

VISCOUS EDDIES NEAR A SHARP CORNER

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1. Introduction

If viscous fluid flows near a corner between two fixed rigid walls intersecting at an acute angle, then sufficiently near the stagnation point at the intersection \mathcal{O} , the flow will be slow enough to be described by the Stokes (slow motion) equations. In terms of the stream function $\Psi(r, \theta)$ (in plane polar coordinate) this is simply the biharmonic equation

(1.1) $\nabla^4 \Psi = 0,$

and the boundary conditions on the two planes, $\theta = \pm\alpha$ say, are that the two components of velocity must vanish:

(1.2) $u = \frac{1}{r} \frac{\partial \Psi}{\partial \theta} = 0, \quad v = -\frac{\partial \Psi}{\partial r} = 0 \quad \text{on} \quad \theta = \pm\alpha.$

It might be anticipated that, sufficiently near the corner, the flow pattern may be independent of the stirring forces far from the corner that agitate the fluid. Rayleigh (1920) attempted to describe this flow by assuming that the stream function near the corner was of the form

(1.3) $\Psi \sim r^\lambda f(\theta),$

but he assumed that λ must be an integer, and he was consequently unable to find a function $f(\theta)$ satisfying the four boundary conditions (1.2). DEAN and MONTAGNON (1948) reconsidered the problem, and showed that if the angle 2α is less than a certain critical angle (approximately 146°) then λ is necessarily complex, but they did not explore the consequences of this rather surprising conclusion. The problem as posed by DEAN and MONTAGNON was incomplete to the extent that the geometry of the stirring forces far from the corner was not specified. Part of my aim in this talk is to find a complete solution corresponding to rather special outer boundary conditions and to verify that (1.3) is the correct asymptotic form for Ψ near the corner: and part is to show that the complex exponent λ implies the existence of a geometrical progression of eddies as the corner is approached.

I shall suppose that the fluid motion is generated by the insertion of two moving "sleeves" inserted in the walls $\theta = \pm\alpha$ in the region $a \leq r \leq b$ (Fig. 1). The sleeves

move with velocity $\pm V$ as indicated, and it is assumed that the Reynolds number Re is small:

$$(1.4) \quad Re = Vb/\nu \ll 1,$$

where ν is the kinematic viscosity of the fluid. One would expect such an arrangement to generate an eddy as indicated in the space between the planes. The question posed here is: what is the nature of the flow for $r \rightarrow 0$ and for

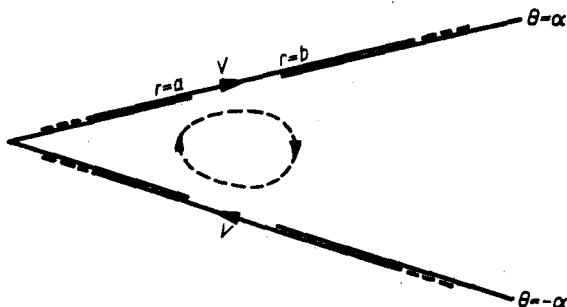


FIG. 1. Flow in a corner induced by the motion of two sleeves in the region $a < r < b$ of the walls $\theta = \pm \alpha$.

$r \rightarrow \infty$? The boundary conditions (1.2) now need to be modified in the following manner:

$$(1.5) \quad \frac{1}{r} \frac{\partial \Psi}{\partial \theta} = \begin{cases} 0 & (r < a), \\ \pm V & (a \leq r \leq b), \\ 0 & (r > b), \end{cases}$$

$$\frac{\partial \Psi}{\partial r} = 0, \quad \text{on} \quad \theta = \pm \alpha.$$

For both small and large r the velocity tends to zero and this imposes further conditions on Ψ . We shall assume such conditions as we require and verify that they are satisfied by the (unique) solution when we have found it.

