Magnetic eddies in an incompressible viscous fluid of high electrical conductivity

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It is shown that in an incompressible fluid in which the magnetic diffusivity λ is much less than the kinematic viscosity ν, certain magnetic field distributions of limited spatial extent (conveniently described as magnetic eddies) can exist on a length scale such that the associated Reynolds number and magnetic Reynolds number are respectively small and large compared with unity. The Lorentz forces are in equilibrium with the dynamic forces associated with the fluid motion. The boundary condition imposed on this motion is that at a large distance from a magnetic eddy the velocity field should be a uniform axisymmetric irrotational straining motion. The eddies are steady in the limit $\lambda \to 0$, but decay slowly in a fluid of finite conductivity. Two particular eddies are considered in detail: a disk-shaped eddy in a compressive straining motion, and a spherical eddy in an extensive straining motion. Possible applications to turbulence in interstellar gas clouds are qualitatively considered.

1. Introduction

In this paper, a fairly general motion of an incompressible fluid of very high conductivity is considered. It will be supposed that the magnetic diffusivity λ is very much smaller than the kinematic viscosity ν, or more strongly, that

$$(\lambda/\nu)^{\frac{1}{4}} \ll 1. \tag{1.1}$$

(It may be necessary to impose a more stringent condition in special cases; see equation (2.35).) In the absence of moving boundaries it is useful to characterize the motion by a typical rate of strain α rather than by a typical velocity. The restriction (1.1) is strong enough to ensure that there exists a range of length scales $l_1 < l < l_2$ within which the Reynolds number $R = \alpha l^2/\nu$ and the magnetic Reynolds number $R_m = \alpha l^2/\lambda$ satisfy the inequalities

$$R^{\frac{1}{2}} \ll 1, \quad R_m^{\frac{1}{2}} \gg 1. \tag{1.2}$$

This combination of circumstances means that the lines of force of any ambient magnetic field are convected with the fluid and can diffuse only very slowly relative to it, whereas vorticity diffuses rapidly so that the motion of the fluid is controlled by viscous rather than by inertial forces.

The restriction on the Reynolds number R is not an essential part of the physical theory to be presented, but it does allow considerable mathematical simplifications. Moreover, the types of solution available in this extreme case may indicate methods of approaching similar problems when R as well as R_m is large compared with unity. The only application that is made of the results of this paper is to those small-scale features of a turbulent velocity field for which the associated Reynolds number *is* small.

In the absence of any electric current or magnetic field distributions, the velocity relative to any point moving with the fluid is approximately a linear function of position throughout any region of dimension l, i.e.

$$u_i = \alpha_{ij} x_j. \tag{1.3}$$

This is an accepted approximation in turbulent flows where l is any length small compared with the scale of the smallest eddies of the turbulence (i.e. $l \ll (\nu^3/\epsilon)^{\frac{1}{4}}$ where ϵ is the rate of dissipation of energy per unit mass of fluid). In a steady laminar flow, the velocity distribution (1.3) may be a reasonable approximation for a large variety of flows, even if the length scale of the region under consideration is such that the associated Reynolds number is of order, or larger than, unity. In general the tensor α_{ij} will not be symmetric and will be time-dependent. Here, for simplicity, attention is restricted to steady flows, irrotational in the absence of electric currents and symmetrical about an axis, so that, referred to the principal axes of strain, equation (1.3) becomes (using the incompressibility condition, $\nabla . \mathbf{u} = 0$)

$$\mathbf{u} = (\alpha x, \alpha y, -2\alpha z). \tag{1.4}$$

This will be taken as the outer boundary condition (as $x_i x_i \to \infty$) in what follows. If $\alpha > 0$, material volumes tend to be flattened and oriented parallel to the (x, y)-plane, while if $\alpha < 0$, they tend to be drawn out into filaments parallel to the z-axis. The strain field may be conveniently described as 'compressive' or 'extensive' in these respective cases.

The main purpose of this paper is to enquire what happens when an axisymmetric 'blob' of magnetic field is present at the centre of such a strain field. Specifically, we suppose that at time $t = 0$ the velocity field is given by equation (1.4) (which is subsequently maintained as an outer boundary condition), and that there is a magnetic field $\mathbf{H}(r, z)$ (in cylindrical polar co-ordinates (r, θ, z)) which is non-zero in the volume V_0 inside a closed surface $|z| = f(r)$, and identically zero outside this surface (on which there may flow a surface current). We suppose in the first instance that the conductivity is infinite ($\lambda = 0$), so that the magnetic field is subsequently confined to the material volume $V(t)$ that was initially V_0.

The Lorentz force associated with the magnetic blob acts as a perturbing force on the uniform strain field. If the Lorentz force is negligible, the strain distorts the volume $V(t)$ into a disk whose dimensions increase as $e^{\alpha t}$ if $\alpha > 0$, or into a filament whose length increases as $e^{-2\alpha t}$ if $\alpha < 0$, and in either case the magnetic lines of force in $V(t)$ are stretched without limit. It seems likely that eventually it will not be possible to ignore the Lorentz forces and that these will ultimately so distort the strain field that the magnetic energy of the blob will not continue to grow without limit. It is not easy to prove this assertion; there is an infinite energy available in a uniform strain field that extends to infinity, and it is not impossible that an infinite energy is transferred from the velocity field to the magnetic field despite the 'back-reaction' of the Lorentz force. In this paper, only a search for possible steady states in which the magnetic energy is finite is attempted; whether these states are the inevitable asymptotic states consequent upon the above initial conditions is left an open question.

