $\mathrm{O}(\mathrm{R_m^{-1/2}})$ on the streamlines lines $\psi = 0$, i.e. on the square network $kx = n\pi, ky = m\pi$ for $m, n = 0, \pm 1, \pm 2, \dots$. This is consistent with the considerations of the previous paragraph: the field in the limit $\mathrm{R_m} \to 0$ is singular, being confined in this case to these layers of vanishing thickness. Similar behaviour has been found for the ABC flow: for example, in the case $A = B = C$, Jones & Gilbert (2013) have exploited symmetries to determine dynamo growth rates up to $\mathrm{R_m} = 10^4$ for modes having the same periodicity as the flow; even at $\mathrm{R_m} = 10^2$, the growing dynamo mode shows a characteristic tube-like structure, the flux-tubes being concentrated around separatrices (heteroclinic orbits) connecting stagnation points of the flow. It is to be expected that these flux tubes will become more and more concentrated with increase of $\mathrm{R_m}$, being singular in the limit $\mathrm{R_m} = \infty$. (Of course, a field $\mathbf{B}$ everywhere parallel to $\mathbf{u}$ is a steady solution of (3.1) when $\eta = 0$, but this has little relevance to the dynamo process that requires an exponentially growing field.)

Despite these difficulties, much effort has been devoted to the fast dynamo problem in terms of iterated mappings that replicate operations like the 'stretch-twist-fold' process that can double magnetic field intensity, albeit with modest change of structure at each stage of the iteration (see Childress & Gilbert 1995 for a systematic account of results obtained in this way).

5. Back-reaction of the Lorentz force

Whenever a large-scale magnetic field grows through dynamo action, the Lorentz force ultimately reacts back upon the turbulent flow that is responsible for this, in such a way as to establish statistical equilibrium. It is usually a combination of mean flow and turbulence that drives the dynamo, and the Lorentz force may influence either or both of these. As far as the turbulence is concerned, the obvious saturation mechanism is through suppression of helicity and of the associated α-effect, a mechanism known as 'α-quenching'. This effect may be analysed in terms of the simple model of §3.3 above. If we suppose that the flow is driven by a constant force

$$\mathbf{f}(\mathbf{x}) = f_0(\sin kz, \cos kz, 0) \tag{5.1}$$

and controlled in amplitude by kinematic viscosity ν, then with

$$\mathbf{u} = \hat{u}(t)(\sin kz, \cos kz, 0), \quad \mathbf{b} = \hat{b}(t)(\cos kz, -\sin kz, 0), \quad \mathbf{j} = \nabla \times \mathbf{b} = k\mathbf{b}, \tag{5.2}$$

the Lorentz force is $\mathbf{j} \times (\mathbf{b} + \mathbf{B}_0) = \mathbf{j} \times \mathbf{B}_0$. Inclusion of this Lorentz force in the model leads, after decay of transients, to a mean emf $\boldsymbol{\mathcal{E}} = < \mathbf{u} \times \mathbf{b} > = \alpha \mathbf{B}_0$, where now

$$\alpha = \frac{\alpha_0}{\left(1 + B_0^2/\eta\nu k^2\right)^2}\,, \tag{5.3}$$

where $\alpha_0 = -(k^5\eta\nu^2)^{-1} f_0^2$ is the value that pertains when the Lorentz force is neglected. The result (5.3) is easily obtained by solving the (linear) equations for $\hat{u}$ and $\hat{b}$ in terms of f_0 when steady conditions are established.

As in §3(c), this type of result can be extended with confidence to three-dimensional ABC flows only when $\mathrm{R_m} = \hat{u}/\eta k \ll 1$. In this situation, the mean field (here B_0) will grow by dynamo action until $B_0^2 = \mathrm{O}(\eta\nu k^2)$, at which stage suppression of the α-effect becomes significant. This level of magnetic energy density may well be large compared with the level of kinetic energy density: there is no question of 'equipartition of energy' in a dissipative system of this kind.