2. Mathematical Solution of the Problem

A method suitable for solving the problem defined above has been suggested by TRANTER (1948) in the context of plane-strain wedge problems in elasticity. The method is to use the Mellin transform $\bar{\Psi}(p, \theta)$ defined by

$$(2.1) \quad \bar{\Psi}(p, \theta) = \int_0^{\infty} r^{p-1} \Psi(r, \theta) dr,$$

in which p may be complex, its real part being such that the integral exists. Assuming Ψ is such that

$$r^{p+n} \frac{\partial^n \Psi}{\partial r^n}, \quad n = 0, 1, 2, 3; \quad r^p \frac{\partial^n \Psi}{\partial \theta^n}, \quad n = 1, 2$$

and

$$r^{p+1} \frac{\partial^3 \Psi}{\partial r \partial \theta^2}$$

all tend to zero both as $r \rightarrow 0$ and as $r \rightarrow \infty$, integration by parts gives:

$$(2.2) \quad \int_0^\infty r \frac{\partial^{n+1} \Psi}{\partial r \partial \theta^n} r^{p-1} dr = -p \frac{d^n \bar{\Psi}}{d\theta^n}, \quad n = 0, 1, 2,$$

$$\int_0^\infty r^2 \frac{\partial^{n+2} \Psi}{\partial r^2 \partial \theta^n} r^{p-1} dr = p(p+1) \frac{d^n \bar{\Psi}}{d\theta^n}, \quad n = 0, 2,$$

$$\int_0^\infty r^3 \frac{\partial^3 \Psi}{\partial r^3} r^{p-1} dr = -p(p+1)(p+2) \bar{\Psi},$$

$$\int_0^\infty r^4 \frac{\partial^4 \Psi}{\partial r^4} r^{p-1} dr = p(p+1)(p+2)(p+3) \bar{\Psi}.$$

In plane polar coordinates, (1.1) becomes

$$(2.3) \quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 \Psi = 0.$$

Multiplication by r^{p+3} and integration from $r = 0$ to $r = \infty$, give

$$(2.4) \quad \frac{d^4 \bar{\Psi}}{d\theta^4} + [(p+2)^2 + p^2] \frac{d^2 \bar{\Psi}}{d\theta^2} + p^2(p+2)^2 \bar{\Psi} = 0.$$

The boundary conditions on $\bar{\Psi}$ are obtained by multiplying Eqs. (1.5) by r^p and integrating from $r = 0$ to $r = \infty$, giving

$$(2.5) \quad \bar{\Psi} = 0, \quad \frac{d\bar{\Psi}}{d\theta} = \pm \frac{V}{p+1} (b^{p+1} - a^{p+1}) = \pm U(p),$$

say, on $\theta = \pm \alpha$.

Now the flow is clearly antisymmetric about the line $\theta = 0$, so that Ψ , and $\bar{\Psi}$, are symmetric functions of θ . The general symmetric solution of (2.4) is

$$\bar{\Psi} = A \cos p\theta + B \cos (p+2)\theta,$$

where A and B are constants, and if these are chosen so that (2.5) is satisfied, then, after some reduction,

$$(2.6) \quad \bar{\Psi} = \frac{U(p)}{W(p)} [\cos p\theta \cos (p+2)\alpha - \cos p\alpha \cos (p+2)\theta],$$

where

$$(2.7) \quad W(p) = (p+1) \sin 2\alpha + \sin 2(p+1)\alpha.$$

The stream function Ψ is now obtained by inverting equation (2.1) (TITCHMARSH, 1937):

$$(2.8) \quad \Psi(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-p} \overline{\Psi}(p, \theta) dp,$$

wherein c is any real constant such that $\int_0^\infty r^{c-1} \Psi(r, \theta) dr$ exists. It is evident that

if Ψ tends to zero rapidly enough for both small and large r , $\int_0^\infty r^{-1} \Psi dr$ exists,

and we may take $c = 0$. It will become apparent that this is certainly justified when the angle 2α is less than $\pi/2$. Otherwise more care is needed in the choice of c . From (2.8) and (2.6) we now have

$$(2.9) \quad \Psi(r, \theta) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} r^{-p} \frac{U(p)}{W(p)} [\cos p\theta \cos(p+2)\alpha - \cos p\alpha \cos(p+2)\theta] dp.$$

