Chiral Transfer of Angular Momentum

H. K. Moffatt[1] and V. A. Vladimirov[2,3,4]
[1]*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK*
[2]*Sultan Qaboos University, Oman,* [3]*University of York, UK,* [4]*University of Leeds, UK*

(Dated: September 3, 2019)

Suppose that viscous fluid is contained in the space between a fixed sphere S_2 and an interior sphere S_1 which moves with time-periodic velocity $\mathbf{U}(t)$ and angular velocity $\mathbf{\Omega}(t)$, with $\langle \mathbf{U}(t)\rangle = \langle \mathbf{\Omega}(t)\rangle = 0$. It is shown that, provided this motion is chiral in character, it can drive a flow that exerts a non-zero torque on S_2. Thus angular momentum can be transferred through this mechanism. In the Appendix, it is shown why lubrication theory does not apply to this problem, even in the limit when the spheres make instantaneous contact.

I. INTRODUCTION

Consider the following simple problem: suppose that a sphere S_1 of radius r_1 is contained inside a fixed sphere S_2 of radius $r_2 > r_1$, the space between being filled with viscous fluid. Suppose that S_1 is moved with velocity $\mathbf{U}(t)$ and angular velocity $\mathbf{\Omega}(t)$, both time-periodic with zero time average $\langle \mathbf{U}(t)\rangle = \langle \mathbf{\Omega}(t)\rangle = 0$, as in the sketch of Fig. 1(a). Is it possible that such a motion can generate a mean torque on the fixed sphere S_2? We shall show by explicit example that the answer is positive, the effect arising only if the flow between the spheres has the property of chirality (lack of reflection symmetry) and provided $\mathbf{U}(t)$ and $\mathbf{\Omega}(t)$ are not in phase; in this respect it is somewhat analogous to the helicity effect that is responsible for the self-excitation of magnetic field in a conducting fluid. We emphasise however that it is chirality in its general sense, rather than the more limited property of helicity, that is relevant in our present context.

The question naturally arises as to how such a system could be realised in practice. The motion of the sphere S_1 must be driven by an appropriate force distribution, which we assume to be provided by some battery system internal to S_1; [the intriguing toy known as the 'perpetual top' <https://www.arborsci.com/products/perpetual-top>, powered by an off-centre motor driven by a battery, provides an example of how such an internal force distribution, effectively a torque, may be produced]. Anticipating that for our problem a mean torque $\langle \mathbf{G}\rangle$ is transmitted to the fixed sphere S_2, an equal and opposite mean torque $-\langle \mathbf{G}\rangle$ must be experienced by S_1; this in turn must be compensated by a mean torque $+\langle \mathbf{G}\rangle$ provided by the internal battery system.

It should be noted that, under the conditions considered (with S_2 fixed), the mean position of the centre of mass of S_1 and its internal battery system is the same from one periodic cycle to the next. The internal battery system cannot therefore provide a non-zero mean force to S_1 and its contents, whatever the driven time-periodic motion of S_1 may be. Thus, although *mean torque* (and its associated angular momentum) may be generated and transferred by the mechanism to be considered, there is no possibility within a closed system of generating and transferring *mean force* (and its associated linear momentum) in this way.

Problems of this kind arise in micro-fluidics, where the scale is such that inertia forces are completely negligible and flows are controlled by viscosity [1]. With this possible application in mind, we shall neglect fluid inertia in the following, so that the flow is in effect governed by the low-Reynolds-number quasi-static Stokes equations. Some motivation comes also from biological applications, e.g. the dynamics of swimming micro-organisms [2], or at the cellular level where intriguing problems involving a deforming embryo within a parent spherical membrane are encountered [3]. Equally, motivation comes from micro-robotics, where motion of an internally mounted sphere can through rotary motion induce controlled rolling of the parent body [4].

A general formulation for the problem of a body moving and deforming in an arbitrary manner in a viscous fluid has been developed in [5], and the problem that we address falls in principle within this class. The problem of the dynamical interaction of two squirming spheres has been treated in [6] using bispherical coordinates, and in §III below we adopt this method also. A related problem of a "treadmilling" sphere far from a fixed plane has been treated in [7] using alternative techniques that exploit the Lorentz reciprocal theorem (as used also in [6]); the reciprocal theorem could be adapted to our problem also, but we have not found this to be necessary for our present limited objective.

The problem that we present here is one that can be treated analytically without recourse to heavy computation and as such may serve as a simple prototype for more complex systems. However, no practical application of this mechanism of mean-torque transfer has as yet been identified.

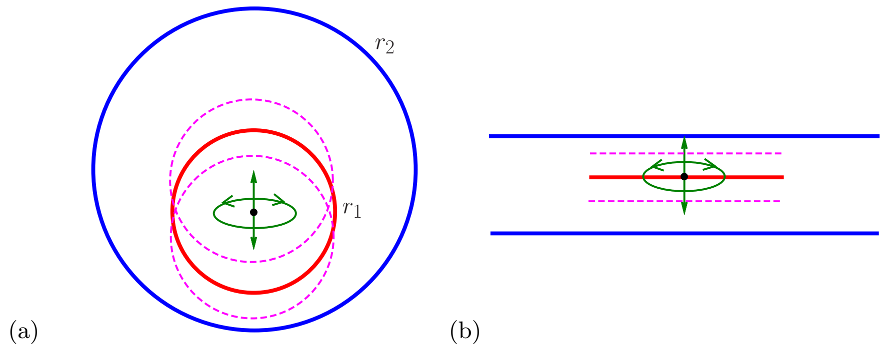

FIG. 1. (Color online.) (a) Two-sphere configuration; the outer sphere is fixed and the inner sphere oscillates between the (dashed) limits indicated, and rotates about the axis of symmetry with time-periodic angular velocity $\omega(t)$ with $\langle\omega(t)\rangle = 0$. (b) Corresponding oscillating-disc model.

