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Electrically driven steady flows in magnetohydrodynamics

By

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Abstract

The flow of electric currents between electrodes immersed in a conducting fluid permeated by a uniform magnetic field is considered. The current interacts with the field to generate a velocity field which modifies the steady current flow pattern. The magnetic Reynolds number of the motion is assumed small, and inertia forces are assumed negligible compared with either magnetic or viscous forces. Under these circumstances the flow pattern depends only on the Hartmann number M in addition to the boundary conditions imposed. When $M \to \infty$, layers of discontinuity may develop in the fluid. Across these layers the tangential component of electric field is discontinuous, the normal component within the layer being of order δ^{-1} where δ is the layer thickness. Two particular geometries are discussed in detail. In the first, the magnetic field is perpendicular to the electrodes, and a layer of discontinuity, with the character of a non-spreading jet, develops as $M \to \infty$. In the second, the magnetic field is parallel to the electrodes, and the discontinuities that develop in the limit are confined to the boundaries of the fluid.

1. Introduction

If electrodes are put in electrical contact with a viscous conducting fluid permeated by a uniform magnetic field $\boldsymbol{B}_0$ (they may either be immersed in the fluid or be part of its boundary), and if these electrodes are raised to different electric potentials through connection with external batteries, then a current distribution $\boldsymbol{j}$ will tend to flow from regions of high potential to regions of low potential. The interaction of the current and the applied field will produce a Lorentz force $\boldsymbol{j} \wedge \boldsymbol{B}_0$ which will in general drive a velocity $\boldsymbol{u}$ in the fluid. The motion of the fluid will cause a modification of the current pattern and an associated modification of the resistances between different electrodes. It may be anticipated that if the boundary potentials are kept steady, then a steady distribution of current and velocity will be established.

This situation is of considerable interest both in the dynamics of a not too hot plasma (which is often activated in just the way described above, by the application of electric and magnetic fields), and in molten metal technology (in particular in the design of electromagnetic pumps). Moreover, the situation lends itself to fairly straightforward experimental investigation, it being not difficult to obtain dramatic effects in the laboratory using mercury as a working fluid, and applied fields of the order of 1000 gauss.

We shall assume

(i) that the flow is incompressible;

(ii) that the magnetic Reynolds number of the flow is small, so that the magnetic field due to the current distribution is negligible compared with $\boldsymbol{B}_0$;

(iii) that inertia forces are negligible compared with either magnetic or viscous forces [see Eqs. (1.7), (2.24), (2.25)]; (inertia forces are identically zero for the particular two-dimensional geometries considered in Secs. 2 and 3). In effect these three conditions all put an upper limit

on the potential differences that may be applied (for given magnetic field and fluid properties) for the validity of the theory. Condition (iii) is usually the most restrictive; conditions (i) and (ii) could be violated only in a plasma.

The governing equations for steady flow (in *emu*) are then

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \wedge \boldsymbol{B}_0), \quad \boldsymbol{E} = -\nabla \phi, \tag{1.1}$$

$$0 = -\nabla p + \boldsymbol{j} \wedge \boldsymbol{B}_0 + \eta \nabla^2 \boldsymbol{u}, \tag{1.2}$$

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{j} = 0, \tag{1.3}$$

where p is the induced pressure distribution, σ the electrical conductivity and η the dynamical viscosity of the fluid. From (1.1) and (1.3),

$$\nabla^2 \phi = \nabla \cdot (\boldsymbol{u} \wedge \boldsymbol{B}_0) = \boldsymbol{\omega} \cdot \boldsymbol{B}_0, \tag{1.4}$$

where $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$ is the fluid vorticity. From (1.2) and (1.3),

$$\eta \nabla^2 \boldsymbol{\omega} = -\nabla \wedge (\boldsymbol{j} \wedge \boldsymbol{B}_0) = -(\boldsymbol{B}_0 \cdot \nabla) \boldsymbol{j},$$

and the component of this equation in the $\boldsymbol{B}_0$ direction is

$$\eta \nabla^2 (\boldsymbol{\omega} \cdot \boldsymbol{B}_0) = -(\boldsymbol{B}_0 \cdot \nabla)(\boldsymbol{B}_0 \cdot \boldsymbol{j}) = \sigma (\boldsymbol{B}_0 \cdot \nabla)^2 \phi. \tag{1.5}$$

