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TURBULENCE IN CONDUCTING FLUIDS

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SOMMAIRE

Maintenant que l'extrême complexité de la théorie de la turbulence dans les fluides ordinaires a été révélée, il peut apparaître à beaucoup une extravagance téméraire d'aborder l'examen des fluides conducteurs de l'électricité. La situation est déjà assez mauvaise, pourquoi la rendre encore pire en autorisant les électrons, aussi bien que les molécules, à se mouvoir sans entraves ? A première vue, étendre toute théorie bien connue des fluides ordinaires à quelques-uns possédant cette dernière propriété semble refléter une autre explosion de cette panique contagieuse. Sur quelques points, l'accusation est justifiée. Une tendance s'est révélée de présenter des extensions directes de quelques-unes parmi les plus connues des théories mathématiques de la turbulence, nécessairement lourdes de formalisme mathématique, soutenues par des hypothèses d'une validité discutable et impliquant une série de conclusions dont la signification n'est comprise que partiellement. Mais l'aspect physique du sujet n'est pas encore suffisamment clarifié pour justifier une approche exclusivement mathématique.

Il est important à cette étape d'essayer de définir les sortes de situations physiques susceptibles de se produire, et c'est en partie mon but dans cette conversation.

Il y a relativement peu de publications à ce sujet et aucun travail expérimental n'a pratiquement été réalisé. Néanmoins, il y a deux raisons encourageantes de poursuivre ce sujet à fond.

— D'abord, dans les recherches concernant l'astrophysique, et la physique des plasmas, la présence de la turbulence est souvent supposée lorsqu'on ne peut pas expliquer les observations par une théorie « bien carénée ».

Il est cependant trop facile d'user, ou plutôt d'abuser, du mot « turbulence », comme d'une baguette magique, pour faire disparaître ce qui ne peut être interprété autrement.

Il est important d'arriver à des conclusions précises, quant à savoir quels phénomènes, dans des fluides conducteurs, peuvent être vraiment attribués à la turbulence, et quels phénomènes ne le peuvent pas.

— La seconde raison est peut-être plus académique. L'action de la turbulence sur une grandeur scalaire, telle que la température, qui est à la fois transmise et diffusée dans le fluide, est maintenant bien connue. Pour compléter le tableau, il serait intéressant de bien comprendre l'action de la turbulence sur une grandeur vectorielle, qui est de même transmise et diffusée. Le champ rotationnel est un exemple, mais il est trop particulier, car intimement lié au champ des vitesses.

Le champ magnétique dans un fluide conducteur est le parfait exemple de sujet de travail. Les lignes de force d'un champ magnétique, dans un fluide de conductivité infinie, sont transportées avec le fluide. Dans les fluides de conductivité finie, elles se diffusent à un taux dépendant de la grandeur de cette conductivité.

La situation est compliquée du fait que le champ magnétique exerce une force sur le fluide; il n'est généralement pas passif dynamiquement; mais dans certaines circonstances il sera possible de négliger cette force, et de se concentrer sur l'effet combiné de la convection et de la diffusion, dans un fluide turbulent, de propriétés statistiques connues.

SUMMARY

Now that the extreme complexity of the theory of turbulence in ordinary fluids has been revealed, it may seem to many a rash extravagance to admit to consideration fluids which conduct electricity. The situation is bad enough already — why make it worse by allowing electrons as well as molecules to move unfettered ? At first sight it seems to reflect another outburst of that infectious stampede to extend every known theory of ordinary fluids to those few with this " latest " property. To some extent, the accusation is justified. A tendency has revealed itself to present direct extensions of some of the better-known mathematical theories of turbulence, necessarily heavy with mathematical formalism, bolstered with assumptions of debatable validity, and carrying a trail of conclusions of partially understood significance. But the physics of the subject is not yet sufficiently clarified to justify an all-out mathematical approach. It is important at this stage to attempt to define the types of physical situation that may arise, and this is partly my aim in this talk.