The nonlinear regime of the ABC dynamo has been investigated by a number of authors. The paper of Podvigina (2003) is of particular interest. By numerical experimentation, Podvigina found a sequence of bifurcations for $\mathrm{R_m}$ in the range $14 \lesssim \mathrm{R_m} \lesssim 60$, to a variety of states, either steady or periodic or chaotic. This work revealed an extraordinary richness of behaviour,

The situation is less clear when, as in astrophysical contexts, $\mathrm{R_m} = \hat{u}/\eta k \gg 1$. Here there is no guarantee that the result (3.13) may be extended to the three-dimensional ABC situation, or to isotropic turbulence, because of the non-uniformity of $\mathbf{u} \times \mathbf{b}$; nevertheless, it does suggest that the α-quenching effect may become much stronger as $\eta \to 0$, i.e. as $\mathrm{R_m} \to \infty$. Such a result has been obtained by direct numerical simulation by Cattaneo & Hughes (1996), who find behaviour compatible with quenching of the form

$$\alpha = \frac{\alpha_0}{1 + \mathrm{R_m} B_0^2}\,, \tag{5.4}$$

Again, this indicates dramatic quenching when $\mathrm{R_m} = \hat{u}/\eta k \gg 1$, and a correspondingly low level of mean magnetic field that can be attained. This poses an acute problem for mean-field theory in astrophysical contexts, perhaps only to be resolved by adopting an eddy-diffusivity η_e of order $u_0 l_0$ in place of the molecular diffusivity η, the 'turbulent magnetic Reynolds number' $u_0 l_0/\eta_e$ being then always of order unity. The need for an adequate theory of the small-scale turbulence that gives rise to such an eddy diffusivity (and may also contribute to the α-effect) then remains as a problem of key importance.

Of course, in order to achieve equilibrium, energy must be supplied to the turbulence in some way, e.g. through gravitation and thermal convection modified by Coriolis forces. In such a situation, the ultimate dynamic balance is one in which Lorentz forces achieve statistical equilibrium with buoyancy and Coriolis forces, while inertial and viscous forces play a subsidiary role. Such a force balance may be described as 'magnetostrophic'. Much computational effort has been devoted to this kind of statistical equilibrium (see, for example, Brandenburg & Subramanian 2005).

6. Magnetostrophic turbulence

The situation is most easily illustrated in the context of the Earth's liquid conducting outer core. Here, slow growth of the inner solid core leads to the release of buoyant blobs and plumes from a 'mushy zone' at the inner core boundary; these buoyant structures rise through the liquid core generating convective flow that is strongly influenced both by Coriolis effects due to the Earth's rotation and Lorentz-force effects due to the magnetic field that is generated by dynamo action. This dynamo action in the case of the Earth involves both differential rotation (the mean-flow effect) and an anisotropic α-effect associated with a state of magnetostrophic turbulence. This type of turbulence is very different from conventional 'Kolmogorovian' turbulence, in that the dominant nonlinearity is due to convection of buoyant elements rather than to fluid inertia. Nevertheless, a cascade of energy is to be expected, as argued by Moffatt (2008); see also Moffatt & Loper (1994).

The depth of the liquid core is about 2300 km, but the rising elements that erupt from a mushy zone at the inner core boundary and contribute to the turbulence in the core are evidently on a much smaller scale. If this scale is of the order of 100 km or less, as seems quite likely, then the magnetic Reynolds number based on this scale is at most of order unity, and the 'low-$\mathrm{R_m}$' approximation is available at least as a first approximation to the small-scale dynamics. Moreover, this scale is still large enough for the Coriolis force to dominate over other contributions (except pressure gradient and Lorentz force) in the equation of motion. The governing equations then take the form

$$2\boldsymbol{\Omega}\times\mathbf{u} = -\nabla P + \rho^{-1}(\mathbf{B_0}\cdot\nabla)\mathbf{b} - \theta\,\mathbf{g}, \quad \nabla\cdot\mathbf{u}=0, \tag{6.1}$$

$$\frac{\partial\mathbf{b}}{\partial t} = (\mathbf{B_0}\cdot\nabla)\mathbf{u} + \eta\nabla^2\mathbf{b}, \quad \nabla\cdot\mathbf{b}=0, \tag{6.2}$$

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = S + \kappa\nabla^2\theta, \tag{6.3}$$

where ρ is now mean density, θ is buoyancy (of either thermal or compositional origin), and S represents the source of buoyancy in the mushy zone, or possibly distributed throughout the liquid core. The term $\partial\mathbf{b}/\partial t$ is retained (although $\nabla\times(\mathbf{u}\times\mathbf{b})$ is discarded), because it turns out (Moffatt 2008) that the time-lag associated with this term is needed to account for the helicity that is responsible for dynamo action. Shearing of the buoyant structures by the differential rotation in the core could conceivably have a similar effect.