The integrand has a removable singularity at $p = -1$ (provided $a \neq 0$), and simple poles at the other zeros of $W(p)$. By writing $p+1 = \xi+i\eta$ and separating the real and imaginary parts of (2.7), it is not difficult to show that there is one and only one such zero, p_n say, in each strip

$$(2.10) \quad (2n-1)\pi < \xi < \left(2n - \frac{1}{2}\right)\pi, \quad n = 1, 2, 3, \dots,$$

and that if p_n is a zero having $\text{Re}(p_n) > 0$, then $p_{-n} = -p_n - 2$ is a zero having $\text{Re}(p_{-n}) < 0$. The integral may now be evaluated by the theorem of residues by completing the contour at a large distance in an appropriate way. Choice of the contour is governed by the factor $b(b/r)^p - a(a/r)^p$ in the integrand (substituting for $U(p)$). If $r < a (< b)$, the contour may be closed by the remaining three sides of a large square in the half plane $\text{Re}(p) < 0$. If $r > b (> a)$, the square in the half plane $\text{Re}(p) > 0$ must be chosen. If $a < r < b$, the part of the integral involving $(b/r)^p$ must be evaluated round the first square and the part involving $(a/r)^p$ round the second. Applying the theorem of residues, and noting that for $n \leq 0$, $\text{Re}(p_n) < 0$, while for $n > 0$, $\text{Re}(p_n) > 0$, the solution $\Psi(r, \theta)$ follows in the form

$$(2.11) \quad \Psi(r, \theta) = \begin{cases} V \sum_{n < 0} \frac{1}{p_n + 1} \left[b \left(\frac{b}{r} \right)^{p_n} - a \left(\frac{a}{r} \right)^{p_n} \right] f_n(\theta, \alpha), & r \leq a, \\ V \sum_{n < 0} \frac{1}{p_n + 1} b \left(\frac{b}{r} \right)^{p_n} f_n(\theta, \alpha) + V \sum_{n > 0} \frac{1}{p_n + 1} a \left(\frac{a}{r} \right)^{p_n} f_n(\theta, \alpha), & a \leq r \leq b, \\ -V \sum_{n > 0} \frac{1}{p_n + 1} \left[b \left(\frac{b}{r} \right)^{p_n} - a \left(\frac{a}{r} \right)^{p_n} \right] f_n(\theta, \alpha), & r \geq b, \end{cases}$$

where

$$(2.12) \quad f_n(\theta, \alpha) = \frac{\cos p_n \theta \cos (p_n + 2)\alpha - \cos p_n \alpha \cos (p_n + 2)\theta}{\sin 2\alpha + 2\alpha \cos 2(p_n + 1)\alpha}.$$

It is now possible to check that the various assumptions made at the outset about Ψ , for large and small r , were justified. For, from (2.11), with $\text{Re}(p_n) = \xi_n - 1$ say,

$$\Psi(r, \theta) = O(r^{-(\xi_n - 1)}) = O(r^{\xi_n + 1}) \quad \text{as } r \rightarrow 0$$

and

$$\Psi(r, \theta) = O(r^{-\xi_n + 1}) \quad \text{as } r \rightarrow \infty.$$

From Table 1, it is evident that, when 2α is acute, $\xi_1 > 2$, and it is now easy to check that the integrations by parts (2.2) are valid when $\text{Re}(p)$ is in a small neighbourhood of zero, and that $\int_0^\infty r^{-1} \Psi dr$ exists, as was required in putting $c = 0$ in (2.8). The solution (2.11) has therefore been obtained by a valid process, and, by the well-known uniqueness theorem for viscous flows inside solid boundaries, it is the solution that we require.