There are relatively few published papers on the subject and practically no experimental work has been done. Nevertheless two compelling reasons can be given for pursuing the subject to its limit. Firstly, in astrophysics and in plasma physics research, the presence of turbulence is often inferred when observations cannot be explained by a streamlined theory. However it is too facile to use, or rather abuse, the word " turbulence ", like a magic wand, to dispel what cannot otherwise be understood. It is important to arrive at some precise conclusions as to what phenomena in conducting fluids can truly be attributed to the presence of turbulence, and what cannot. The second reason is perhaps more academic. The action of turbulence on a scalar quantity, such as temperature, which is both convected and diffused in the fluid is now fairly well understood. To complete the picture it would be satisfying to understand fully the action of turbulence on a vector quantity which is likewise convected and diffused. The vorticity field is an example, but it is too special, being closely related to the velocity field. The magnetic field in a conducting fluid is the perfect working example. The lines of force of a magnetic field in a fluid of infinite conductivity are convected with the fluid. In fluids of finite conductivity, they diffuse at a rate determined by the magnitude of this conductivity. The situation is complicated by the fact that the magnetic field exerts a force on the fluid — it is not in general dynamically passive; but in certain circumstances it will be possible to neglect this force, and concentrate on the combined effect of convection and diffusion in a turbulent fluid with known statistical properties.

2. The turbulent dynamo

The standard equations of magnetohydrodynamics can be conveniently written in terms of the fluid velocity $\mathbf{u}(\mathbf{r}, t)$ and the Alfvèn velocity at each point $\mathbf{h}(\mathbf{r}, t)$, which is simply proportional to the magnetic field $\mathbf{H}(\mathbf{r}, t)$:

$$\mathbf{h} = \sqrt{\frac{\mu}{4\pi\rho}}\,\mathbf{H} \tag{1}$$

where μ and ρ are the constant magnetic permeability and density of the fluid. In this notation the kinetic energy and the magnetic energy per unit mass are $\frac{1}{2}u^2$ and $\frac{1}{2}h^2$ respectively. The total pressure $\chi(\mathbf{r}, t)$ is the sum of the fluid pressure $p(\mathbf{r}, t)$ and the magnetic pressure $\frac{1}{2}\rho h^2$,

$$\chi = p + \frac{1}{2}\rho h^2. \tag{2}$$

The equations for $\mathbf{u}(\mathbf{r}, t)$ and $\mathbf{h}(\mathbf{r}, t)$ are then

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla\chi + \mathbf{h}\cdot\nabla\mathbf{h} + \nu\nabla^2\mathbf{u} \tag{3}$$

$$\frac{\partial \mathbf{h}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{h} = \mathbf{h}\cdot\nabla\mathbf{u} + \lambda\nabla^2\mathbf{h} \tag{4}$$

together with $\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{h} = 0$. (5)

Two diffusive constants appear, the kinematic viscosity ν, and the magnetic diffusivity λ. When $\lambda = 0$, one can deduce from equations (4) and (5) the well-known result that the flux of magnetic field through any circuit moving with the fluid remains constant, or equivalently that the lines of force move with the fluid and the strength of the magnetic field at any point moving with the fluid is proportional to the length of an element of the line of force through that point. I shall, for simplicity, restrict attention to homogeneous turbulence, and shall use the spectrum tensors $\Phi_{ij}(\mathbf{k})$ and $\Gamma_{ij}(\mathbf{k})$ of velocity and magnetic fields to describe the energy distributions in any steady state. Let it be our first aim to describe the development and steady state form of these spectra, given certain gross conditions defining the various situations that may arise.

In 1950, two irreconcilable theories were proposed, the one by BATCHELOR [1], the other by BIERMANN and SCHLUTER [2], to predict the development of an initially weak random magnetic field in a fluid in turbulent notion. To be fair, it must be stated that no fully convincing argument has yet been given to prove or disprove either theory. The matter is of fundamental importance and it seems highly appropriate that the theories should be reviewed at this meeting at any rate to clarify the points at which they diverge, and perhaps to suggest some critical problem whose solution might finally distinguish between the two standpoints. Let me therefore recall the main points of these theories.

BATCHELOR exploited the analogy between equation (4) for the magnetic field and that for vorticity $\omega\,(= \nabla_\wedge \mathbf{u})$ in a non-conducting fluid, viz.,

$$\frac{\partial \omega}{\partial t} + \mathbf{u}\cdot\nabla\omega = \omega.\nabla\mathbf{u} + \nu\nabla^2\omega \tag{6}$$