From (1.4) and (1.5),

$$\nabla^4 \phi = \frac{M^2}{a^2} (\boldsymbol{i} \cdot \nabla)^2 \phi, \tag{1.6}$$

where $\boldsymbol{i}$ is a unit vector in the direction of $\boldsymbol{B}_0$, and M is the Hartmann number,

$$M = \left(\frac{\sigma}{\eta}\right)^{1/2} B_0 a, \tag{1.7}$$

a being a typical length scale (e.g. the distance between two electrodes). Eq. (1.6) was derived by Braginskii (1960) who also discussed some of its consequences. Clearly $\boldsymbol{\omega} \cdot \boldsymbol{B}_0$ satisfies the same equation.

Flows that are generated dynamically (either by the application of pressure gradients or by the motion of solid boundaries) are also governed by (1.6), although then more appropriately expressed in terms of the velocity distribution. For example Hasimoto (1960), Chester (1961) and others have considered the steady motion of a cylinder in a uniform applied magnetic field, various configurations of the three directions of motion, applied field, and cylinder generators being possible. In this paper, we deliberately restrict attention to the complementary situation (which seems to have received very little attention) in which the sole source of energy is electrical (from the working of external batteries) all the solid boundaries of the fluid being fixed. The appropriate boundary conditions are then

$$\left.\begin{array}{l} \boldsymbol{u} = 0 \text{ on solid boundaries}, \\ \phi \text{ is prescribed on electrodes (regarded as perfectly conducting)}, \\ \text{and } \boldsymbol{j} \cdot \boldsymbol{n} = 0 \text{ on insulators with unit normal } \boldsymbol{n}. \end{array}\right\} \tag{1.8}$$

One could also consider boundaries of finite conductivity σ', anticipating that they would behave like insulators if $\sigma'/\sigma \ll 1$ and like perfect conductors if $\sigma'/\sigma \gg 1$. It is sometimes mathematically convenient, though physically somewhat artificial, to adopt the condition that $\boldsymbol{j} \cdot \boldsymbol{n}$ rather than ϕ is prescribed on electrodes (see Sec. 3); this could be achieved in practice only by insulating different small parts of an electrode from each other.

From Eqs. (1.1)—(1.3) and the boundary conditions (1.8), it is a straightforward matter to derive the energy equation for the whole fluid, viz.

$$\sum \phi_i J_i = \int (\sigma^{-1} j^2 + \eta \omega^2)\, dV, \tag{1.9}$$

the sum being over all the conductors, and the integral being throughout the fluid. J_i is the total current leaving the i'th conductor at potential ϕ_i. The uniqueness of the solution for the given boundary conditions can be easily established using (1.9). It is also possible to prove certain "minimum dissipation" theorems similar to Korteweg's theorem for slow viscous flow (Lamb 1962), but the details are of limited interest and will be omitted here.

When M is large, (1.6) suggests that throughout the bulk of the fluid (i.e. except possibly in singular sheets and boundary layers),

$$(\boldsymbol{i} \cdot \nabla)^2 \phi = 0, \tag{1.10}$$

$$\text{i.e. } \phi = x f(y, z) + g(y, z), \tag{1.11}$$

where f and g are piecewise continuous functions, x being the coordinate along the direction of $\boldsymbol{B}_0$. The lines of current flow must be parallel to $\boldsymbol{B}_0$ in the limit $M = \infty$ [since otherwise the force $\boldsymbol{j} \wedge \boldsymbol{B}_0$ in (1.2) would give rise to infinite velocity gradients and so infinite dissipation], so that

$$j_x = -\sigma \frac{\partial \phi}{\partial x} = -\sigma f(y, z), \; j_y = j_z = 0. \tag{1.12}$$

This implies the existence of certain curious layers of discontinuity within the fluid.