II. A SIMPLE PLANAR MODEL

Consider first the simple planar model of Fig. 1(b). Here fluid of viscosity μ and kinematic viscosity $\nu = \mu/\rho$ fills the space $|z| < 1$ between two rigid plane boundaries $z = \pm 1$, and a circular disc of radius $a \gg 1$ and negligible thickness is immersed in the fluid parallel to the boundaries at position $z = d_0$ with $0 < d_0 < 1$. We adopt cylindrical polar coordinates (r, φ, z), where r is the radial distance from the axis of the disc. Suppose now that the disc is caused to oscillate vertically so that its position at time t is $d(t) = d_0 + \zeta(\tau)$ where $\tau = \sigma t$ and σ is a frequency. The disc is simultaneously caused to rotate about its axis with angular velocity $\omega(\tau)$. Both $\zeta(\tau)$ and $\omega(\tau)$ are assumed to be 2π-periodic with zero average, $\langle\zeta(\tau)\rangle = \langle\omega(\tau)\rangle = 0$. We ignore here the mechanism by which such motion could in practice be realised.

Let U_0 be the maximum speed of the disc during its periodic motion. We assume that the Reynolds number $U_0 a/\nu$ and the Stokes number $\sigma a^2/\nu$ are small, so that inertia effects in the fluid may be neglected; the methods of 'thin-film' lubrication theory (e.g. [8]) are then applicable. In these circumstances, the poloidal (r, z) and toroidal (φ) components of the equations of motion are decoupled. We need here consider only the toroidal component of velocity $v(r, z, \tau)$ within the gaps $-1 < z < d$ and $d < z < 1$. This is essentially a "Couette-flow" situation in each gap, with boundary conditions $v = 0$ at $z = \pm 1$ and $v = \omega r$ at $z = d(\tau) \pm 0$. A simple calculation of the total instantaneous torque $G(\tau)$ exerted on the plates yields $G(\tau) = \kappa\,\omega(\tau)\,D(\tau)$, where

$$\kappa = \pi\mu a^4 \quad\text{and}\quad D = \frac{1}{2}\left(\frac{1}{1-d} + \frac{1}{1+d}\right) = \frac{1}{1-d^2}\,. \tag{1}$$

Hence the average torque on the plates over a period of the disc motion is

$$\langle G(\tau)\rangle = \kappa\,\chi \quad \text{where} \quad \chi = \langle\omega(\tau)D(\tau)\rangle. \tag{2}$$

The fluid exerts an equal and opposite mean torque on the moving disc.

By way of example, let

$$\zeta(\tau) = \zeta_0\cos\tau, \quad \omega(\tau) = \omega_0\cos(\tau+\phi_0), \tag{3}$$

where ζ_0, ω_0 and ϕ_0 are non-negative constants and $|d_0+\zeta_0| < 1$. The trajectory of any point of the disc then has parametric equation $r = r_0$, $\varphi = \varphi_0 + \int_0^\tau \omega(\tau')\,d\tau'$, $z = d(\tau)$, and, being periodic in τ with period 2π, is a closed curve on the cylinder $r = r_0$. Three examples of such trajectories, with $r_0 = 1$, $d_0 = 0.2$, $\zeta_0 = 0.76$, $\omega_0 = 2.5$, and phase differences $\phi_0 = 0, \pi/4$ and $\pi/2 - 0.05$, are shown in the top row of Fig. 2. Each such trajectory winds almost once round the cylinder $r = r_0$, then reverses and winds back so that the closed curve makes zero net turns round the cylinder, as most clearly seen in Fig. 2(a). For the limiting case $\phi_0 = \pi/2$, the trajectory is a portion of a helix traversed up and down periodically. The projections of the same three figures on the plane $x = 0$ are shown in the bottom row; note that these projections enclose a finite area (with positive or negative contributions respectively from regions whose boundaries are traversed in an anticlockwise or clockwise sense) which decreases in proportion to $\cos\phi_0$ as ϕ_0 increases from 0 to $\pi/2$.