The important thing to note here however is that the only surviving nonlinearity of this model is the term $\mathbf{u}\cdot\nabla\theta$ in (6.3). Here $\mathbf{u}$ is a linear functional of θ, $\mathbf{u}=\mathbf{u}\{\theta\}$, which can be obtained by solution of the linear equations (6.1) and (6.2), treating $-\theta\,\mathbf{g}$ as a 'source' term. Thus, eqn. (6.3) must be considered, not as the usual equation for a passive scalar field, but here as an equation for a very active scalar field $\theta(\mathbf{x},t)$. We do not attempt to analyse these equations further here, but merely note that the 'magnetostrophic turbulence' that they describe is clearly very different from the usual Kolmogorovian turbulence, for which the nonlinear inertia term $\mathbf{u}\cdot\nabla\mathbf{u}$ (here discarded) is responsible for the cascade of energy to small scales. The system (6.1)-(6.3) is simpler in one important respect: global regularity (in time) of solutions of such equations has been established by Friedlander & Vicol (2011). No such controlling theorem is as yet available for the Navier-Stokes equations.

7. The VKS experiment

A number of attempts have been made to replicate the geodynamo under controlled laboratory conditions with liquid sodium as the working fluid. Two pioneering experiments deserve particular mention: first, the Riga dynamo experiment (Gailitis et al. 2000, 2001), the result of a long period of prior development based on the theoretical model of Ponomarenko (1973); here, a helical flow is driven through a cylindrical channel, returning through a surrounding cylindrical annulus. Second, the Karlsruhe dynamo experiment (Müller et al. 2004), designed by Busse et al. (1996) on the basis of the space-periodic dynamo of G.O.Roberts (1970, 1972); here 52 parallel cylindrical flow units, each structured like that of the Riga facility, were set in periodic array, with helical flow driven alternately up and down through adjacent units. In both these experiments, the flow geometry was deliberately contrived in order to maximise mean helicity and thereby maximise the chance of dynamo action; and indeed, in both experiments, dynamo amplification of a very weak applied magnetic field was detected.

A less contrived geometry was adopted in the so-called Von Karman Sodium (VKS) experiment carried out at the CEA centre in Cadarache, France, (Monchaux et al. 2009). Here a helical mean flow is driven by two counter-rotating propellors in a large cylindrical container filled with liquid sodium. The flow is vigorously turbulent, and the turbulence presumably inherits mean helicity through instability of the mean flow; magnetic Reynolds numbers up to 45 (based on the propellor rotation rate) were achieved, and dynamo action was observed for $\mathrm{R_m} \gtrsim 30$. As argued by Pétrélis et al. (2007), the dynamo appears to be of '$\alpha\omega$' type, i.e. toroidal field is generated from poloidal field by the differential rotation induced by the propellors, and poloidal field is generated from the toroidal field by a turbulent α-effect, this dual action providing a regenerative feedback cycle. Polarity reversals, mimicking those of the Earth's field, are also observed, but only when asymmetry is deliberately introduced by rotating the propellors at different speeds.

However, there is a significant problem with this experiment, in that dynamo action has been achieved only when the propellors are made from ferromagnetic (soft iron) material, which is susceptible to magnetisation by the dynamo-generated magnetic field. If copper propellors are used, no dynamo action has been observed within the range of $\mathrm{R_m}$ so far achieved. It is perhaps not surprising that soft iron should aid the dynamo process, if only through augmenting the magnetic field near the propellors where the differential rotation that generates toroidal field is operative. There is no external source of magnetic field in the experiment, so it is fair to describe this as a genuine dynamo driven purely by an input of mechanical energy, and one for which the mean helicity of the turbulent flow is crucial.