$$\nabla\cdot\omega = 0. \tag{7}$$

Vorticity is generated by the stretching of vortex lines as they are convected by the fluid motion and it is destroyed by viscous diffusion at high wave-numbers. These two processes are approximately in equilibrium. In the same way, magnetic energy is generated by the stretching of magnetic lines of force in so far as they are convected by the turbulent motion. It is to be expected therefore that those statistical properties of the magnetic field that depend only upon this stretching mechanism will in time approximate to the corresponding statistical properties of the vorticity field. If $\lambda = \nu$, the conductive diffusion of lines of force is then just rapid enough for the magnetic field spectrum (like the vorticity spectrum) to remain approximately steady. If $\lambda > \nu$, conduction wins over stretching and the field decays to zero, while if $\lambda < \nu$, conduction is less important and the field increases in intensity. When λ is only slightly less than ν, it is not clear whether an all-round decrease of scale together with increased Ohmic dissipation limits the growth of the field, or whether it is the Lorentz force which modifies the straining motion and so limits the growth. Buth when $\lambda \ll \nu$, BATCHELOR argued that conduction alone would be of small importance and that the magnetic energy level must increase until magnetic stresses are comparable with the dynamic stresses governing the smallest turbulent eddies in which most of the vorticity is concentrated, that is until the mean magnetic energy per unit mass $\frac{1}{2} h^2$ is comparable with the kinetic energy per unit mass of the small-scale motion, $(\varepsilon\nu)^{1/2}$ (ε being the usual rate of dissipation of energy per unit mass).

Now BIERMANN and SCHLÜTER did not explicitly discuss the criterion for growth, but in any case they were considering a fluid, the interstellar gas, to which the condition $\lambda \ll \nu$ certainly applied, and they agreed with BATCHELOR that the mean magnetic energy would increase in these circumstances. However, it was their opinion that magnetic field components of all wave-numbers would be intensified, not only those with wave number near the viscous cut-off $\left(\frac{\varepsilon}{\nu^3}\right)^{1/4}$ where the vorticity is concentrated, as suggested by BATCHELOR. Briefly, they argued as follows. Consider eddies of dimension l larger than $\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ but small compared with the dimension L of the energy-containing eddies. The time-scale of such eddies by Kolmogorovian analysis is $l^{2/3}\varepsilon^{-1/3}$. When $\lambda \ll \nu$ the magnetic lines of force are to a very good approximation carried by these eddies as well as by all the smaller eddies that are superimposed on them. One might therefore expect loops of magnetic field of dimension l to be, say, doubled in intensity in a time of order $l^{2/3}\varepsilon^{-1/3}$. Such intensification could then continue until equipartition of energy was established at this length-scale. Equipartition would in this way be established by degrees at smaller and smaller wave-numbers until finally the whole spectrum was thus partitioned. Investigation of this argument reveals that although BIERMANN and SCHLÜTER agreed that the condition $\lambda < \nu$ was sufficient for initial growth they would not admit its necessity. The criterion most appropriate to their type of argument was stated explicity by SYROVATSKY [3] in 1957, who argued that magnetic eddies of size l would grow provided the stretching term $\mathbf{h}\cdot\nabla\mathbf{u}$ of equation (4) was greater in order of magnitude than the conduction term $\lambda\nabla^2\mathbf{h}$.

i. e., if

$$\frac{u_l}{l} > \frac{\lambda}{l^2} \quad \text{or} \quad R_m(l) \equiv \frac{u_l\, l}{\lambda} > 1 \tag{8}$$

where $u_l\ (= (\varepsilon l)^{1/3})$ is the velocity in an eddy of size l and $R_m(l)$ is the magnetic Reynolds number for that length-scale. Substituting for u_l this condition gives

$$l > \left(\frac{\lambda^3}{\varepsilon}\right)^{1/4}, \tag{9}$$

a situation that can arise only if

$$L \gg \frac{\lambda^3}{\varepsilon}. \tag{10}$$

The test case on which the two theories really collide is therefore when

$$\frac{1}{L} \ll \left(\frac{\varepsilon}{\lambda^3}\right)^{1/4} \ll \left(\frac{\varepsilon}{\nu^3}\right)^{1/4}. \tag{11}$$

For, by the first of these inequalities, the BIERMANN and SCHLÜTER attack predicts magnetic intensification at all wave numbers in the range $\left[\frac{1}{L}, \left(\frac{\varepsilon}{\lambda^3}\right)^{1/4}\right]$, while by the second (tantamount to $\lambda \gg \nu$) the BATCHELOR attack predicts that all random magnetic field fluctuations ultimately decay to zero. If I use the semi-empirical formula for ε,

$$\varepsilon = \frac{u'^3}{L} \tag{12}$$

where u' is the r. m. s. velocity, and define the Reynolds number R and magnetic Reynolds number R_m of the turbulence by