Evaluation of the mean $\langle G\rangle$ from (3) now yields

$$\langle G\rangle = \kappa\,\omega_0\,g(d_0, \zeta_0)\cos\phi_0, \tag{4}$$

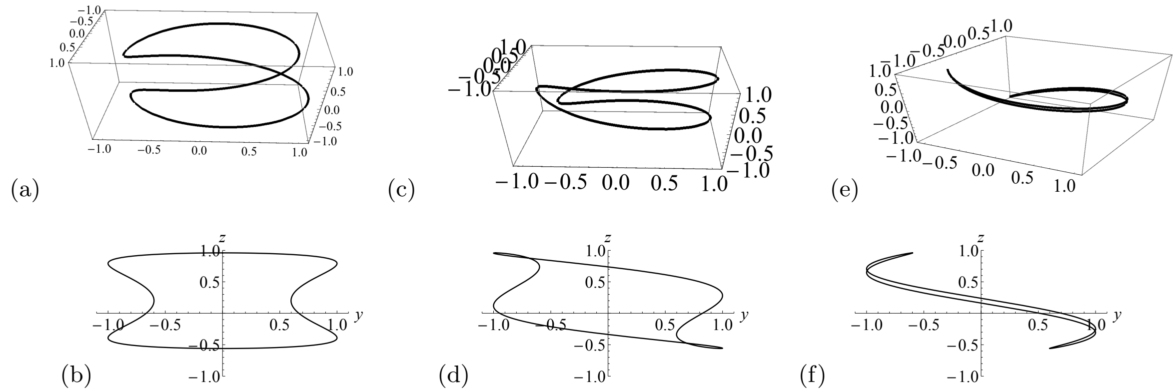

FIG. 2. (a,c,e) Trajectories of a point on the disc with velocity (3) and (b,d,f) corresponding projections on the plane $x = 0$; $r_0 = 1$, $d_0 = 0.2$, $\zeta_0 = 0.76$, $\omega_0 = 2.5$; (a, b) $\phi_0 = 0$, (c, d) $\phi_0 = \pi/4$, (e, f) $\phi_0 = \pi/2 - 0.05$.

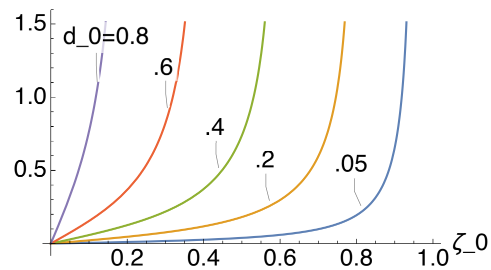

FIG. 3. (Color online.) The functions $g(d_0, \zeta_0)$ defined by (5) for five values of d_0 and for $0 < \zeta_0 < 1 - d_0$ in each case.

where

$$g(d_0,\zeta_0) = \frac{1}{2\zeta_0}\left[\frac{1-d_0}{\sqrt{(1-d_0)^2-\zeta_0^2}} - \frac{1+d_0}{\sqrt{(1+d_0)^2-\zeta_0^2}}\right]. \tag{5}$$

The functions $g(d_0, \zeta_0)$ are shown in Fig. 3 for $d_0 = 0.2, 0.4, 0.6, 0.8$ and the relevant range in each case, $0 < \zeta_0 < 1 - d_0$. Note the singular behaviour as $\zeta_0 \to 1 - d_0$; this simply reflects the torque singularity that is to be expected when the gap between the disc and the boundary $z = 1$ tends to zero.

It is easy to understand the physical origin of this mean torque. Consider just the time interval $-\pi < \tau \leq \pi$. Since by assumption $d_0 > 0$ and $\zeta_0 > 0$, the disc is at its nearest to the upper fixed boundary when $\tau = 0$ (as most evident in the projection of Fig. 2(a)); if $\phi_0 = 0$, the instantaneous torque on this boundary is then maximal and in the positive direction (i.e. the direction of increasing φ); when $\tau = \pi$, the disc is at its furthest from the upper boundary and the instantaneous torque is then minimal and in the negative direction. Hence, the averaged torque on the upper boundary is positive. A similar consideration for the more remote lower boundary results in a smaller average torque in the negative direction. The time-averaged total torque is therefore positive when $\phi_0 = 0$, as confirmed by (4) and (5) and Fig. 3. Note the $\cos\phi_0$ dependence on the phase factor in (5), the mean effect being maximal when $\phi_0 = 0$, i.e. when $\zeta_0(\tau)$ and $\omega(\tau)$ are in phase.

Similarly, the mean effect is minimal (and actually vanishes) when $\phi_0 = \pi/2$, i.e. when $w(\tau) \equiv \mathrm{d}\zeta/\mathrm{d}\tau$ and $\omega(\tau)$ are in phase. If $\phi_0 = \pi/2$ the disc motion reverses from one half-period to the next. By the reversibility theorem for Stokes flow, the fluid flow is similarly reversed, and the contributions to the mean torque are therefore equal and opposite in the two half-periods. The projected area enclosed by the trajectory (like that of Fig. 2(f)) goes to zero as $\phi_0 \to \pi/2$. This effect of time-reversibility is similar to that described by Purcell's 'scallop' theorem [9], which indicates that a body that undergoes a time-reversible periodic deformation cannot achieve a non-zero mean swimming velocity when fluid inertia is negligible.