Measurement of the flow properties in the interior of the liquid sodium in this experiment poses great difficulties and has not been attempted. However, such measurements were made in earlier experiments using water as the working fluid (Dernoncourt et al. 1998; Fauve et al. 1993). These experiments confirmed the turbulent character of the flow, and identified the presence of intermittent concentrated vortex filaments, a feature detected somewhat earlier in a different turbulent context by Douady et al. (1991).

8. Magnetic relaxation

When a strong magnetic field is generated by dynamo action, there is a natural tendency for it to seek a minimum energy state compatible with its topology. This may be viewed as an aspect of equilibration; in a sense, magnetic relaxation is the counterpart of dynamo action, of comparable importance in the 'swings and roundabouts' of the cosmic dynamo. This dual process is present also in thermonuclear fusion devices like the tokamak: here, a helical magnetic field is maintained in the plasma region by external currents, but plasma instabilities may give rise to an α-effect, while at the same time the magnetic field seeks a quasi-static force-free equilibrium. This is the sort of consideration that underlies the conjecture of Taylor (1974) in regard to the 'reversed field pinch', namely, that the magnetic field relaxes to a minimum energy state subject to the single constraint that the global magnetic helicity be conserved. This assumption, applied to a plasma confined to a cylinder, leads to Bessel function solutions, which exhibit axial field reversal near the boundary. The conjecture has great appeal because of its simplicity and has attracted much attention, although convincing justification is elusive.

Taylor's conjecture is relevant to a very low density plasma in which magnetic pressure p_M dominates over fluid pressure p ($\beta \equiv p_M/p \gg 1$). If β is not small, then relaxation to a magnetostatic equilibrium

$$\mathbf{j} \times \mathbf{B} = \nabla p, \tag{8.1}$$

is to be expected. Such an 'equilibrium' will generally evolve slowly under the effect of weak magnetic diffusivity $\eta \neq 0$; for it to remain in quasi-static equilibrium, there must be an alternative dissipative mechanism through which energy (magnetic plus internal) is minimised subject to conserved field topology. As soon as (8.1) is violated, the Lorentz force drives fluid motion whose kinetic energy is dissipated by viscosity $\mu = \rho\nu$ (and also possibly indirectly by thermal conduction); through this mechanism, magnetic energy is converted to kinetic energy and immediately dissipated.

Here we note that, as first recognised by James Clerk Maxwell, the dynamic viscosity μ of a tenuous gas is effectively independent of density ρ in the limit as $\rho \to 0$. This is essentially because, although the molecular collision frequency decreases, this is compensated by increase of the mean-free path between collisions, so that net transport of momentum due to molecular collisions is little affected. It follows that kinematic viscosity $\nu \equiv \mu/\rho$ increases without limit as $\rho \to 0$, and that the magnetic Prandtl number ν/η is therefore large in a sufficiently low density (i.e. high-β) plasma. Relaxation can therefore occur on a much shorter time-scale than that of the slow evolution due to magnetic diffusion alone.

(a) Relaxation of a single-component field

The resulting behaviour is well illustrated by the simple one-dimensional model developed by Bajer & Moffatt (2013). Here, an initial unidirectional magnetic field periodic in x, $\mathbf{B}_0 = (0, 0, B_0(x))$, is allowed to relax in an extremely tenuous plasma. The relevant dimensionless parameter is $S = \langle |B_0(x)| \rangle\, L/\sqrt{\mu\eta}$, where L is the scale of periodicity of $B_0(x)$, and the angular brackets represent a space average; S is assumed large, by virtue of the small diffusivity η. A one-dimensional flow $\mathbf{u} = (u(x,t), 0, 0)$ is driven by the gradient of non-uniform magnetic pressure $p_M(x,t) = \frac{1}{2}\mathbf{B}^2$, and the kinetic energy of this flow is dissipated by viscosity. Field lines are initially nearly frozen in the plasma, and the field relaxes until the magnetic pressure is nearly uniform. In approaching this force-free state, current sheets form at null points of the magnetic field, and it is here that subsequent slow diffusive evolution occurs. The plasma density builds up at the null points, and continues to grow in spectacular manner during the diffusive stage (see Fig. 2).