$$R = \frac{u'L}{\nu}, \qquad R_m = \frac{u'L}{\lambda} \tag{13}$$

then the case (11) is defined in more fundamental terms by the inequalities

$$1 \ll R_m^{3/4} \ll R^{3/4} \tag{14}$$

Let me digress for a moment in order to consider this problem afresh in the light of related work on the spectrum of a scalar solute which is convected and diffused in a turbulent fluid. It is well known and understood that if a variation of temperature, say, is initially present in such a fluid, the turbulence rapidly mixes the temperature distribution, increasing temperature gradients without limit until molecular conduction finally erases all trace of variation. If a steady distribution of heat sources is present on a large length scale, so that in effect a pulse of temperature variation is emitted in each small time interval, then there is established a steady spectrum of temperature variation whose form at large wave-numbers has been discussed and determined in divers circumstances by OBUKHOV [4], CORRSIN [5], BATCHELOR [6], and BATCHELOR, HOWELLS and TOWNSEND [7]. Can the development of magnetic field variations be followed in the same way ? Let us concentrate first on the most controversial case described by (11) or (14) and suppose again that magnetic variations are present at a length scale l. The ability of turbulence to mix the convected quantity (now a vector) is in no way reduced. The new feature is the intensification through stretching of the convected lines of force. In other words the field may initially increase, but the claim that its length scale at the same time on average decreases is no stronger than the same claim that is accepted for the scalar field. When the length scale of the magnetic field is reduced below $\left(\frac{\lambda^3}{\varepsilon}\right)^{1/4}$, conduction outweighs intensification, and converts all the magnetic energy into Joule heat, no matter how much intensification may have initially taken place. Thus the magnetic pulse disappears in this case like the scalar pulse although it initially grew in strength for the reason underlying Syrovatsky's argument. Of course here again if a large scale magnetic field is maintained (e. g. a constant magnetic field may be externally applied, or a random large-scale distribution of electromotive forces may be supposed present) then the turbulence will generate fluctuations whose intensity will be proportional to the applied field and whose spectrum should be easily obtainable. Knowing the spectrum, it is possible to calculate the increased dissipation and the eddy diffusivity of the turbulent fluid. Thus if the magnetic fluctuation spectrum increases as $k^{+1/3}$ like the vorticity spectrum in the range where neither viscosity nor conductivity is important, i. e. up to the wave number $\left(\frac{\varepsilon}{\lambda^3}\right)^{1/4}$, and falls off rapidly beyond this wave number, then it can be shown that the eddy diffusivity is approximately equal to the ordinary diffusivity (λ) multiplied by the 5/2 — power of the magnetic Reynolds number.

3. Fluctuations at low magnetic Reynolds number when a uniform field is applied

It is noteworthy that the fluctuations will not be small compared with the applied magnetic field when the magnetic Reynolds number is large compared with unity. Hence any perturbation method which assumes that the fluctuations are small compared with the applied field can be valid only when the magnetic Reynolds number is smaller, and preferably much smaller, than unity. The perturbation approach was used by LIEPMANN [8] in 1952 and by GOLITSYN [9] in 1960, and although the condition $R_m \ll 1$ was not stated explicitly in either paper, it is apparently only to this case that the theories can be applied. Liepmann supposed that at time $t = 0$ a constant field $\mathbf{h}_0$ is switched on in a fluid in turbulent motion, and he derived the time development of the spectrum of the field fluctuations $\mathbf{h}_1$, that are generated, on the assumption that these always remain small compared with $\mathbf{h}_0$:

$$|\mathbf{h}_1| \ll |\mathbf{h}_0| \tag{15}$$

If this is true, then the equation for $\mathbf{h}_1$ becomes approximately

$$\frac{\partial \mathbf{h}_1}{\partial t} = \mathbf{h}_0 \cdot \nabla \mathbf{u} + \lambda \nabla^2 \mathbf{h}_1 \tag{16}$$

In terms of the Fourier coefficients of the fields $\mathbf{u}$ and $\mathbf{h}_1$, defined by

$$u_i = \int p_i(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \mathbf{dk} \tag{17}$$

$$h_{1i} = \int q_i(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \mathbf{dk} \tag{18}$$

equation (16) may be written

$$\left(\frac{\partial}{\partial t} + \lambda k^2\right) q_i = (\mathbf{h}_0\cdot\mathbf{k})\, p_i \tag{19}$$

with solution

$$q_i(\mathbf{k}, t) = (\mathbf{h}_0\cdot\mathbf{k}) \int_0^t e^{-\lambda k^2 (t-\sigma)}\, p_i(\mathbf{k}, \sigma)\, d\sigma. \tag{20}$$