When $d_0 = 0$, then from (4) and (5), $\langle G \rangle = 0$ also, i.e. the total averaged torque generated on both boundaries taken together is zero, as might be expected from symmetry. The above physical description suggests however how torque may be generated through modification of $\omega(\tau)$. Thus suppose for example that

$$\omega(\tau) = \omega_0 \cos n\tau, \quad d(\tau) = \zeta_0 \cos\tau, \tag{6}$$

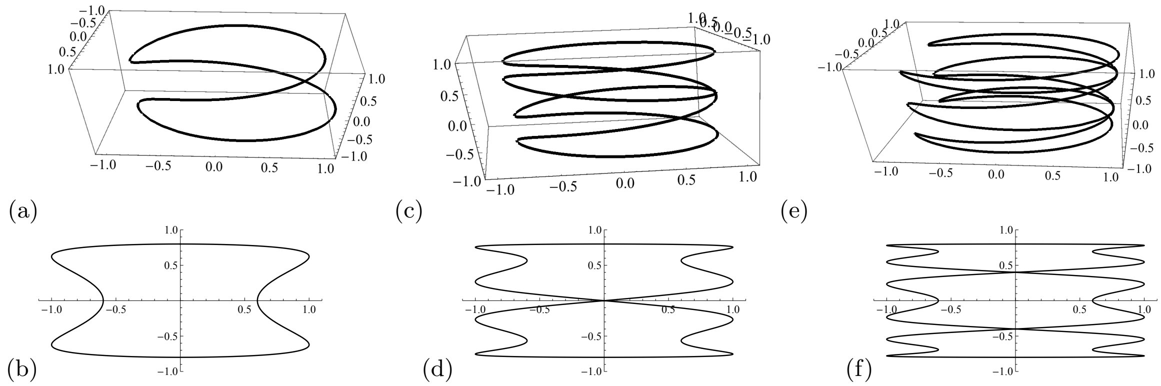

FIG. 4. (a,c,e) Trajectories of a point on the disc with velocity (6) and (b,d,f) corresponding projections on the plane $x = 0$; $r_0 = 1, d_0 = 0, \zeta_0 = 0.8, \omega_0 = 2.5$; (a, b) $n = 1$, (c, d) $n = 2$, (e, f) $n = 3$.

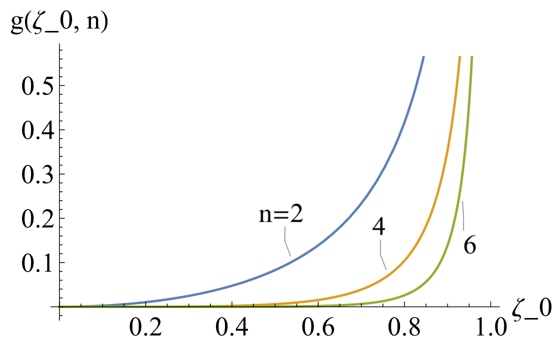

FIG. 5. (Color online.) The functions $g(\zeta_0, n)$ for $n = 2, 4, 6$, defined by (8).

where n is a positive integer. Fig. 4 shows three possibilities for the trajectories of a point on the disc (with corresponding projections below) with $\omega(t)/\omega_0 = \cos\tau$, $2\cos 2\tau$, and $3\cos 3\tau$ (so $\varphi(\tau)/\omega_0 = \sin\tau$, $\sin 2\tau$, $\sin 3\tau$ respectively). For $n = 1$ or 3 (and more generally for any odd n) such a trajectory has the opposite sense (clockwise/anticlockwise) near the boundaries $z = \pm 1$, whereas for $n = 2$ (and more generally for any even n) it has the same sense near both boundaries. We may then expect that $\langle G \rangle$ should be non-zero only if n is even. This is confirmed by evaluation of (2) for the motion (6), which yields (with the help of [10], p.391, formula 3.613)

$$\langle G(\zeta_0, n)\rangle = \kappa\, \omega_0\, g(\zeta_0, n), \tag{7}$$

where

$$g(\zeta_0, n) = \frac{1 + (-1)^n}{2\sqrt{1-\zeta_0^2}} \left(\frac{1 - \sqrt{1-\zeta_0^2}}{\zeta_0} \right)^n . \tag{8}$$

For n even, $g(\zeta_0, n) \sim (\zeta_0/2)^n$ for small ζ_0; the first three ($n = 2, 4, 6$) are plotted in Fig. 5. Note again the (expected) singular behaviour as $\zeta_0 \to 1$, when the disc makes instantaneous contact with each boundary $z = \pm 1$ once in each period of the motion.

The quantity $\chi = \langle \omega(\tau) D(\tau) \rangle$ that appears in the above examples is a pseudo-scalar, which can be non-zero only if the motion of the disc is chiral in character (i.e. lacking reflection symmetry). It is in this respect that the phenomenon of mean-torque generation is analogous to the phenomenon of spontaneous dynamo-generation of magnetic field in a conducting fluid, resulting from non-zero mean helicity of a random wave field. In that context, a phase shift between velocity and magnetic perturbations is required to provide a non-zero α-effect ([11], §4). Similarly, in the present context, there must (in (3)) be a phase shift between the vertical velocity $w(\tau) = \mathrm{d}\zeta/\mathrm{d}\tau$ of the disc and its angular velocity $\omega(\tau)$, (i.e. $\phi_0 \neq \pi/2$) to provide a non-zero mean torque. The pseudo-scalar χ is non-zero for both cases (3) (with $\phi_0 \neq \pi/2$) and (6) (with n even) and provides the appropriate measure of chirality of the disc motion.