The problem posed above arose naturally in a detailed study of the turbulent dynamo problem. The condition $\lambda < \nu$ was first put forward by Batchelor (1950) as the condition that an initially weak random magnetic field should be intensified by a turbulent fluid motion. The intensification is chiefly associated with the small-scale straining, and the approximation (1.3) to the velocity field, though not ideal, is at least a reasonable assumption over regions of small enough dimension. In the limit $\lambda \to 0$, the intensification must ultimately be checked by the Lorentz forces which resist the stretching. (This may still be true when λ is small but non-zero, although Saffman 1963 has recently stressed that the increase in this case may be checked by the increased Ohmic dissipation associated with the decrease in scale of the strained magnetic field.) There is still considerable doubt as to the level attained by the magnetic field even in the limit $\lambda \to 0$. Batchelor estimated that approximate equipartition of energy would be established between the magnetic energy and the kinetic energy of the smallest turbulent eddies. Biermann & Schlüter (1950, 1951) on the other hand estimated that equipartition between magnetic and kinetic modes would ultimately be established at all length scales. These estimates differ by a factor of order the square-root of the overall Reynolds number of the turbulence, which may be huge in astrophysical applications. Hence the importance of examining in more detail the types of balance that can persist between magnetic and velocity fields.

Interest in the problem is not, however, derived solely from the turbulence context. An enormous effort has been devoted in recent years to the problem of determining magnetostatic equilibrium configurations in a perfect conductor, that is, magnetic field configurations that can be supported by pressure gradients in a fluid at rest (see, for example, Grad & Rubin 1959; Kruskal & Kulsrud 1956). The chief stimulus for these investigations comes from thermonuclear fusion research, although the results are also of great interest in astrophysics. If the conductivity is large but finite, localized magnetic field distributions diffuse outwards in a fluid at rest. The dynamo problem is usually understood to signify the problem of determining a fluid motion which can maintain a steady magnetic field against persistent Ohmic decay. Cowling (1934) showed that no axisymmetric motion could reproduce this type of dynamo effect. Bullard (1949), Herzenberg (1958) and others have aimed at determining suitable non-axisymmetric velocity fields, but no attempt (other than qualitative) has been made to satisfy the dynamical equation of motion. In this contribution, the dynamical equation *is* satisfied in an extreme situation ($R \ll 1$), but the motion is axisymmetric and the field decays slowly if the conductivity is finite. By approaching the dynamo problem from this complementary angle, it is hoped that a little light may be thrown on some of the inherent difficulties involved.

2. The disk eddy in a region of compressive strain

In a region of compressive strain ($\alpha > 0$) the volume $V(t)$, as observed above, tends to be flattened into a disk-shaped region near the plane $z = 0$. The tension in the azimuthal lines of force provides an inward force resisting the outward

15-2

straining motion. As this tension increases, the field may eventually succeed in reversing the straining motion within V and thus in generating a disk-shaped eddy in which the streamlines are closed and (as $t \to \infty$) the motion becomes steady as suggested schematically in figure 1. This possibility is now investigated in detail.

It is supposed that the magnetic field $\mathbf{B}$ is purely toroidal

$$\mathbf{B} = \left(\frac{4\pi\rho}{\mu}\right)^{\frac{1}{2}} (0, h(r,z), 0) = \left(\frac{4\pi\rho}{\mu}\right)^{\frac{1}{4}} \mathbf{h}, \quad \text{say,} \tag{2.1}$$

where ρ is the fluid density. If the velocity field $\mathbf{u} = (u, 0, w)$ is expressed in terms of a stream function $\psi(r,z)$ by the usual relations

$$u = -r^{-1}\psi_z, \quad w = r^{-1}\psi_r, \tag{2.2}$$

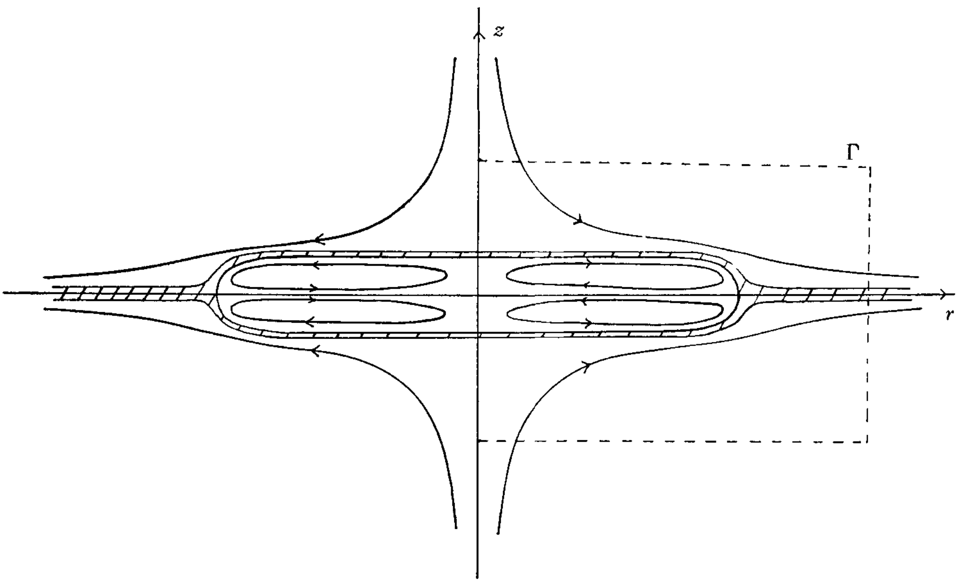

Figure 1. Schematic representation of the disk eddy. The shaded region is the magnetic boundary layer and the magnetic wake. The flux of magnetic field through the material circuit Γ is constant.

then it is readily verified that the solution of the steady induction equation (with $\lambda = 0$),

$$\nabla \wedge (\mathbf{u} \wedge \mathbf{h}) = 0, \tag{2.3}$$

is

$$\mathbf{h} = (0, rH(\psi), 0), \tag{2.4}$$

where $H(\psi)$ is an arbitrary function of ψ.