If we now suppose that the kinetic turbulence remains statistically steady so that Φ_{ij} is independent of t (thus requiring that the energy transferred to the magnetic field must remain small compared with the total energy of the turbulence) then the spectrum $\Gamma_{ij}(\mathbf{k}, t)$ of the field fluctuation can be explicitly derived :

$$\begin{aligned}
\Gamma_{ij}(\mathbf{k}, t) &= \overline{q_i(\mathbf{k}, t)\, q_j^*(\mathbf{k}, t)} \\
&= (\mathbf{h}_0\cdot\mathbf{k})^2 \int_0^t \int_0^t e^{-\lambda k^2 (2t-\sigma-\sigma')}\, p_i(\mathbf{k}, \sigma)\, p_j(\mathbf{k}, \sigma')\, d\sigma\, d\sigma' \\
&= (\mathbf{h}_0\cdot\mathbf{k})^2 \int_0^t \int_0^t e^{-\lambda k^2 (2t-\sigma-\sigma')}\, \Phi_{ij}(\mathbf{k}, \sigma - \sigma')\, d\sigma\, d\sigma' \\
&= (\mathbf{h}_0\cdot\mathbf{k})^2 \int_0^t \int_0^t e^{-\lambda k^2 (\rho+\rho')}\, \Phi_{ij}(\mathbf{k}, \rho' - \rho)\, d\rho\, d\rho'
\end{aligned}$$

$$\text{(writing } \rho = t - \sigma,\ \rho' = t - \sigma'). \tag{21}$$

The star indicates a complex conjugate, and the overbar an ensemble average. $\Phi_{ij}(\mathbf{k}, t)$ here represents the Fourier transform of the space-time velocity correlation and its time dependence is not in general known. Howver, in the case considered here ($R_m \ll 1$) the integral is dominated by values of ρ and ρ' smaller than any characteristic time of the turbulence, so that $\Phi_{ij}(\mathbf{k}, \rho'-\rho)$ may be replaced by $\Phi_{ij}(\mathbf{k})$, the velocity spectrum tensor, (the first term in an expansion in powers of $(\rho'-\rho)$).
Then

$$\Gamma_{ij}(\mathbf{k}, t) = (\mathbf{h}_0 \cdot \mathbf{k})^2 \frac{(1-e^{-\lambda k^2 t})^2}{\lambda^2 k^4} \Phi_{ij}(\mathbf{k}). \tag{22}$$

Golitsyn independently derived with an equivalent approximation the asymptotic form of this relationship assuming the Kolmogorov spectrum for isotropic turbulence,

$$\Phi_{ij}(\mathbf{k}) = C\, \varepsilon^{2/3} k^{-5/3} \cdot \frac{1}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \tag{23}$$

where C is a constant of order unity. Substitution in equation (22) with $t = \infty$, gives

$$\Gamma_{ij}(\mathbf{k}) = C\, h_0^2\, \varepsilon^{2/3} \cos^2\theta \cdot k^{-11/3} \cdot \frac{1}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \tag{24}$$

The factor $\cos^2\theta$, where θ is the angle between $\mathbf{h}_0$ and $\mathbf{k}$, represents the anisotropy that is to be expected because of the preferred direction along $\mathbf{h}_0$. The spectrum, averaged over any sphere in wave number space, falls off as $k^{-11/3}$, i. e., more rapidly than the $k^{-5/3}$ fall-off the velocity spectrum, because the conductive damping of fluctuations increases with wave number more rapidly than the intensification through stretching.

4. The uniform strain attack when $R_m \gg R$

The above analysis is, as already observed, valid only when $R_m \ll 1$. Let us now return to the other extreme case $R_m \gg R$ (i. e. $\lambda \ll \nu$), that is, the case in which an instability to small magnetic perturbation is to be expected, and examine the consequences of applying methods that have already succeeded when applied to scalar fields. In this case of high conductivity the length scale at which conduction becomes important must be small compared with the length scale at which viscous forces control the smallest turbulent eddies. Any element of volume of dimension small compared with $\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ is simultaneously convected, rotated and uniformly strained in the fluid motion, and it is plausible to suppose that the uniform strain is the chief agent in modifying any magnetic field distribution within the element. Batchelor (6) has determined the form of the scalar spectrum at wave-numbers large compared with $\left(\frac{\varepsilon}{\nu^3}\right)^{1/4}$ by considering the effect of a uniform straining motion,

$$\mathbf{u} = (\alpha x, \beta y, \gamma z), \qquad (\alpha + \beta + \gamma = 0) \tag{25}$$

on a random homogeneous distribution of the scalar. We may now ask, what is the effect of such a uniform strain on an initially weak random homogeneous magnetic field distribution ? The answer has been effectively given by Pearson [10] who showed that the action of uniform strain on a weak random homogeneous vorticity distribution was to increase the mean square vorticity without limit. The same mathematics applied to the present problem shows that uniform strain increases the magnetic energy without limit. This is consistent with the conclusion that when $\lambda \ll \nu$ the magnetic energy increases until the magnetic body force intervenes to restrict the growth. But at the same time it indicates that the assumption of uniform constant strain is perhaps inadequate to represent the effect of turbulence on the highest wave number components of the magnetic field. A thorough examination of the effect of Lorentz forces within such small volume elements undergoing uniform strain and also of the effect of allowing the rate of strain tensor to change slowly in time would throw much light on this problem.