III. THE TWO-SPHERE PROBLEM

Similar effects are to be expected for the two-sphere problem illustrated in Fig. 1(a). Here we shall assume that the outer sphere S_2 with radius r_2 is fixed, and that the inner sphere S_1 with radius $r_1 < r_2$ oscillates on the line of centres so that the separation between the centres is $d(\tau) = d_0 + \zeta_0 \cos\tau$, where

$$0 < d_0 \pm \zeta_0 < r_2 - r_1 \quad \text{and} \quad \tau = \sigma\, t, \tag{9}$$

and simultaneously about the line of centres with time-periodic angular velocity $\omega(\tau)$, with period 2π and $\langle\omega(\tau)\rangle = 0$. The dotted spheres in Fig. 1(a) indicate the range of the up-down oscillation. We further assume as before that the Reynolds and Stokes numbers based on the scale r_2 are small, so that the flow, governed by the Stokes equations, is quasi-static and instantaneously determined by the no-slip boundary conditions. To this extent, the formulation is very similar to that of §II.

Following [12], [13] and [6], we adopt bi-spherical polar coordinates (ξ, φ, η) defined in terms of cylindrical polar coordinates (r, φ, z) by

$$\xi + \mathrm{i}\,\eta = \log\left[\frac{r + \mathrm{i}\,(z+R)}{r + \mathrm{i}\,(z-R)}\right], \tag{10}$$

or equivalently

$$r + \mathrm{i}\,z = R\,\frac{\sin\eta + \mathrm{i}\sinh\xi}{\cosh\xi - \cos\eta}, \tag{11}$$

where R is an arbitrary positive real number. The contours $\xi =$ const. (solid) and $\eta =$ const. (dashed) are shown in Fig. 6(a). From (10), it may be ascertained that

$$r^2 + (z - R\coth\xi)^2 = R^2 \mathrm{cosech}^2\xi\,. \tag{12}$$

Thus the contours $\xi =$ const. are spheres with centres at $r = 0$, $z = R\coth\xi$, and radii $R\,\mathrm{cosech}\,\xi$. For given $(r_1,\, r_2,\, d)$, it follows that ξ_1, ξ_2 and R satisfy

$$R = r_1 \sinh\xi_1 = r_2 \sinh\xi_2, \tag{13}$$

and

$$r_1 \sinh\xi_1 - r_2\sinh\xi_2 = 0, \quad r_1\cosh\xi_1 - r_2\cosh\xi_2 = -d. \tag{14}$$

These equations may be solved for $\cosh\xi_1$, $\cosh\xi_2$ and R in terms of d, giving

$$\cosh\xi_1 = \frac{r_2^2 - r_1^2 - d^2}{2r_1 d}, \quad \cosh\xi_2 = \frac{r_2^2 - r_1^2 + d^2}{2r_2 d}, \tag{15}$$

and

$$R = +\left[\left((r_1 - r_2)^2 - d^2\right)\left((r_1+r_2)^2 - d^2\right)\right]^{1/2}/2d\,. \tag{16}$$

Note that d must be non-negative for real $\{\xi_1, \xi_2\}$. We may take $0 < d < r_2 - r_1$; these inequalities ensure that $\xi_1 > \xi_2 > 0$, and that R is real and positive. Fig. 1(a) actually shows the situation when $r_1 = 0.5$, $r_2 = 1$, $d = 0.25$ and $\zeta_0 = 0.2$, the dotted spheres indicating the limiting positions in the up-down oscillations. Fig 6(b) shows the functions $\xi_1(d)$, $\xi_2(d)$ and $R(d)$ when $r_1 = 0.5$, $r_2 = 1$. More generally, $\xi_1(d)$ and $\xi_2(d)$ both vanish when $d = r_2 - r_1$, i.e. when the spheres are in contact. From (16), we then have asymptotically

$$r_1\xi_1 \sim r_2\,\xi_2 \quad \text{as } d \to r_2 - r_1. \tag{17}$$

In this contact limit, one might expect the torque to be dominated by local conditions near the contact point, but this is not in fact the case (see Appendix).