(b) Relaxation of a two-component field

The model has been extended to the very different situation of a two-component initial field $\mathbf{B}_0 = (0, b_{y0}(x), b_{z0}(x))$, which generally has nonzero helicity density $\mathcal{H}_0(x)$ (Moffatt 2015). Here the peculiar behaviour associated with null points does not in general arise, but the field still relaxes via the viscous mechanism to a state of nearly uniform magnetic pressure. The evolution corresponding to a particular initial condition for which the field energy is peaked near boundaries $x = \pm\pi$, assumed perfectly conducting, is shown in Fig. 3. The total flux of each component is conserved; here, we have chosen axes so that the net flux is in the z-direction.

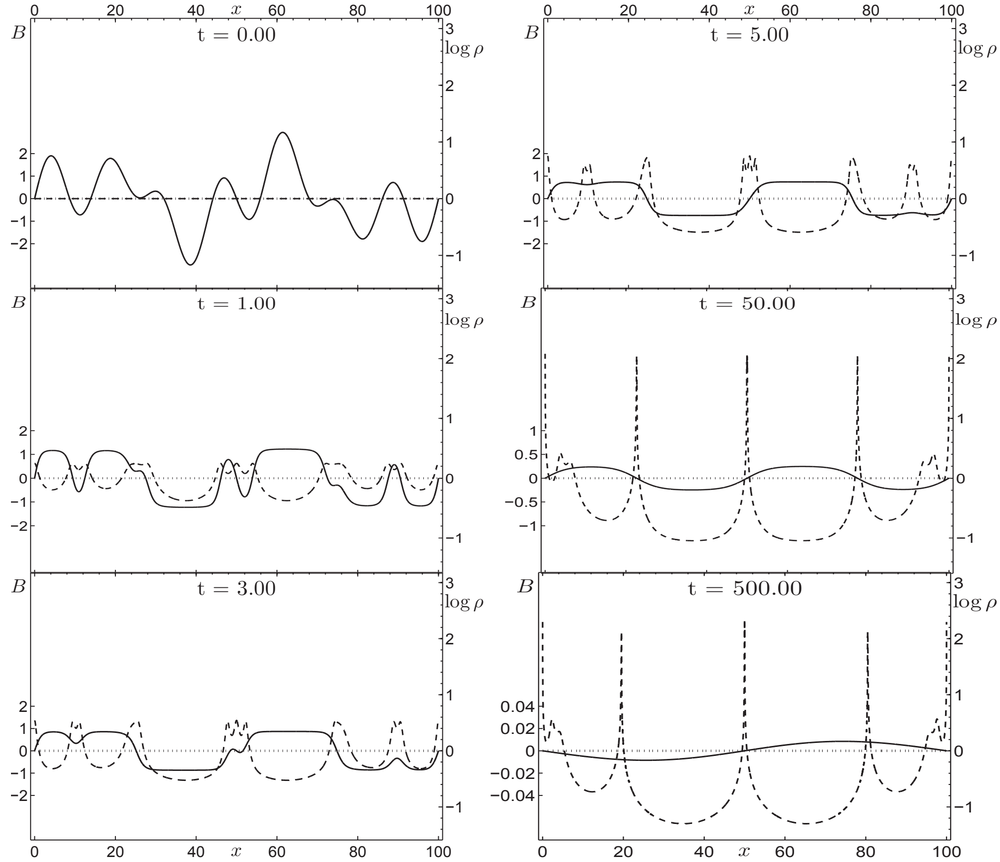

Figure 2. Coupled evolution of magnetic field B (solid) and log of density $\log_{10}\rho$ (dashed) from an initial condition for which B has 10 null points and ρ is uniform. Note the changes of scale of B at $t=50$ and $t=500$. The evolution is weakly diffusive on a short relaxation time-scale on the left, strongly diffusive on a long time-scale on the right. For $0<t\lesssim 1$, plateaus of B develop between the nulls. For $t\gtrsim 1$, the current sheets at the nulls slowly decay, but continuing inflow to the nulls where p_M remains minimal leads to dramatic concentrations of density in their neighbourhoods. (Adapted from Bajer & Moffatt 2013).