5. Turbulence driven by magnetic forces

The essential problem of stationary magnetohydrodynamic turbulence is to follow the flow of energy in wave number space from the two sources (kinetic and magnetic) at low wave-numbers to the two sinks (viscous and conductive) at high wave-numbers. The relative strength of the two sinks is controlled by the ratio $\frac{\nu}{\lambda}$ which is therefore vital in determining the kinetic and magnetic spectra at large wave numbers. Similarly the relative strength of the two sources is equally critical in the specification of the problem. In those problems considered so far the kinetic source (K) has been supposed strong compared with magnetic source (M). Indeed even when $M = 0$ it is likely that a steady state with non-vanishing magnetic field can be maintained if $\lambda \ll \nu$. The other extreme case for which $K = 0$ and only a magnetic source is present is equally interesting, and indeed more relevant to plasma experiments in which strong applied magnetic fields are the only obvious source of energy for the turbulence that is inferred from photographs. This situation has been discussed for a geometry with cylindrical symmetry by Kovasznay [11] who considered extensions of Reynold's equation for mean quantities derivable from equations (3) and (4). The velocity fluctuations were estimated from the balance between Reynolds stress and magnetic stress terms from equation (3), and this led to an estimate of the induced mean electric field, $\overline{\mathbf{u} \wedge \mathbf{h}}$, due to motion across applied field lines. Knowing the mean current, the effective eddy conductivity of the plasma follows; the value obtained by Kovasznay compared favourably with experimental estimates.

The situation considered by Kovasznay ($K = 0$, $\lambda \gg \nu$) is in a sense complementary to that considered by Batchelor ($M = 0$, $\nu \gg \lambda$). Kovasznay's work was motivated by observations of spontaneous turbulence in the presence of applied fields; Batchelor's by the widespread astrophysical phenomenon of spontaneous magnetic fields in the presence of background turbulence. The kinetic and magnetic spectra for Batchelor's case and for the isotropic analogue of Kovasznay's case are sketched in figures 1 and 2, in which this complementarity is pronounced.

To make the foregoing picture of magnetohydrodynamic turbulence less impressionistic, some experimental results are very much required. For example the determination of the amplification factor of a weak applied field in the case $R \gg R_m \gg 1$ would be sufficient to distinguish between the Batchelor standpoint and that of Biermann and Schlüter. The condition $R_m \gg 1$ is unfortunately hard to realise in laboratory conditions, but it