The dynamic problem that remains is to find a closed surface S (the surface of $V = \lim_{t\to\infty} V(t)$) and a velocity distribution $\mathbf{u}$ satisfying

$$\mathbf{u}.\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} \quad \text{outside } S, \tag{2.5}$$

and

$$\mathbf{u}.\nabla\mathbf{u} = -\frac{1}{\rho}\nabla\chi + \mathbf{h}.\nabla\mathbf{h} + \nu\nabla^2\mathbf{u} \quad \text{inside S}, \tag{2.6}$$

where $\chi = p + \frac{1}{2}\rho h^2$ is the sum of fluid and magnetic pressures inside S. The boundary conditions are that

$$\mathbf{u} \sim (\alpha r, 0, -2\alpha z) \tag{2.7}$$

at large distances, and that u_i and the total force on a surface element of S (with unit normal n_i),

$$-(p+\tfrac{1}{2}\rho h^2)\,n_i+\nu\rho(\partial u_i/\partial x_j+\partial u_j/\partial x_i)\,n_j,$$

should be continuous across S. (The magnetic field has no component across S, so that the contribution $\rho h_i h_j n_j$ to the force on a surface element of S vanishes identically.)

Since the eddy is flattened by the external strain, it is reasonable to look for a solution in which (at least in a first approximation) the volume V is bounded by the planes $z = \pm b$ and the cylinder $r = a$, where $a \gg b$. For large z the stream function is

$$\psi = -\alpha r^2 z, \tag{2.8}$$

and hence in the region $r < a$ sufficiently far from the edge of the disk for edge effects to be negligible, it may be possible to express the stream function in the form

$$\psi = \begin{cases} -r^2 f(z) & \text{for} \quad |z| > b, & (2.9)\\ -r^2 g(z) & \text{for} \quad |z| < b. & (2.10)\end{cases}$$

Substitution of equation (2.9) in equations (2.2) and (2.5) leads to the equation, well known in the context of axisymmetric stagnation flow (Homann 1936; Schlichting 1960),

$$f'^2 - 2ff'' = \alpha^2 + \nu f'''. \tag{2.11}$$

The pressure p is independent of z throughout the region $z > b$. There is a sudden drop in fluid pressure, however, on passing into the eddy region $z > b$, since the total pressure is continuous. (The normal viscous stress is continuous on $z = b$ because the velocity and tangential stress are continuous there—see equation (2.20).) Substitution of equation (2.10) in equations (2.2) and (2.6), using the linking pressure condition,

$$\chi = p \quad \text{on} \quad |z| = b \quad (\text{all } r) \tag{2.12}$$

yields the equation for g

$$g'^2 - 2gg'' = \alpha^2 - H^2(\psi) + \nu g'''. \tag{2.13}$$

Clearly the assumed form (2.10) of ψ is self-consistent only if $H(\psi)$ is equal to a constant, κ say, independent of ψ (otherwise equation (2.13) still involves the variable r). In this case, the magnetic field within V assumes the simple form

$$\mathbf{h} = (0, \kappa r, 0) \tag{2.14}$$

and the equation for g is

$$g'^2 - 2gg'' = \alpha^2 - \kappa^2 + \nu g'''. \tag{2.15}$$

Other magnetic field distributions may also yield steady states, but this is certainly the simplest possibility, and it is the one to which attention is now restricted. The electric current distribution that gives rise to the field (2.14) is a uniform current through V parallel to the z-axis, the current loops being completed by a surface current on S.

It can be shown that the effect of slow Ohmic diffusion within the eddy is (as might be expected) to smooth out current variations within the eddy so that the uniform current distribution chosen above is in a sense the preferred one. A relevant result (in which vorticity, rather than magnetic field, was the slowly diffused quantity) was proved by Batchelor (1956). By integrating the equation of motion round a closed streamline in a region of closed streamlines in axisymmetric flow, he showed that the azimuthal component of vorticity must be proportional (in the steady state) to the distance r from the axis. The same reasoning applies here to magnetic field, so that the distribution (2.14) is the one which Ohmic diffusion slowly selects. Unfortunately this same diffusion is also responsible for a slow leakage of flux from the eddy (to be estimated later in this section) and it seems likely that, starting from an arbitrary initial distribution of the form (2.4), the eddy would disappear before the particular distribution (2.14) could be established by Ohmic diffusion alone. It is, nevertheless, possible to imagine a situation in which magnetic flux is continuously supplied to the eddy in which case the solution preferred by Ohmic diffusion is likely to be the relevant one. (In the situations envisaged by Batchelor, some vorticity would likewise leak from the region of closed streamlines, to be replaced by an input of vorticity generated at a solid boundary.)

Now the Reynolds number of the flow within V is assumed small so that the non-linear terms of equation (2.15) may be neglected. The simplified equation may then be integrated with the boundary conditions that the velocity profile is symmetrical about the plane $z = 0$, and that the plane $z = b$ is a stream-surface, i.e.

$$g(0) = g(b) = 0, \quad g''(0) = 0, \tag{2.16}$$

to give

$$g(z) = (\kappa^2 - \alpha^2)\, z(z^2 - b^2)/6\nu. \tag{2.17}$$

The velocity inside V therefore has components

$$u = rg'(z) = (\kappa^2 - \alpha^2)\, r(3z^2 - b^2)/6\nu, \tag{2.18}$$

$$w = -2g = -(\kappa^2 - \alpha^2)\, z(z^2 - b^2)/3\nu. \tag{2.19}$$

The profile within the disk is thus parabolic, and, since the stress must be continuous at the surface of the disk, it can be matched to the solution outside only if $\kappa^2 > \alpha^2$, when the profile is convex to the origin; this is a necessary condition on the strength of the magnetic field if a disk eddy of this type with closed streamlines is to be possible.