Panel (b) shows the rapid relaxation of magnetic energy density $M(x,t)$ to a nearly uniform state. Panels (c) and (d) show the surface swept out by a $\mathbf{B}$-line as x increases through its range, at $t=0$ and $t=5$ respectively, suggesting quite a complex evolution process. Panels (e) and (f) show the coupled evolution of density $\rho(x,t)$ and helicity density $\mathcal{H}(x,t)$, ($\mathcal{H}(x,t)/\rho(x,t)$ is actually a Lagrangian invariant when $\eta=0$); panel (g) shows the initial rapid relaxation of magnetic energy and the near invariance of helicity during the early relaxation phase. Although total helicity is effectively constant, there is no tendency here for helicity density to become uniform, as would be a consequence of Taylor's conjecture if applicable.

Here however it must be noted that instabilities of tearing-mode type may occur in regions where the field gradient is large (Furth et al. 1963), and such instabilities may well inherit helicity from the applied magnetic field. In this situation, an α-effect will be generated and this will affect the mean-field relaxation process.

Note further that if the field $\mathbf{B}_0(x)$ is localised (rather than periodic) and unconstrained, then in the low-β limit it will expand without limit in the x-direction. When considering the situation

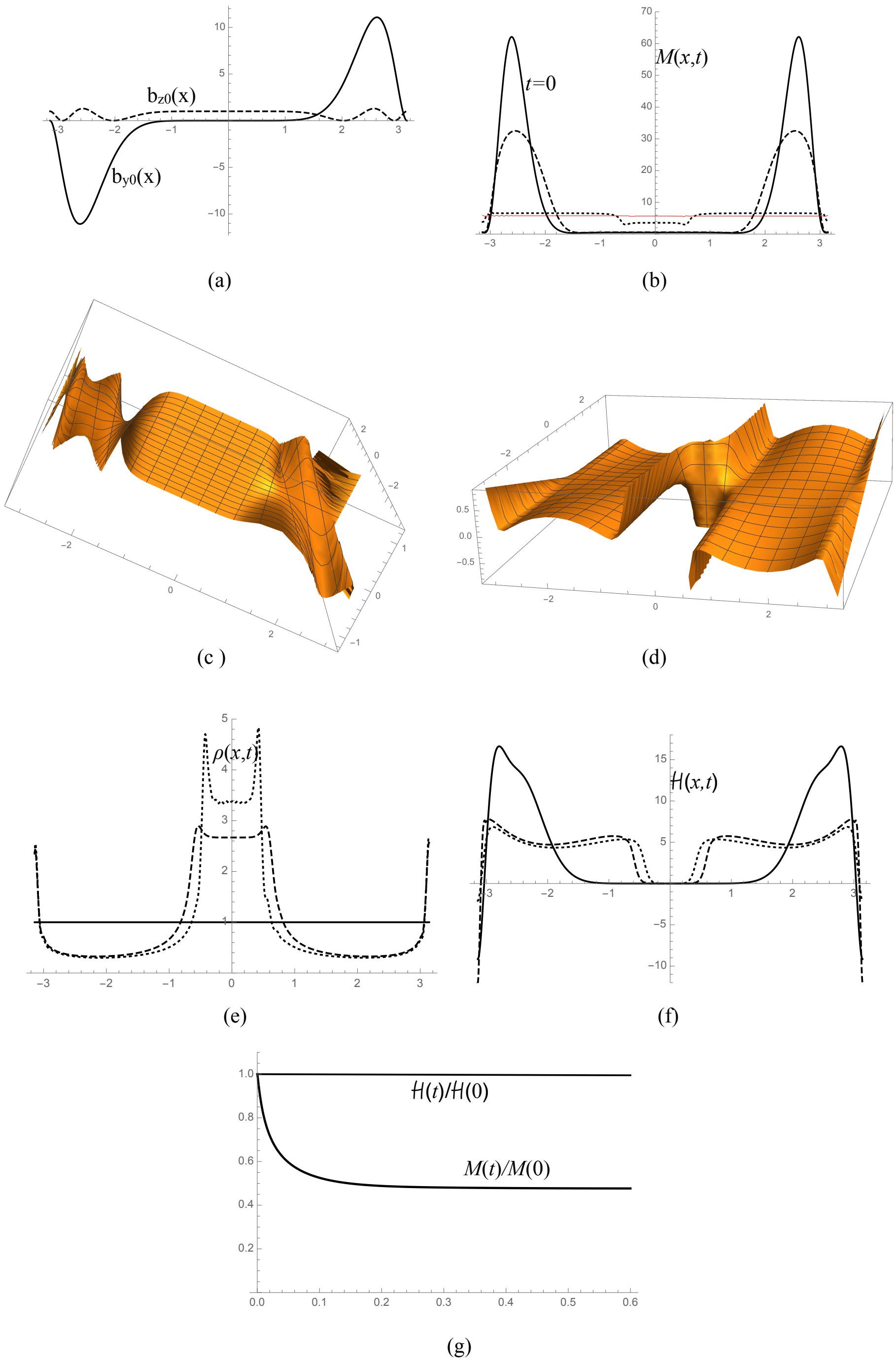

Figure 3. (a) Initial field components b_{y0} (solid) and b_{z0} (dashed); (b) associated magnetic pressure distribution at non-dimensionalised times $t=0$ (solid), 0.01 (dashed), 0.2 (dotted) and 0.6 (red/faint); (c) ruled surface containing $\mathbf{B}$-lines at $t=0$; (d) same at $t=5$; (e) density $\rho(x,t)$ and (f) helicity density $\mathcal{H}(x,t)$ at times $t=0$ (solid), 0.2 (dashed) and 3 (dotted); (g) early evolution of mean magnetic energy $M(t)$ and mean helicity $\mathcal{H}(t)$, relative to their initial values. (Adapted from Moffatt 2015).

on a galactic length-scale, it would be interesting to incorporate self-gravitation in the model, as a natural means of containing this expansion.

9. Conclusions