Table 1. Geometrical and Dynamical Scale Factors for Corner Eddies

$2\alpha^\circ$	$\xi_1 = \text{Re}\mu_1$	$\eta_1 = \text{Im}\mu_1$	$\ln \rho_1 = \pi/\eta_1$	$\ln \omega_1 = \pi\xi_1/\eta_1$
0	∞	∞	0	5.87
10	24.75	13.25	0.24	5.87
30	8.12	4.23	0.75	6.02
50	4.88	2.42	1.30	6.32
70	3.49	1.62	1.94	6.79
90	2.74	1.13	2.79	7.63

3. Interpretation of the Solution

Sufficiently near the origin $r = 0$, from (2.11) we have

$$\Psi \sim \frac{V}{p-1+1} (b^{p-1+1} - a^{p-1+1}) r^{-p-1} f_1(\theta, \alpha),$$

or

$$(3.1) \quad \Psi \sim -\frac{V}{\mu_1} (b^{-\mu_1} - a^{-\mu_1}) r^{\mu_1+1} f_1(\theta, \alpha),$$

where, from (2.7), μ_1 is the root of

$$\mu \sin 2\alpha + \sin 2\mu\alpha = 0$$

having smallest positive real part. Clearly μ_1 is a function of α ; its value has been calculated numerically for a range of values of α , and is displayed in Table 1.

The most interesting feature of the asymptotic behaviour (3.1) is that it describes a geometric progression of eddies near the corner. The simplest way to see this is to calculate the transverse velocity component on the line $\theta = 0$:

$$(3.2) \quad v_{\theta=0} = -\left(\frac{\partial \Psi}{\partial r}\right)_{\theta=0} \sim \frac{V}{\mu_1} (\mu_1 + 1) \left(\frac{r}{a}\right)^{\mu_1} f_1(0, \alpha),$$

(supposing for simplicity that $b \gg a$).

Now, if $i\mu_1 = \xi_1 + i\eta_1$, then

$$\left(\frac{r}{a}\right)^{\mu_1} = \left(\frac{r}{a}\right)^{\xi_1} \left[\cos\left(\eta_1 \log \frac{r}{a}\right) + i \sin\left(\eta_1 \log \frac{r}{a}\right) \right],$$

so that (3.2) reduces to the form

$$(3.3) \quad v_{\theta=0} \sim AV \left(\frac{r}{a}\right)^{\xi_1} \sin\left(\eta_1 \log \frac{r}{a} + \varepsilon\right),$$

where

$$A = (1 + \mu_1^{-1}) f_1(0, \alpha), \quad \varepsilon = \arg i(1 + \mu_1^{-1}) f_1(0, \alpha).$$

Clearly the expression (3.3) changes sign infinitely often as the point $r = 0$ is approached. In fact $v_{\theta=0} = 0$ for values of r satisfying

$$\eta_1 \log \frac{r}{a} + \varepsilon = -s\pi, \quad s = 0, 1, 2, \dots,$$

i.e.

$$(3.4) \quad r = (ae^{-\varepsilon/\eta_1}) e^{-s\pi/\eta_1} = r_s \quad \text{say.}$$

It is clear that r_s is the distance of the centre of the s th eddy (counted from any chosen eddy) from the corner. Note that the dimensions of successive eddies fall off in geometric progression with common ratio

$$(3.5) \quad \rho_1 = \frac{r_s}{r_{s+1}} = \frac{r_s - r_{s+1}}{r_{s+1} - r_{s+2}} = e^{\pi/\eta_1},$$

a quantity which depends only on the angle 2α .