Equation (2.11) must now be solved for $f(z)$ subject to the boundary conditions

$$\left.\begin{aligned} f'(\infty) &= \alpha, \\ f(b) &= 0, \\ f'(b) &= g'(b) = (\kappa^2 - \alpha^2)\, b^2/3\nu, \\ f''(b) &= g''(b) = (\kappa^2 - \alpha^2)\, b/\nu. \end{aligned}\right\} \tag{2.20}$$

The last two follow from the continuity of velocity and tangential stress on the surface $z = b$. Three of these conditions determine a solution of equation (2.11); the fourth will determine the thickness b of the disk. The non-linear terms cannot be neglected in this equation, because they dominate for large z.

Now Homann determined the particular solution $f_1(z)$ of equation (2.11) that satisfies the stagnation flow boundary conditions

$$f_1'(\infty) = \alpha, \quad f_1(0) = 0, \quad f_1'(0) = 0,$$

by expanding the solution in powers of z (for small z) and z^{-1} (for large z) and matching the two solutions. The first few coefficients in the following expansion for small z

$$f_1(z) = (\nu\alpha)^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n (\alpha/\nu)^{\frac{1}{2}n} z^n, \tag{2.21}$$

were found to be

$$a_0 = a_1 = 0, \quad a_2 = 0{\cdot}66, \quad a_3 = 0{\cdot}16, \quad a_4 = a_5 = 0, \dots. \tag{2.22}$$

A more general solution satisfying the condition $f'(\infty) = \alpha$ is

$$f(z) = c + f_1(z - z_0), \tag{2.23}$$

where c and z_0 are constants; these can be chosen so that the remaining conditions of the set (2.20) are satisfied. Clearly the condition $f(b) = 0$ determines c. In the remaining two conditions only the first non-vanishing term of the series in the expression (2.23) need be retained, since when z is of order b (and provided z_0 is also of order b, as the subsequent analysis confirms (see equation 2.26)), the ratio of each term to its successor is of order $(\nu/\alpha)^{\frac{1}{2}} b^{-1}$, and this is at least of order $R^{-\frac{1}{2}}$, i.e. much greater than unity by the assumption (1.2) (assuming that b lies in the range of length scales for which (1.2) is valid).

The conditions are therefore

$$\left.\begin{aligned} 2a_2(b - z_0)(\alpha^3/v)^{\frac{1}{2}} &= (\kappa^2 - \alpha^2)\, b^2/3\nu \\ \text{and} \qquad 2a_2(\alpha^3/\nu)^{\frac{1}{2}} &= (\kappa^2 - \alpha^2)\, b^2/\nu, \end{aligned}\right\} \tag{2.24}$$

which may be solved immediately to give

$$b = 1{\cdot}32(\alpha^3\nu)^{\frac{1}{2}} (\kappa^2 - \alpha^2)^{-1} \tag{2.25}$$

and

$$z_0 = \tfrac{2}{3}b. \tag{2.26}$$

It is now apparent that the solution is self-consistent only if $\kappa^2 \gg \alpha^2$, for only then is $b \ll (\nu/\alpha)^{\frac{1}{2}}$ as the model requires. In fact from (2.25), b is then given approximately by

$$b \approx 1{\cdot}32(\nu/\alpha)^{\frac{1}{2}} (\alpha^2/\kappa^2), \tag{2.27}$$

or equivalently

$$\kappa^2/\alpha^2 \approx 1{\cdot}32 R_b^{-\frac{1}{2}}, \tag{2.27a}$$

where R_b is the Reynolds number based on the length scale b.
If the total volume of the eddy is τ, its radius a is then given by

$$\pi a^2 b = \tau. \tag{2.28}$$

If κ^2 is *not* very much greater than α^2 an eddy with closed streamlines may still exist but clearly the flat disk approximation is not then appropriate.

The simplest interpretation of the ratio κ/α is that κ^2/α^2 is the ratio of the magnetic energy in the disk

$$\frac{1}{2}\int_{-b}^{b}\int_{0}^{a} \kappa^2 r^2 \,.\, 2\pi r\, dr\, dz$$

to the kinetic energy of the fundamental strain field

$$\frac{1}{2}\int_{-b}^{b}\int_{0}^{a} \alpha^2 r^2 \,.\, 2\pi r\, dr\, dz$$

(neglecting b^2 compared with a^2) that would exist in the same volume if the eddy were not present to distort it. Alternatively, since the electric current inside the eddy is

$$j = \frac{1}{4\pi}\left(\frac{4\pi\rho}{\mu}\right)^{\frac{1}{2}} \nabla \wedge \mathbf{h} = \left(0, 0, \left(\frac{\rho}{4\pi\mu}\right)^{\frac{1}{2}} \kappa\right),$$

the ratio κ/α may be interpreted as giving a measure of the strength of this current compared with the rate of strain constant α.