Consider for example the simple geometry of the apparatus of an experiment recently performed by J. C. R. HUNT at the Engineering Laboratories of Cambridge University (Fig. 1a). Two circular electrodes are inserted in the end walls of an otherwise insulating cylindrical container filled with mercury. A potential difference is applied to the electrodes, and the whole container is immersed in the uniform magnetic field $\boldsymbol{B}_0$ of a solenoid, with the axis of symmetry parallel to $\boldsymbol{B}_0$. In effect the total current passing through the circuit was measured as a function of the strength of the applied field. It was found that the current decreased as the magnetic field increased, but approached a steady asymptotic value (corresponding roughly to the current that would be obtained if the fluid occupied only the space directly between the electrodes) when the inequality

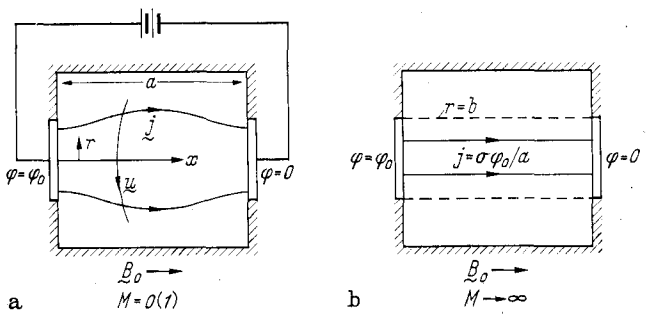

Fig. 1 a and b. Current flow through mercury in an applied magnetic field. As $M \to \infty$, the surface $r = b$ becomes a surface of discontinuity of E_x; at the same time $E_r \to \infty$ in the layer. The double line represents an electrode, the dashed line an insulator

$$M^{1/2} \gg a/b$$

began to be satisfied. Here a is the distance between the electrodes, b the electrode radius, and M the Hartmann number based on a. Clearly as $M \to \infty$, the electric field and current are channelled along the x-axis (Fig. 1b), and the fluid does behave like a linear conductor. However this implies a discontinuity in the tangential component of $\boldsymbol{E}$ across the cylindrical surface $r = b$. The condition $\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$ for a circuit threading this surface must not be violated, and this implies the existence of a very large [actually $0(M^{1/2})$] normal electric field through the layer. The condition $j_r \equiv 0$ within the layer gives

$$B_0 w = \partial \phi / \partial r, \tag{1.13}$$

and it is evident that there is a net flux in the layer given by

$$B_0 W(x) = B_0 \int_{\text{across layer}} w \, dr = [\phi] = \phi_0 \left(1 - \frac{x}{a}\right). \tag{1.14}$$

The layer therefore behaves locally like a non-spreading jet in the azimuth direction whose intensity is linear in the transverse coordinate. The divergence of the electric field gives the charge distribution (effectively a dipole sheet of possibly complex structure) across the layer.

The above type of discontinuity clearly arises for more general electrode geometries. The above simple arguments give useful information about the overall properties of the sheet, but no details of its structure. The most satisfactory method of determining the structure would be to solve (1.6) subject to the appropriate boundary conditions, and then examine the asymptotic form of the solution for large M.

2. Electrodes perpendicular to applied field

The situation of Fig. 1 is too complicated to find an analytic solution. The simplest possible situation which must give rise to an "E_t-discontinuity" is sketched in Fig. 2a. The fluid is bounded by perfectly conducting planes $x = 0$, $x = a$ on which the potential is prescribed as follows:

$$\left.\begin{aligned} &\text{on } x = 0,\ \phi = 0\ (y > 0),\\ &\qquad\qquad\quad \phi_0\ (y < 0),\\ &\text{on } x = a,\ \phi = 0\,. \end{aligned}\right\} \tag{2.1}$$

This could almost be achieved in practice by having a thin strip of insulating material along the line $x = y = 0$ separating the parts of the electrode that are maintained at different electric potentials[1]. For this two dimensional geometry, there is only one component of velocity w in the z-direction, and in addition to (2.1) we must satisfy the conditions

$$w = 0 \quad \text{on} \quad x = 0, a. \tag{2.2}$$

There are no applied pressure gradients.

Clearly since the governing equation and boundary conditions are linear, the solution (when found) can immediately be extended by superposition to more complicated geometries such as in Fig. 2b.