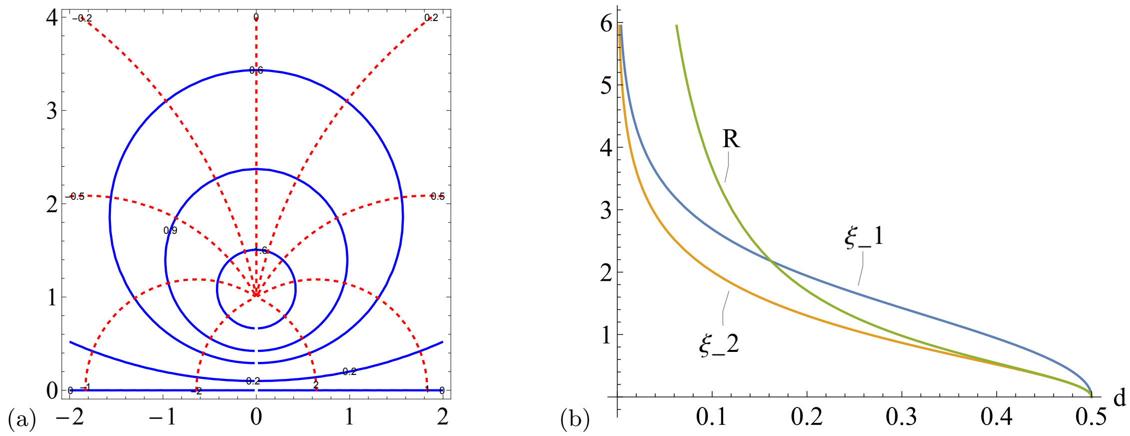

FIG. 6. (Color online.) (a) Contours $\xi = $ const. (solid) and $\eta = $ const. (dashed) in the half-plane $z > 0$, as given by (10) for $R = 1$; (b) the functions $\xi_1(d)$, $\xi_2(d)$ and $R(d)$ for $r_1 = 0.5$, $r_2 = 1$, as given by (15) and (16). ($\xi_1(d)$, $\xi_2(d)$ tend logarithmically to infinity as $d \to 0$.)

IV. TRANSFER OF ANGULAR MOMENTUM

In the quasi-static Stokes approximation, and under axisymmetric conditions, the azimuthal component of velocity $v(r, z)$ is decoupled from the r and z components, and the pressure is independent of φ; hence $v(r, z)$ satisfies the equation

$$(\nabla^2 - r^{-2})v = 0. \tag{18}$$

The general solution of this equation in the above bi-spherical coordinates [13] is

$$v = \sqrt{W}\sum_{n=0}^{\infty}\{a_n \exp[(n+1/2)(\xi-\xi_1)] + b_n \exp[-(n+1/2)(\xi-\xi_2)]\}P_n^1(\cos\eta), \tag{19}$$

where $W = R/(\cosh\xi - \cos\eta)$. The constants a_n and b_n are determined through the boundary conditions

$$v(\xi_1) = \omega(\tau)\, r_1\,, \quad v(\xi_2) = 0. \tag{20}$$

The instantaneous torque $G(\tau)$ exerted on the outer sphere is then given by equation (11) from [13], which may here be expressed in the form

$$G(\tau) = 8\pi\mu r_1^3 \omega(\tau)\sum_{m=0}^{\infty}\frac{\sinh^3\xi_1(\tau)}{\sinh^3[(m+1)\xi_1(\tau) - m\xi_2(\tau))]}\,, \tag{21}$$

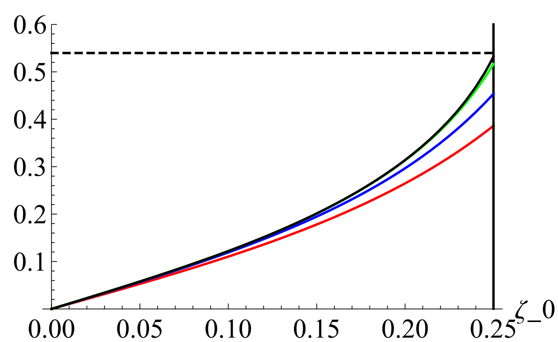

FIG. 7. (Color online.) With $g_m(r_1, d_0, \zeta_0)$ defined by (24), the partial sums $\sum_{m=1}^{n} g_m(0.5, 0.25, \zeta_0)$ are shown for $n = 2$ (red), 3 (blue), 6 (green) and 10 (black); the sum to infinity is uniformly convergent, and in particular $\sum_{m=1}^{\infty} g_m(0.5, 0.25, 0.25) \approx 0.5397$.

and the instantaneous torque experienced by the inner sphere is $-G(\tau)$ [14]. Since each term of the sum (21) is positive, it follows that $G(\tau)$ has the same sign as $\omega(\tau)$ at each instant τ, in accord with physical intuition. It is remarkable that the result (21) has been known for the steady situation since Jeffery's seminal 1915 paper, but that its consequences for the time-periodic but quasi-static situation considered here has not previously been recognised.

We are again concerned with the mean torque exerted on the outer sphere,

$$\langle G\rangle = 8\pi\mu r_1^3 \sum_{m=0}^{\infty} \left\langle \frac{\omega(\tau)\sinh^3\xi_1(\tau)}{\sinh^3[(m+1)\xi_1(\tau) - m\,\xi_2(\tau)]} \right\rangle . \tag{22}$$

With $\omega(\tau) = \omega_0\cos(\tau+\phi_0)$, this becomes

$$\langle G\rangle\,(r_1, d_0, \zeta_0) = 4\mu r_1^3\omega_0\cos\phi_0 \sum_{m=0}^{\infty} g_m(r_1, d_0, \zeta_0)\,, \tag{23}$$

where

$$g_m(r_1, d_0, \zeta_0) = \int_\Delta \frac{\cos\tau\sinh^3\xi_1(\tau)}{\sinh^3[(m+1)\xi_1(\tau) - m\,\xi_2(\tau))]}\,\mathrm{d}\tau, \tag{24}$$

Δ is any 2π-period of τ, and $\xi_1(\tau)$ and $\xi_2(\tau)$ are given by (15) with $d(\tau) = d_0 + \zeta_0\cos\tau$. Fig. 7 shows the functions $\sum_{m=1}^{n} g_m(r_1, d_0, \zeta_0)$ $(n = 2, 3, 6, 10)$ with (by way of illustration) $r_1 = 0.5, d_0 = 0.25$, and for the relevant range of ζ_0, i.e. $0<\zeta_0<d_0$. The sum (23) is uniformly convergent, even at the limiting value $\zeta_0 = d_0$ for which the spheres make contact and for which $\sum_{m=1}^{\infty} g_m(1) \approx 0.539667$.