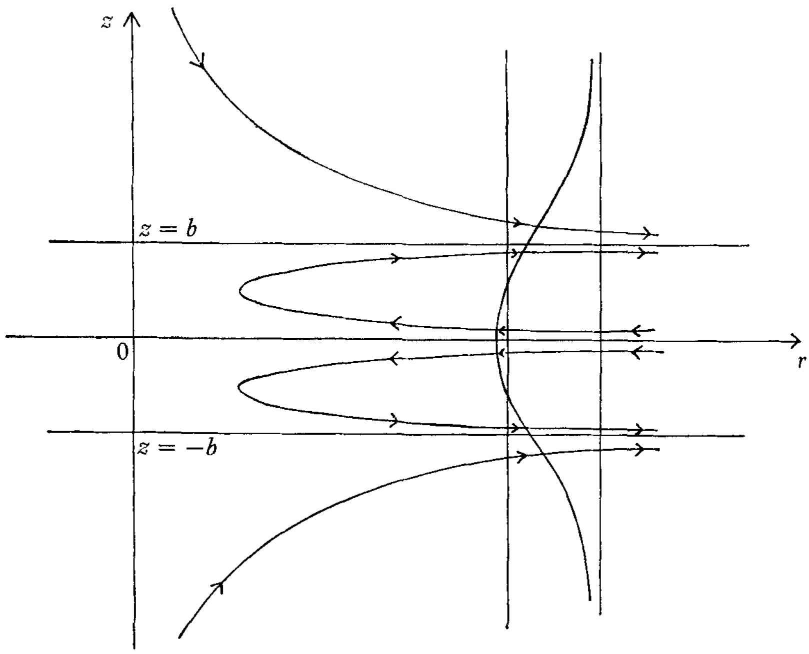

FIGURE 2. Streamlines and velocity profile for a disk eddy of infinite radius.

The streamlines and radial velocity profile determined by the above solution are sketched in figure 2. The solution of course gives no information about the flow near the rim of the disk, where, as suggested in figure 1, the streamlines must close. The solution strictly describes only a disk of infinite radius; it is not certain that a similar steady solution describing a disk of finite radius exists at all. The difficulties encountered in searching for such a solution are similiar to, if not greater than, those encountered in investigations of the flow due to a rotating disk of finite radius, and they have not been overcome in either problem.

The real significance of the above solution is that it demonstrates the possibility of a balance between magnetic and dynamic forces. If Lorentz forces are negligible, the intensity of any magnetic field in a perfect conductor increases exponentially under uniform strain. Here, in one case, albeit specialized, it has been demonstrated that the effect of Lorentz forces is to admit the possibility of a steady state; and it seems likely (although this has not been proved) that any magnetic field initially confined near the (x, y)-plane in a region of compressive strain will either approach a steady state similar to that of the disk eddy, or at least will ultimately fluctuate about such a state.

Effects of finite conductivity

If the conductivity is large but finite, the discontinuity in the magnetic field across the surfaces $z = \pm b$ of the disk will be smoothed out in some form of magnetic boundary layer, and, if the disk is of finite radius, some magnetic flux will be convected radially outwards into a magnetic wake. These features may be analysed simply as follows.

Near both sides of the surface $z = b$ of the disk, the velocity distribution (2.18) and (2.19) (continuous across the surface), with neglect of α^2 compared with κ^2, is

$$\mathbf{u} = \frac{\kappa^2 b^2}{3\nu}(r, 0, -2(z-b)). \tag{2.29}$$

In the magnetic boundary layer, the field $\mathbf{h}$ satisfies the equation

$$\nabla \wedge (\mathbf{u} \wedge \mathbf{h}) + \lambda \nabla^2 \mathbf{h} = 0, \tag{2.30}$$

with boundary conditions

$$\mathbf{h} = 0 \quad \text{and} \quad \mathbf{h} = (0, \kappa r, 0)$$

at the outer and inner edges of the layer respectively. A trial solution of the form

$$\mathbf{h} = (0, rF(z), 0) \tag{2.31}$$

gives an equation for $F(z)$

$$2\frac{\kappa^2 b^2}{3\nu}(z-b)\frac{dF}{dz} + \lambda \frac{d^2F}{dz^2} = 0, \tag{2.32}$$

with relevant solution

$$F(z) = \frac{\kappa^2 b}{3\pi\lambda\nu}\int_z^\infty \exp\left\{\frac{-\kappa^2 b^2}{3\pi\lambda\nu}(\zeta - b)^2\right\} d\zeta, \tag{2.33}$$

an error function tending to a step-function as $\lambda \to 0$. The layer on $z = -b$ is of course the mirror image in the plane $z = 0$ of that on $z = +b$. The thickness of the layer is of order

$$\delta = (\lambda\nu/\kappa^2 b^2)^{\frac{1}{2}} \approx (\lambda/\alpha)^{\frac{1}{2}}(\kappa/\alpha), \tag{2.34}$$

on using the expression (2.27) for the thickness b. The condition $\delta \ll b$ is then satisfied provided

$$(\lambda/\nu)^{\frac{1}{2}} \ll \alpha^3/\kappa^3, \tag{2.35}$$

or equivalently, using (2.27*a*) and the fact that $(\lambda/\nu)^{\frac{1}{2}} = (R/R_m)^{\frac{1}{2}}$,

$$R_m^{\frac{1}{2}} \gg R^{-\frac{1}{4}}. \tag{2.35a}$$

It is worth remarking that the contribution $\mathbf{h}.\nabla\mathbf{h}$ to the Lorentz force, that cannot be compensated by the steep pressure gradient that exists in the layer, is of the form $-r[F(z)]^2\,\mathbf{u}_r$ (where $\mathbf{u}_r$ is a unit vector in the radial direction). This increases smoothly from zero outside the eddy to its value $-r\kappa^2\mathbf{u}_r$ inside, so that no wild departures from the velocity distribution (2.29) are to be expected within the layer.