There are two methods of solving this type of problem. Either we may use a Fourier integral with respect to the y-variable, or we may use a Fourier series with respect to the x-variable. The former method is straightforward, but the problem of interpreting the answer is rather severe. The latter method runs into convergence difficulties due to the nature of the singularity at the origin, but it leads to an answer that is plausible, informative, and easy to interpret; it is the method that we adopt here.

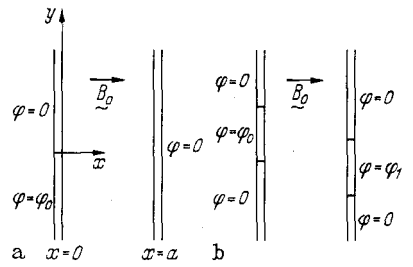

Fig. 2a and b. Electrodes perpendicular to applied field. In (a) a thin strip of insulator at 0 separates the electrodes at different potentials. Four such strips are needed in (b)

Let

$$\left.\begin{aligned} \phi &= \phi_0 \sum_1^\infty f_n(y) \sin k_n x && (y > 0),\\ &= \phi_0\left(1 - \frac{x}{a}\right) + \phi_0 \sum_1^\infty g_n(y) \sin k_n x && (y < 0). \end{aligned}\right\} \tag{2.3}$$

The corresponding velocity $w(x, y)$ (in the z-direction) from (1.4) is given by

$$B_0 \frac{\partial w}{\partial y} = \nabla^2 \phi = \sum \left(\frac{d^2}{dy^2} - k_n^2\right) \left\{\begin{matrix} f_n \\ g_n \end{matrix}\right\} \sin k_n x, \quad (y \gtrless 0), \tag{2.4}$$

so that this expansion of ϕ permits both ϕ and w to satisfy the boundary conditions (2.1) and (2.2).

From (1.6), f_n satisfies the equation

$$\left(\frac{d^2}{dy^2} - k_n^2\right)^2 f_n = -\frac{M^2}{a^2} k_n^2 f_n, \tag{2.5}$$

and the solution which tends to zero as $y \to +\infty$ may be derived by elementary methods in the form

$$f_n = e^{-\alpha_n y}[A_n^{(1)} \cos \beta_n y + A_n^{(2)} \sin \beta_n y], \tag{2.6}$$

where

$$\left\{\begin{matrix} \alpha_n \\ \beta_n \end{matrix}\right\} = \frac{k_n}{\sqrt{2}} \left[\pm 1 + \left\{1 + \left(\frac{M}{a k_n}\right)^2\right\}^{1/2}\right]^{1/2}, \tag{2.7}$$

and $A_n^{(1)}$ and $A_n^{(2)}$ are constants.

[1] A less artificial situation would be that in which an insulator occupies the position $x = 0$, $y > 0$ (the rest being as above), but this would introduce mixed boundary conditions on $x = 0$, and would be much more difficult to solve analytically.

Similarly,

$$g_n = e^{\alpha_n y}[B_n^{(1)} \cos \beta_n y + B_n^{(2)} \sin \beta_n y], \tag{2.8}$$

where $B_n^{(1)}$ and $B_n^{(2)}$ are constants.

The existence of the factors $e^{\pm\alpha_n y}$ in (2.6) and (2.8) ensure that the Fourier series in (2.3) are uniformly convergent in any y-neighbourhood excluding $y = 0$, [assuming that the A_n and B_n do not involve an exponential increase with n — see (2.11) below]. The derivatives ϕ_y, ϕ_{yy} and ϕ_{yyy}, may then be evaluated by term-by-term differentiation of the Fourier series for ϕ. Let us assume that the series so obtained are valid right up to $y = 0$. We may then evaluate the constants A_n and B_n by demanding that ϕ, ϕ_y, ϕ_{yy} and ϕ_{yyy} be continuous across $y = 0$ [the differential equation (1.6) then ensuring continuity of higher derivatives]. By writing

$$1 - \frac{x}{a} = \sum_1^\infty C_n \sin k_n x \qquad (0 < x \leq a), \tag{2.9}$$

where

$$C_n = 2/n\pi, \tag{2.10}$$

these conditions on the smoothness of ϕ, from (2.3), lead to

$$A_n^{(1)} = -B_n^{(1)} = \frac{1}{2} C_n, \; A_n^{(2)} = B_n^{(2)} = (k_n a/2M) C_n. \tag{2.11}$$