The situation here is to be contrasted with that of Fig. 3 which shows a divergent torque when the amplitude of the oscillation is maximal. In the spherical case, the influence of the inner sphere on the outer sphere is much weaker in the contact limit, so weak in fact that the resulting torque is finite in this limit.

The above treatment requires minor modification if the centre C_1 of S_1 rises above the centre C_2 of S_2 in the course of its oscillations: while C_1 is above C_2, it is necessary to 'switch' the coordinate system by simply replacing z by $-z$ in (10); the procedure is straightforward, and we need not labour the details here.

V. CONCLUSIONS AND DISCUSSION

For the Cartesian geometry of Fig. 1(b), we have shown by straightforward analysis that time-periodic axisymmetric motion of the disc with zero mean can generate a mean torque on the two fixed plates bounding the fluid domain; in this, we have assumed quasi-static Stokes flow in which fluid inertia is negligible. Mean torque generation requires that the disc motion should have a chiral character, as measured by the pseudo-scalar quantity $\chi = \langle\omega(\tau)D(\tau)\rangle$, where $D(\tau) = (1 - d(\tau)^2)^{-1}$, and also that there should be a phase shift between the vertical and rotational components of velocity. The phase shift is required, because without it the disc motion is time-reversible, and the reversibility theorem for Stokes flow ensures that, although the disc motion is still chiral, the instantaneous torque generated necessarily averages to zero. In these respects, the generation of a mean torque is analogous to the dynamo generation in a conducting fluid of a mean (large-scale) magnetic field by a random-wave field that is chiral and for which a phase shift exists between the velocity and magnetic-perturbation fields.

For the spherical geometry of Fig. 1(a), we have demonstrated a similar phenomenon when the outer sphere S_2 is fixed and the inner sphere S_1 is subject to a time-periodic motion with zero mean velocity. The clear physical interpretation of the effect is that when S_1 is near to S_2 the rotational torque transmitted to S_2 is stronger than when it is far from S_2. The spherical problem is significantly different from the planar problem in that the mean torque transmitted remains finite even in the limit when the spheres make instantaneous contact during a period of the up-down oscillation.

Although the analysis of the paper is restricted to Stokes flow, there seems little doubt that the effect must persist when fluid inertia is taken into account, because the above physical explanation is still applicable. It would be reasonably straightforward to take account of increasing frequency of the disc motion (i.e. increase of the Stokes number) because the problem then remains linear. It would be much more difficult to take account of increasing Reynolds number, because there is then a nonlinear interaction between the poloidal and toroidal ingredients of the flow. Again, the problem has some similarity with the dynamo problem, in that explicit calculation of the α-effect in dynamo theory is easy only when the magnetic Reynolds number Re_m based on the random-wave scale is small, and very difficult when $Re_m \gg 1$ ([15]).

If the sphere S_2 were free to rotate (e.g. if it were suspended by a thread from a fixed point), then the mean torque $\langle G\rangle$ would drive a mean angular velocity (superposed on oscillations). If its moment of inertia about the vertical is large, this angular velocity would be small, and could easily be taken into account in the foregoing analysis. There is moreover no need to restrict the boundaries S_1 and S_2 to be spherical (see, for example, [16]); the chiral-transfer effect will still obviously be present if the boundaries are axisymmetric with a common axis of symmetry; and indeed the general principle of chiral transfer of angular momentum may be expected to have wide generality.

Appendix A: Comment on the contact limit

We comment here on the inapplicability of conventional lubrication theory when the minimum gap between the two spheres tends to zero. It is sufficient to consider just the situation when $r_2 \to \infty$ so that S_2 becomes the plane boundary $z = 0$, and when the sphere S_1 makes contact with this plane; the situation is shown in Fig. 8. If in this situation S_1 rotates about the diameter through the point of contact with *steady* angular velocity ω_0, then as observed in [6], the torque G (experienced by the sphere) given by (21) reduces to

$$G \sim G_0 = -8\pi\, \mu\, r_1^3\, \omega_0\, \hat{\zeta}(3), \tag{A1}$$

where $\hat{\zeta}(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. When the sphere S_1 is remote from the plane, the well-known expression for the torque is

$$G \sim G_\infty = -8\pi\, \mu\, r_1^3\, \omega_0, \tag{A2}$$

so that

$$G_0/G_\infty = \hat{\zeta}(3) \approx 1.20205. \tag{A3}$$

Jeffery (1915) actually tabulated the value of the torque for various values of the ratio of the radius r_1 to the distance $r_1 + \epsilon$ where ϵ is the minimum gap between the sphere and the plane, and noted that, if $\epsilon/r_1 = 0.02$, “the couple required to maintain the rotation is only increased [from G_∞] by 17%”. We see from the above that in the contact limit $\epsilon = 0$, the couple is increased by only 20.2%.