For a disk eddy of finite radius, the inner part of this layer is presumably swept round into the interior of the disk giving rise to some kind of magnetic shear layer on the plane $z = 0$. The outer part is swept radially outwards by the external strain into a magnetic wake. Far downstream, equation (2.30) with $\mathbf{u} = (\alpha r, 0, -2\alpha z)$ admits the similarity solution

$$\mathbf{h} = (0, h_0 a r^{-1} \exp(-\alpha z^2/\lambda), 0), \tag{2.36}$$

where h_0 is a measure of the maximum field strength that leaks from the eddy. It has already been remarked that an eddy can form only if $\kappa > \alpha$; it therefore seems possible that the outer part of the boundary layer (2.31) satisfying $F(z) < \alpha$ will escape into the wake and that the inner part where $F(z) > \alpha$ will be drawn back into the eddy. If this is true, then $h_0 = O(\alpha a)$ and the field in the wake

$$\mathbf{h} = (0, C\alpha a^2 r^{-1} \exp(-\alpha z^2/\lambda), 0), \tag{2.37}$$

where C is a constant of order unity. This will hold only for $r \gg a$.

The eddy decays slowly as the toroidal magnetic flux escapes into the wake. It is possible to estimate the rate of decay knowing the form of the magnetic wake (2.37). Consider the flux Φ of the field $\mathbf{h}$ (proportional to the usual magnetic flux) through the material circuit Γ in the plane $\theta = 0$, sketched in figure 1, consisting of the axis of the disk, two lines $|z| = z_1 \gg b$, and a line $r = r_1 \gg a$, cutting the wake. The diffusive transport of flux across Γ is $-\lambda \oint_\Gamma \nabla \wedge \mathbf{h} . \mathbf{dl}$ and this is zero since $\nabla \wedge \mathbf{h}$ is radial in the wake and zero elsewhere on Γ. Hence Φ is constant as Γ moves with the fluid. The contribution to Φ through the eddy itself is of order $a^2 b\kappa$ which may decrease through a decrease in a^2 or b or κ. In a time dt, the part of Γ crossing the wake moves a distance $\alpha r_0 \, dt$, so that, using equation (2.37), the contribution to Φ from the wake increases by an amount

$$C \frac{\alpha a^2}{r_0} \alpha r_0 \, dt \int_{-\infty}^{\infty} e^{-\alpha z^2/\lambda} dz = C a^2 \alpha^2 dt \left(\frac{\pi\lambda}{\alpha}\right)^{\frac{1}{2}}.$$

This must be compensated by the leakage of flux from the eddy, so that

$$\frac{d}{dt}(a^2 b\kappa) = -2C' a^2 \alpha^2 \left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}},$$

where C' is another constant of order unity. Since the magnetic field is convected away from the rim of the eddy, it seems likely that only a^2 decreases in time and that κ (and consequently b from equation (2.27)) remain constant. Then

$$\frac{da^2}{dt} = 2C'(\lambda\alpha)^{\frac{1}{2}} \frac{a^2 \alpha}{b\kappa},$$

so that the radius of the disk decreases according to the law

$$a(t) = a(0) \exp[-C'(\lambda\alpha)^{\frac{1}{2}} \alpha \kappa^{-1} b^{-1} t]. \tag{2.38}$$

The volume of the eddy therefore decreases, which is not surprising, because as the magnetic field leaks away from the eddy its power to stop the straining motion becomes less and less effective. The half-life of the eddy, again using the relation (2.27), is of order $(\nu/\lambda)^{\frac{1}{2}} \kappa^{-1}$.

3. Regions of extensive strain; the spherical eddy

In regions of extensive strain magnetic loops will be drawn out parallel to the z-axis. A search for steady cigar-shaped magnetic eddies is unsuccessful because the magnetic lines of force, which must coincide in the steady state with the streamlines within the eddy, are not suitably arranged to hold the eddy together against the disruptive stress of the external strain.

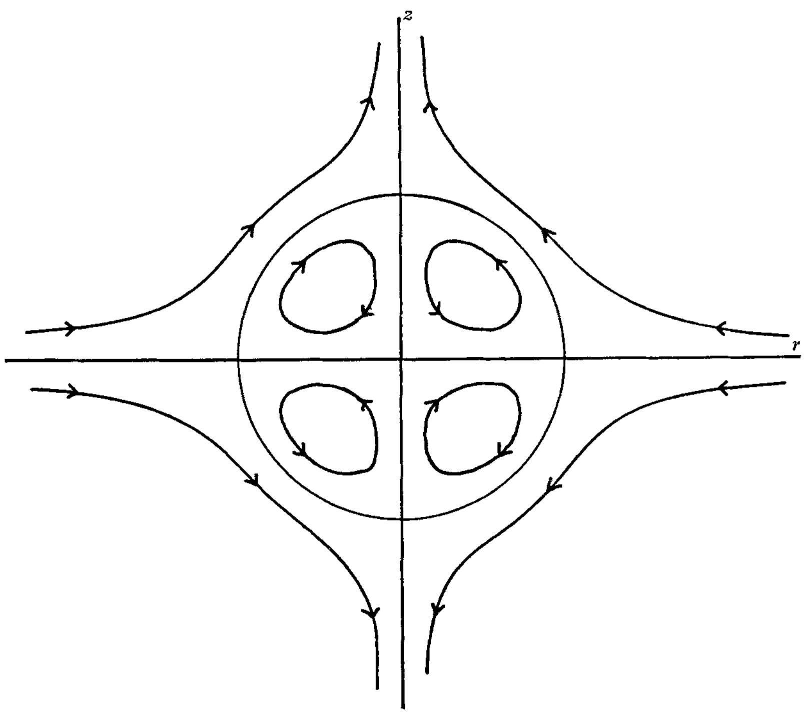

FIGURE 3. Streamlines for the spherical eddy in a perfectly conducting fluid.