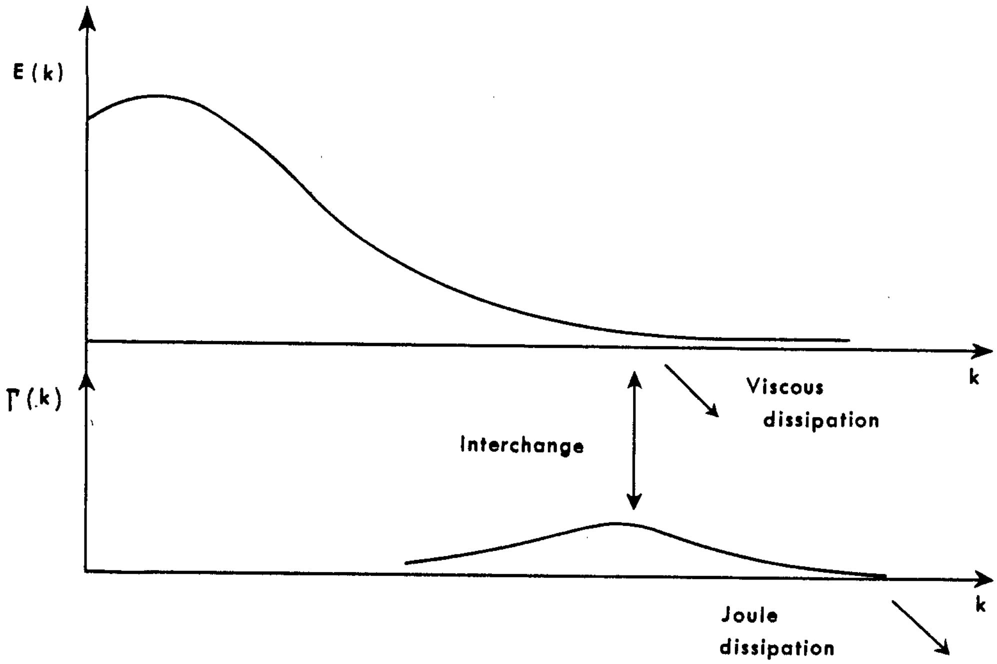

FIGURE 1

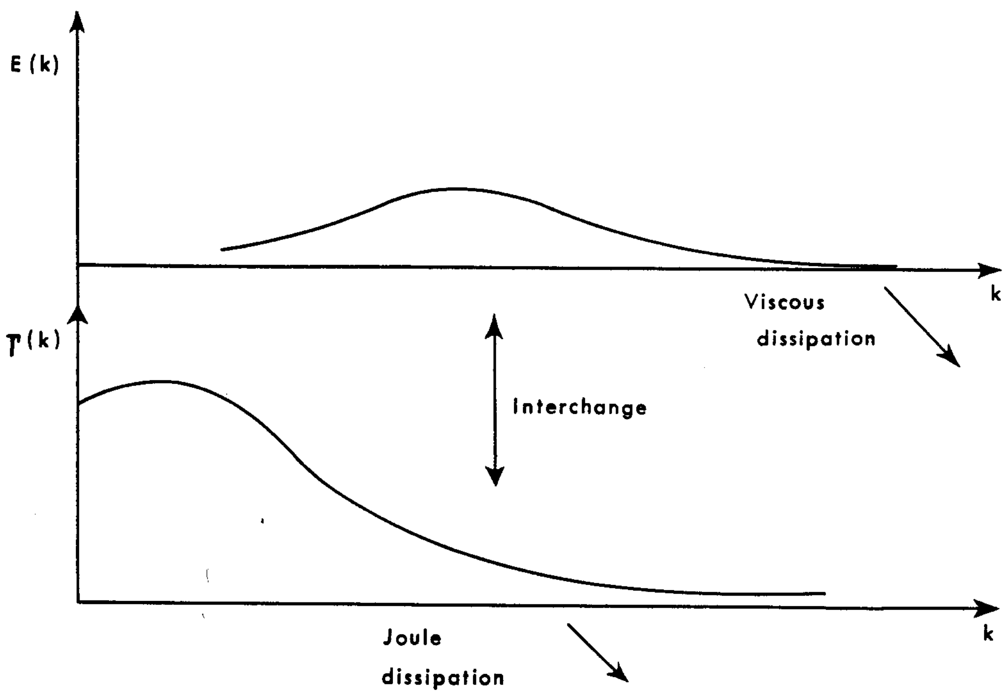

FIGURE 2

may not remain hopelessly beyond experimental technique. The less interesting situation $R_m \ll 1$ offers more scope for experiment. The way has been pioneered by MURGATROYD [12] who demonstrated that turbulence in channel flow of mercury could always be eliminated by applying a sufficiently strong transverse magnetic field. It would be valuable to determine the modification of the turbulence spectrum in the presence of an increasing field before the elimination is complete, and also to repeat the experiment with a longitudinal magnetic field (which does not directly distort the mean velocity profile) as well as with conducting fluids other than mercury. It is a little paradoxical that increasing the magnetic source of energy in Murgatroyd's experiment results in the suppression of turbulence. The reason is that in increasing the applied field a more effective vehicle is supplied for the immediate transfer of energy from the two sources (applied field and pressure drop in this case) to the conductive sink which drains energy efficiently at length scales of the order of the channel diameter. This reasoning only applies when $R_m \ll 1$. Kovasznay's contrasting picture of magnetic-driven turbulence is then relevant to the case $R_m \gg 1$, a condition that did indeed apply in the type of turbulent plasma that he considered.

Let me conclude by summarising the above observations in the following rough classification of types of stationary magnetohydrodynamic turbulence together with the chief situations in which each type may arise.

(a) Kinetic source dominant, weak applied field, $K \gg M$.

(i) $R_m \ll 1$: Small field fluctuations only generated by turbulence (ionosphere, turbulent mercury, liquid sodium etc.)

(ii) $1 \ll R_m \ll R$: Applied field intensified to level controlled by conduction (stellar interiors, regions of the ionosphere)

(iii) $R_m \gg R$: Equipartition at high wave-numbers, even if M is zero (HII regions of interstellar gas)

(b) Magnetic source dominant : strong applied fields, $M \gg K$

(i) $R_m \ll 1$: Suppression of turbulence (experiments on mercury in an increasing field)

(ii) $R_m \gg 1$: Magnetic driven turbulence (hot plasma, stellar interiors)

This is only a tentative scheme of limiting cases. A more thorough examination of the particular cases $K = M$ and $R_m = 1$ might also throw light on the general situation.