Inserting these values in (2.6) and (2.8), and returning to (2.3), the potential is determined everywhere in the form

$$\left.\begin{aligned} \phi &= \phi_0 \sum_1^\infty e^{-\alpha_n y}\left(\frac{1}{n\pi}\cos\beta_n y + \frac{1}{M}\sin\beta_n y\right)\sin k_n x, \; (y > 0) \\ &= \phi_0\left(1 - \frac{x}{a}\right) + \phi_0 \sum_1^\infty e^{\alpha_n y}\left(-\frac{1}{n\pi}\cos\beta_n y + \frac{1}{M}\sin\beta_n y\right)\sin k_n x, \; (y < 0). \end{aligned}\right\} \tag{2.12}$$

These series do indeed converge when $y = 0$, both expressions for ϕ giving

$$\phi = \frac{1}{2}\phi_0\left(1 - \frac{x}{a}\right), \qquad (y = 0). \tag{2.13}$$

However the derived series for $\partial\phi/\partial x$ does not converge at $y = 0$. This is to be expected since $\partial\phi/\partial y$ has a singularity on the boundary $x = 0$ (i.e. at one end of the range of the Fourier series). The current density and the velocity gradient are likewise singular at $x = y = 0$. However this does not seem to invalidate the method used above to find the constants A_n and B_n, since the Fourier coefficients of ϕ_y, ϕ_{yy} and ϕ_{yyy}, regarded as generalised functions, must be the same on $y = 0$, approaching from positive or negative values of y.

The α_n defined in (2.7) form a monotonic increasing sequence, so that the infinite series in (2.12) are dominated for large y by their leading terms (which involve the factors $e^{\pm\alpha_1 y}$). The form of the leading term indicates a damped periodic structure for sufficiently large $|y|$. Similarly all quantities derived from ϕ, in particular the deviation of the electric current from zero for $y > 0$ and from $\sigma\phi_0/a$ for $y < 0$, exhibit the same damped periodicity. The current component j_x for large y is given by

$$j_x = -\sigma\frac{\partial\phi}{\partial x} \sim -\sigma\phi_0 e^{-\alpha_1 y}\left(\frac{1}{\pi}\cos\beta_1 y + \frac{1}{M}\sin\beta_1 y\right)k_1 \cos k_1 x, \tag{2.14}$$

and vanishes on $x = a/2$. The current pattern for $M = 0(1)$ is sketched in Fig. 3a.

For small M, from (2.7),

$$\alpha_n \sim k_n \text{ and } \beta_n \sim M/2a, \tag{2.15}$$

and so from (2.12)

$$\begin{aligned} \phi &\sim \frac{V}{\pi}\left(\cos\frac{My}{2a} + \frac{\pi}{M}\sin\frac{My}{2a}\right)\sum_1^\infty \frac{1}{n} e^{-k_n y}\sin k_n x \\ &= \frac{V}{\pi}\left(\cos\frac{My}{2a} + \frac{\pi}{M}\sin\frac{My}{2a}\right)\tan^{-1}\left[\frac{e^{-\pi y/a}\sin\pi x/a}{1 - e^{-\pi y/a}\cos\pi x/a}\right], \quad (y > 0) \end{aligned} \tag{2.16}$$

and similarly for $y < 0$. The wavelength of the periodicity is $2a/M$, while the damping distance is of order a. The term $\pi/M \sin(My/2a)$ appears irregular in the limit $M \to 0$, but this is illusory since it fluctuates only for $y = 0\,(a/M)$, and for such large values of y the $\tan^{-1}$-factor contributes exponential damping of order $e^{-\pi/M}$.