There is one interesting solution, however, in regions of axisymmetric extensive strain, which describes a spherical eddy in which the electric current is again uniform and parallel to the axis of symmetry (figure 3). The magnetic field in such an eddy (of radius a) in spherical polar co-ordinates (r, θ, ϕ) is

$$\mathbf{h} = (0, 0, \kappa r \sin\theta), \tag{3.1}$$

and the equations of motion, neglecting non-linear terms, are

$$-\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} = 0 \quad (r > a) \tag{3.2}$$

and

$$-\frac{1}{\rho}\nabla\chi + \mathbf{h}.\nabla\mathbf{h} + \nu\nabla^2\mathbf{u} = 0 \quad (r < a). \tag{3.3}$$

The Stokes approximation is justified here, as for the problem of low Reynolds number streaming past a sphere, by the fact that a solution is found which satisfies the outer boundary condition sufficiently accurately within the region in which inertia forces are negligible.

Using the expression (3.1), equation (3.3) may be written in the form

$$-\frac{1}{\rho}\nabla q+\nu\nabla^2\mathbf{u} = 0, \tag{3.4}$$

where

$$q = \chi+\tfrac{1}{2}\rho\kappa^2r^2\sin^2\theta. \tag{3.5}$$

The velocity $\mathbf{u} = (u, v, 0)$ may be derived from a Stokes stream function $\psi(r,\theta)$ by the relations

$$u = (r^2\sin\theta)^{-1}\psi_\theta, \quad v = -(r\sin\theta)^{-1}\psi_r, \tag{3.6}$$

and the outer boundary condition is

$$\psi \sim -\alpha r^3\sin^2\theta\cos\theta \quad \text{as} \quad r\to\infty. \tag{3.7}$$

A solution is therefore sought of the form

$$\psi = \begin{cases} f(r)\sin^2\theta\cos\theta & (r > a) \\ g(r)\sin^2\theta\cos\theta & (r < a) \end{cases} \tag{3.8}$$

with boundary conditions (on $r = a$)

$$f(a) = g(a) = 0, \quad f'(a) = g'(a), \quad f''(a) = g''(a), \tag{3.9}$$

the last reflecting continuity of tangential stress across the surface of the sphere.

Equations (3.2) and (3.5) imply that ψ everywhere satisfies the Stokes equation

$$D^4\psi \equiv \left(\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)^2\psi = 0. \tag{3.10}$$

Hence

$$\left.\begin{aligned} \left(\frac{\partial^2}{\partial r^2}-\frac{6}{r^2}\right)^2 f = 0 \quad (r > a), \\ \left(\frac{\partial^2}{\partial r^2}-\frac{6}{r^2}\right)^2 g = 0 \quad (r < a), \end{aligned}\right\} \tag{3.11}$$

with general solutions

$$\left.\begin{aligned} f &= A_1r^5+B_1r^3+C_1+D_1r^{-2}, \\ g &= A_2r^5+B_2r^3+C_2+D_2r^{-2}. \end{aligned}\right\} \tag{3.12}$$

Conditions at $r = 0$ and $r = \infty$ imply that

$$A_1 = C_2 = D_2 = 0, \quad B_1 = -\alpha;$$

then the four conditions (3.9) determine C_1, D_1, A_2 and B_2, giving

$$\left.\begin{aligned} f &= -\tfrac{1}{4}\alpha r^3(4-7a^3/r^3+3a^5/r^5), \\ g &= \tfrac{3}{4}\alpha r^3(1-r^2/a^2). \end{aligned}\right\} \tag{3.13}$$

We have yet to satisfy the condition that the normal stress should be continuous across $r = a$, i.e.

$$\chi = p \quad \text{or} \quad r = a \quad (\text{all}\,\theta). \tag{3.14}$$

From equations (3.2) and (3.13), the pressure is

$$p/\rho = (21\nu\alpha a^3/2r^3)\sin^2\theta-21\nu\alpha a^3/4r^3+p_\infty/\rho,$$

and, from equation (3.4),

$$q/\rho = -(63\nu\alpha r^2/4a^2)\sin^2\theta+21\nu\alpha r^2/2a^2+q_0/\rho,$$

so that, using the relation (3.5), the condition (3.14) gives

$$2a^2\kappa^2 = -105\nu\alpha, \tag{3.15}$$

once again relating the electric current density (proportional to κ) to the dimension a of the eddy. Only if $\alpha < 0$, i.e. only in regions of extensive strain, can this flow exist. The gross effect of the magnetic forces is to reverse the rates of strain in the neighbourhood of the origin (as for the disk eddy), so that a steady state can be realized. The streamlines for the flow described by the stream function (3.8) and (3.13) are sketched in figure 3.