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COMMENTAIRE DE LA SECTION
TURBULENCE EN MILIEU COMPRESSIBLE ET ÉLECTRO-CONDUCTEUR

Prof. Leslie S. G. KOVASZNAY, Président

Une session spéciale était consacrée aux effets de la compressibilité et de la présence d'un milieu possédant une conductibilité électrique.

Il semble être un peu prétentieux de s'occuper de ces complications quand la turbulence simple d'un fluide incompressible et non-conducteur présente elle-même des difficultés presque insurmontables.

De nombreuses raisons conduisent à exécuter des recherches dans ces domaines quelque peu ésotériques. Le bruit produit par l'écoulement turbulent est un problème pratique en ce qui concerne les avions à réaction. La turbulence magnéto-hydrodynamique semble devenir un obstacle important au développement des réacteurs thermo-nucléaires contrôlés. Mais, même du point de vue de la recherche de base, cette question est intéressante parce que le cas du fluide incompressible et non-conducteur peut être mieux compris en tant que cas limite du fluide compressible et conducteur.

Monsieur MORKOVIN a discuté les résultats obtenus dans la couche limite turbulente supersonique.

Les mesures de turbulence faites à l'anémomètre à fil chaud dans la couche limite nous ont surpris. Même à un nombre de Mach de 1.75 à 2.00, nous avons constaté que le mécanisme interne de la turbulence diffère peu de celui de la couche limite incompressible. Bien entendu, il y a des fluctuations d'entropie, et même des fluctuations de pression (des ondes acoustiques), mais la véritable turbulence qui possède une divergence nulle, c'est-à-dire la partie incompressible du champ de vitesse, change très peu.

Une des questions essentielles est le comportement des tensions de Reynolds en milieu compressible, et un choix convenable des lignes de courant moyennes la ramène au cas incompressible.

Les spectres des fluctuations ressemblent aussi fortement à ceux des couches limites incompressibles.

Ces fluctuations peuvent être décomposées en trois modes : le mode rotationnel, le mode d'entropie et le mode acoustique. Deux de ces modes sont paraboliques, autrement dit, obéissent à des équations du type conduction de la chaleur. Par contre le mode acoustique est hyperbolique, et obéit à une équation de propagation d'ondes.

Dans un écoulement où la région turbulente est bornée, comme par exemple une couche limite turbulente, ou un jet, ou un sillage, les ondes acoustiques engendrées au sein de la portion turbulente se propagent et peuvent être observées dans l'écoulement extérieur non turbulent.

Monsieur LAUFER nous a présenté les résultats de mesures des fluctuations acoustiques obtenues à l'extérieur de la couche limite supersonique, et a fait aussi la critique des théories existantes sur la production de bruit par la couche limite supersonique. La théorie asymptotique de Phillips (valable à un nombre de Mach infini) se trouve approximativement confirmée. D'ailleurs l'énergie rayonnée est très faible par rapport à la dissipation visqueuse, même à un nombre de Mach très élevé, et par exemple à $M = 5$, elle est de l'ordre de 1 %, ce qui constitue un résultat surprenant.

Diverses considérations sur la turbulence magnéto-hydrodynamique ont été présentées par Monsieur MOFFATT et j'ai apporté personnellement quelques preuves expérimentales de l'existence de la turbulence dans un plasma. Quand le milieu possède une conductibilité électrique, les équations dynamiques (de Navier Stokes) comprennent un terme supplémentaire traduisant la force de Lorentz, qui est une fonction quadratique du champ magnétique. Par contre, l'équation qui gouverne le champ magnétique est linéaire.

Le problème essentiel de l'augmentation de l'énergie magnétique totale par l'agitation de la turbulence cinétique n'est pas résolu d'une façon définitive. D'autre part, un progrès considérable a été apporté dans le cas où le Nombre de Reynolds magnétique est très inférieur au Nombre de Reynolds cinétique. Dans ce cas particulier, le champ magnétique peut être traité par une méthode analogue à celle utilisée pour la diffusion turbulente, à cette différence près que le champ magnétique est une quantité vectorielle transportée d'une façon passive, tandis que la chaleur, ou la concentration d'une matière qui diffuse sont des quantités scalaires. La question expérimentale qui s'avère la plus importante est de trouver des moyens pour réaliser un écoulement turbulent de plasma qui soit simple et bien défini.