Conventional lubrication theory ([8]), would approximate the sphere $z/r_1 = 1 - (1 - (r/r_1)^2)^{1/2}$ near the point of contact by the paraboloid

$$z/r_1 = h(r) = \tfrac{1}{2}(r/r_1)^2, \tag{A4}$$

shown in red in Fig. 8. Lubrication theory is justifiable only in the ‘contact region’ $r \ll r_1$. In this small region, the velocity profile is linear, i.e. locally Couette, and the contribution to the torque on S_1 out to radius r in this region is easily calculated as

$$G(r) \sim -2\pi\, \mu\, \omega_0 \int_0^r \frac{r'^3}{h(r')} \mathrm{d}r' = -2\pi\, \mu r_1 \omega_0\, r^2\,. \tag{A5}$$

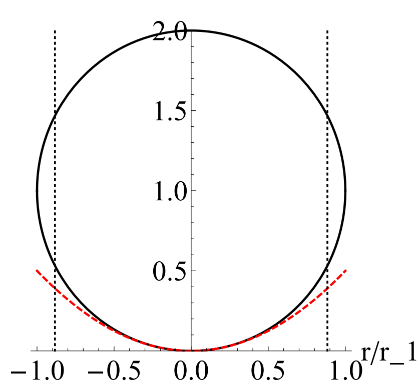

FIG. 8. (Color online.) The limit when the sphere S_1 touches the plane $z = 0$; the sphere may be approximated near $r = 0$ by the paraboloid as shown dashed, but the dominant contribution to the total torque comes from the region indicated by the dotted lines where lubrication theory is not applicable.

This increase with r resulting from the factor r'^3 in the integrand, indicates that the dominant contribution to the total torque in fact comes from outside this contact region, i.e. from the region $r = O(r_1)$ indicated by the dotted lines in Fig. 8; the torque cannot therefore be obtained by lubrication theory.
[*Note added by Editor:* Although the limiting value of the torque G(0) as $h/r_1 \to 0$ cannot be determined by lubrication theory, the asymptotic change in the torque $G(h)$ as h increases from zero *can* be determined by an improved lubrication theory in the form $[G(h) - G(0)]/G_\infty \sim -0.5(h/r_1)\log(r_1/h)$.]

ACKNOWLEDGMENTS

We acknowledge helpful discussion with Prof. Gareth Alexander, University of Warwick, and the suggestions of two referees that have led to improvements of presentation. This research is partially supported by grant IG/SCI/DOMS/18/16 from the Sultan Qaboos University, Oman.

[1] Squires, T.M. & Quake, S.R. 2005 Microfluidics: Fluid physics at the nanoliter scale *Rev. Mod. Phys.* **77**, 977-1026.
[2] Lauga, S. & Powers, T.R. 2009 The hydrodynamics of swimming microorganisms *Rep. Prog. Phys.* **72**, 096601.
[3] Höhn, S., Honerkamp-Smith, A.R., Haas, P.A., Trong, P.K. & Goldstein, R.E. 2015 Dynamics of a volvox embryo turning itself inside out *Phys. Rev. Lett.* **114**, 178101.
[4] Ilin, K.I., Moffatt, H.K. & Vladimirov, V.A. 2017 Dynamics of a rolling robot *Proc. Nat. Acad. Sci.* **114**, 12858-12863.
[5] Shapere, A. & Wilczek, F. 1989 Geometry of self-propulsion at low Reynolds number *J. Fluid Mech.* **198**, 557–585.
[6] Papavasiliou, D. & Alexander, G.P. 2017 Exact solutions for hydrodynamic interactions of two squirming spheres. *J. Fluid Mech.* **813**, 618-646.
[7] Davis, A.M.J. & Crowdy, D.G. 2015 Matched asymptotics for a spherical low-Reynolds-number treadmilling swimmer near a rigid wall *IMA J. Appl. Math.* **80**, 634 650.
[8] Batchelor, G.K. 1967 *An Introduction to Fluid Dynamics*, Cambridge Univ. Press.
[9] Purcell, E.M. 1977 Life at low Reynolds number *Amer. J. Phys.,* **45**, 3–11.
[10] Gradshteyn, I. & Ryzhik, I. 2007 *Table of Integrals, Series, and Products, 7th ed.*, Elsevier/Academic Press, Amsterdam.
[11] Moffatt, H.K. 1970 Turbulent dynamo action at low magnetic Reynolds number, *J. Fluid Mech.* **41**, 435–452.
[12] Jeffery, G.B. 1912 On a form of the solution of Laplace's equation suitable for problem relating to two spheres. *Proc. R. Soc. Lond. A* **87**, 109-120.
[13] Jeffery, G.B. 1915 On the steady rotation of a solid of revolution in a viscous fluid. *Proc. Lond. Math. Soc.* **2**, 327-338.
[14] Jeffery's expression for his G_1 is actually the torque exerted by the sphere S_1 on the fluid, and not as he states the torque experienced by S_1. In our time-dependent situation, a torque $+G(\tau)$ would have to be exerted on the sphere S_1 by some internal battery mechanism to maintain its postulated time-periodic motion.
[15] Moffatt, H.K. & Dormy, E. 2019 *Self-Exciting Fluid Dynamos*, Cambridge Univ. Press.
[16] Kanwal, R.P. 1961 Slow steady rotation of axially symmetric bodies in a viscous fluid. *J. Fluid Mech.* **10**, 17-24.