In this paper, we have sought to identify some key issues in dynamo theory that, despite the great advances that have been made over the last decades, still present a serious challenge for theoreticians. First, the question first raised by Batchelor (1950): in the large R_m situation, in turbulence that is assumed to be homogeneous and stationary, can it be proved (rather than just computationally demonstrated) that magnetic energy grows permanently rather than just in transient manner, and if so, then for what range of magnetic Prandtl number? The manner in which intensification by 'stretching' wins over accelerated dissipation through decrease of scale then requires elucidation.

Second, with attention focussed on the generation of large-scale field, when dynamo action occurs through an α-effect, the saturation mechanism involving 'α-quenching' calls for similar detailed elucidation. This situation is reasonably well understood when $R_m \ll 1$, but not in the astrophysically relevant situation when $R_m \gg 1$. Here, the influence of 'sub-grid-scale' turbulence in providing an effective eddy diffusivity for the magnetic field is presumably crucial.

Third, the nature of magnetostrophic turbulence, for which the dominant (and ideally only) nonlinearity is that associated with convection of buoyant elements, calls for more detailed analysis. Here velocity and magnetic perturbations are linearly related to the buoyancy field, so the problem is essentially one involving advection and diffusion of an active scalar field, for which upward buoyancy flux provides the energy input. Progress in the analysis of this class of problem may be anticipated.

Finally, we have discussed the problem of field relaxation in a highly tenuous medium, like that of the interstellar medium, or of a contained plasma in a thermonuclear device. Here, relaxation is the counterpart of dynamo action, and is the means by which some form of statistical equilibrium is attained. The one-dimensional models described in this paper are perhaps indicative, but the obvious need is to extend such theoretical models to two- and three-dimensional situations, in order to better understand the various numerical simulations that have been reported.

Data accessibility statement The work does not have any experimental data.

Competing interests I have no competing interests.

Authors' contributions This is a single-author paper.

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