4. Application to turbulence in interstellar gas clouds

Although the magnetic eddies described above are certainly of mathematical interest, the conditions (1.2) under which they may materialize are rather extreme, and the physical relevance of the solutions requires some discussion. The condition (1.1) is satisfied in hot HII interstellar gas clouds, for which the usual estimates of temperature T, number density of hydrogen ions n, and length scale L are

$$T \approx 10^4\,°\mathrm{K}, \quad n \approx 1\,\mathrm{cm}^{-3}, \quad L \approx 10^{19}\,\mathrm{cm}. \tag{4.1}$$

Molecular transport theory for a totally ionized hydrogen plasma gives the following expressions for λ and ν (Spitzer 1956; Kantrowitz & Petschek 1957):

$$\lambda = 5{\cdot}19 \times 10^{11} T^{-\frac{3}{2}} \ln \Lambda, \tag{4.2}$$

$$\nu = 1{\cdot}81 \times 10^{9} T^{\frac{5}{2}} (n \ln \Lambda)^{-1}, \tag{4.3}$$

where Λ is a collision parameter, depending only on T and n, and tabulated by Spitzer. Substituting the particular values (4.1) gives

$$\lambda \approx 1{\cdot}2 \times 10^{7}\,\mathrm{cm}^2/\mathrm{sec}, \quad \nu \approx 7{\cdot}8 \times 10^{17}\,\mathrm{cm}^2/\mathrm{sec}, \tag{4.4}$$

$$\lambda/\nu \approx 1{\cdot}5 \times 10^{-9}, \tag{4.5}$$

so that the condition (1.1) is indeed satisfied. The velocity of sound, corresponding to the conditions (4.1), is of order 10 km/sec. An irregular component of fluid velocity U (say) of order 5 to 7 km/sec is observed in these clouds (Burbidge 1959) so that the motion is subsonic, though perhaps not incompressible. At the same time it is likely to be highly turbulent; the overall Reynolds number based on the above estimates is of order 10^8. In this case the kinetic energy cascades through the turbulent spectrum and is dissipated at a rate (per unit mass)

$$\epsilon \approx U^3 L^{-1} \approx 10\,\mathrm{cm}^2\,\mathrm{sec}^{-3}. \tag{4.6}$$

The length scale of the smallest turbulent eddies is

$$l_v \approx (\nu^3/\epsilon)^{\frac{1}{4}} \approx 10^{13}\,\mathrm{cm}. \tag{4.7}$$

Throughout any region of fluid small compared with l_v the strain field is uniform and the associated Reynolds number small. Moreover, the small-scale velocity differences occurring in these regions are an order of magnitude smaller than the large scale velocity U, so that on the small scale the motion may certainly be treated as incompressible. These conditions are then conducive to the formation of magnetic eddies of the type described in §§ 2 and 3. The dimension of these eddies would be so small (on a cosmical scale) that they could not be individually detected. If a large number of eddies were distributed throughout the fluid, however, one eddy (say) in each region of fluid of dimension l_v in which the strain is approximately uniform, then they would all contribute to a stationary random magnetic field whose statistical properties and effects might be observable.

Suppose, for the sake of argument, that there is one disk eddy of radius $a = O(l_v)$ (the maximum possible) within every sphere of compressive strain of radius l_v, and one spherical eddy of radius $a = O(l_v)$ in every sphere of extensive strain of radius l_v. The eddies are then packed as closely as possible at any instant without mutual interference. The magnetic energy within the disk eddy is

$$M_d \approx 2\int_0^b \int_0^a \tfrac{1}{2}\rho h^2 \,.\, 2\pi r\, dr\, dz$$
$$\approx \rho a^4(\alpha^3\nu)^{\frac{1}{2}}$$

using equations (2.14) and (2.27), and similarly the magnetic energy in a spherical eddy of radius a, M_s say, is of order $\rho\nu\alpha a^3$. Under the close packing assumption, the magnetic energy density is

$$M \approx (M_d \text{ or } M_s)/\tfrac{4}{3}\pi l_v^3 = O[\rho(\epsilon\nu)^{\frac{1}{2}}], \tag{4.8}$$

using $a = O(l_v)$, $l_v = O(\nu^3/\epsilon)^{\frac{1}{4}}$, $\alpha = O(\epsilon/\nu)^{\frac{1}{2}}$. Thus the magnetic energy density is of the same order of magnitude as the kinetic energy density of the small-scale motion; as pointed out by Batchelor (1950), this is the only dimensional possibility if it is the average strain which controls the magnetic energy density level.

The particular values for ϵ, ν and ρ ($\approx 10^{-24}n$) given above imply a magnetic energy density.

$$\frac{\overline{B^2}}{8\pi\mu} = M \approx 2\times 10^{-15}\,\text{erg/c.c.},$$

corresponding to a root-mean-square magnetic field (with $\mu \approx 1$)

$$(\overline{B^2})^{\frac{1}{2}} \approx 3\times 10^{-7}\,\text{Gauss}. \tag{4.9}$$

This is weaker than currect estimates of the general galactic magnetic field ($\sim 5\times 10^{-5}$ Gauss) based on polarization measurements. However, the evidence also suggests that this general field is uniform over length-scales much greater than 10^{13} cm and is roughly aligned along the spiral arms of the galaxy, suggesting that it is not generated by the internal turbulence of the galaxy, but that it is either generated by a regular large-scale motion or is of extra-galactic origin. There may, nevertheless, exist certain hot clouds of high conductivity which cannot be penetrated by a general uniform field, but within which a genuine turbulent magnetic field of the type described above, whose r.m.s. intensity is of order 10^{-7} Gauss, may be supported.

The ratio λ/ν, from the expressions (4.2) and (4.3), is proportional to T^{-4} (apart from the weakly varying logarithmic factor), and therefore decreases fairly rapidly with temperature. It may, therefore, be much less than unity in a hot laboratory plasma, but it is hardly realistic to apply the ideas of this paper in such circumstances, because, first, such a plasma is usually permeated by a strong applied magnetic field which controls (rather than is controlled by) the dynamics of any plasma motion, steady or turbulent, and secondly, at the high temperatures at which $\lambda/\nu \ll 1$ (based on the expressions (4.2) and (4.3)) the mean free path of ions and electrons is much greater than the dimensions of the apparatus (a criticism that cannot be levelled at the cosmic laboratory) and the conventional equations of magnetohydrodynamics are of dubious applicability.

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