The velocity $v_{\theta=0}$ has a local maximum at the points

$$r = r_{s+1/2} = (ae^{-\varepsilon/\eta_1}) e^{-(s+1/2)\pi/\eta_1},$$

and the velocity at these points is

$$v = AV \left(\frac{r_{s+1/2}}{a}\right)^{\xi_1} = v_{s+1/2}, \quad \text{say.}$$

This may be taken as a measure of the "intensity" of consecutive eddies. The intensities therefore fall off in geometric progression with common ratio

$$(3.6) \quad \omega_1 = \frac{v_{s+1/2}}{v_{s+3/2}} = \left(\frac{r_{s+1/2}}{r_{s+3/2}} \right)^{\xi_1} = e^{\pi \xi_1 / \eta_1},$$

a quantity which also depends only on the angle 2α .

The parameters ϱ_1 and ω_1 seem to provide the best description of this sequence of eddies; their logarithms are given in Table 1. It will be observed that ϱ_1 is of order unity so that adjacent eddies are of comparable size for values of 2α up to about 40° , but for larger corner angles, ϱ_1 increases to about 10 at $2\alpha = 90^\circ$. The relative intensity of adjacent eddies is always large, its minimum value, $\omega_{\min} = 365$, being attained in the limiting case $2\alpha = 0$. For a right angle, $\omega_1 = 2000$ and the eddy intensity falls off very rapidly as the corner is approached. The relative eddy sizes and intensities are indicated in Fig. 2 for a corner angle $2\alpha = 30^\circ$. The relative

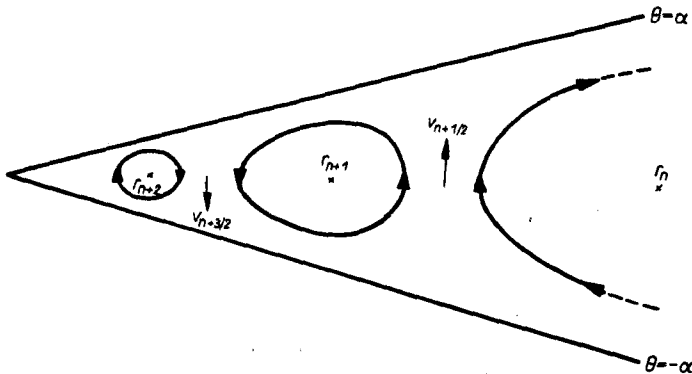


FIG. 2. The sequence of corner eddies when $2\alpha = 30^\circ$; Geometric scale factor $\varrho_1 = 2.1$. Dynamical scale factor $\omega_1 = 403$.

dimensions of the eddies are approximately correct, but their shapes have not been calculated exactly and are only indicated schematically. All the eddies (for a given corner angle) are geometrically and dynamically similar, but with successive changes of length and velocity scales represented by the factors ϱ_1 and ω_1 .

The asymptotic behaviour for large r , from (2.11)₃ is clearly very similar; a sequence of eddies develop, whose intensities fall off in geometric progression with increasing distance from the corner. In the region $a < r < b$, the series in (2.11)₂ converge rather slowly, but it seems unlikely that the solution in this region can represent anything other than a single eddy following the movement of the boundaries as suggested in Fig. 1.

It seems indisputable that the eddying behaviour demonstrated here is in fact quite general and independent of the precise method of agitation of the fluid. Further aspects of the problem have been explored in another paper to be published shortly (MOFFATT, 1964).

References

- [1] W. R. DEAN, P. E. MONTAGNON, *On the steady motion of viscous liquid in a corner*, Proc. Camb. Phil. Soc., **45** (1949), 389.
- [2] H. K. MOFFATT, *Viscous and resistive eddies near a sharp corner*, J. Fluid Mech. **18** (1964).
- [3] Lord RAYLEIGH, *Steady motion in a corner of a viscous fluid*, Scientific Papers, **6** (1920), 18.
- [4] E. C. TITCHMARSH, *Introduction to the theory of Fourier integrals*, Oxford University Press, 1937.
- [5] C. J. TRANTER, *The use of the Mellin transform in finding the stress distribution in an infinite wedge*, Quart. J. Mech. App. Math., **1** (1948), 125.

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