The solution (2.16) in the limit $M \to 0$ does not tend uniformly to the potential function which describes current flow in the absence of a magnetic field (or equivalently current flow in a solid conductor). The latter potential, the solution of $\nabla^2\phi = 0$ satisfying the boundary conditions (2.1), may be obtained in series form as above, and the series may be summed to give

$$\left.\begin{aligned} \phi &= \frac{\phi_0}{\pi}\tan^{-1}\left[\frac{e^{-\pi y/a}\sin \pi x/a}{1 - e^{-\pi y/a}\cos \pi x/a}\right], \qquad (y > 0), \\ &= \phi_0\left(1 - \frac{x}{a}\right) - \frac{\phi_0}{\pi}\tan^{-1}\left[\frac{e^{+\pi y/a}\sin \pi x/a}{1 - e^{+\pi y/a}\cos \pi x/a}\right], \qquad (y < 0). \end{aligned}\right\} \tag{2.18}$$

This form cannot be obtained as the limit of (2.16) as $M \to 0$. The interpretation is that a magnetic field, no matter how weak, inevitably affects the current distribution by a significant factor at a sufficient distance from the plane $y = 0$. The lines of current flow corresponding to the solution (2.18) are sketched in Fig. 3b.

In the other limit, as $M \to \infty$, if we choose some large number $N = o(M)$, then

$$\alpha_n, \beta_n \sim (M k_n/2a)^{1/2} \text{ for all } n < N. \tag{2.19}$$

The corresponding Fourier components of the potential distribution are damped (and oscillating) within a layer of thickness

$$\delta = 0(a/M^{1/2}), \tag{2.20}$$

and the whole pattern of current flow variation is compressed into this layer (Fig. 3c). [Fourier components of wave number $\geq 0(M)$ are largely unaffected by the magnetic field; these components describe the field within a distance $0(a/M)$ of the discontinuity at $x = y = 0$.] SHERCLIFF (1953) in a study of flow through rectangular pipes found boundary layers parallel to a magnetic field of thickness $0(M^{-1/2})$* [in contrast to those perpendicular to the field of thickness $0(M^{-1})$], but the layers described above are unusual in that the velocity distribution is confined entirely to the sheet, vanishing exponentially on either side of it. It has the character of a nonspreading jet, whose intensity varies with the transverse coordinate x.

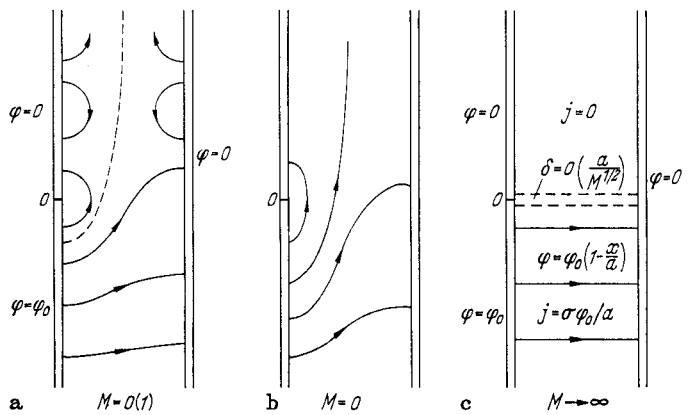

Fig. 3 a — c. Lines of current flow for different values of M; in (c) an "E_t-discontinuity" develops on $y = 0$ as $M \to \infty$

Within the jet, $\partial/\partial y \gg \partial/\partial x$, and the velocity is determined by (1.11) (except in the immediate neighbourhood of 0). The order of magnitude of the velocity is

$$w = 0(\phi_0/B_0\delta) = 0(\phi_0 M^{1/2}/B_0 a) = 0\left(\phi_0\left(\frac{\sigma}{\eta}\right)^{1/2} M^{1/2}\right), \tag{2.21}$$

and the vorticity within the layer is of order

$$\omega = 0(w/\delta) = 0(a^{-1}\phi_0(\sigma/\eta)^{1/2}). \tag{2.22}$$

The viscous and ohmic dissipation rates within the layer are of the same order of magnitude,

$$j^2/\sigma \approx \eta\omega^2 = 0(\sigma\phi_0^2/a^2), \tag{2.23}$$

* Further examples of this type of flow have recently been studied by HUNT (1965), HUNT and STEWARTSON (1965); the boundary layers studied in the former paper have an oscillatory structure similar to that described above.

and the total rate of dissipation of energy integrated throughout the layer is $0(M^{-1/2})$ as $M \to \infty$.

The appropriate Reynolds number for the flow in the layer is

$$R = \frac{(\phi_0/B_0\delta)\,\delta}{\nu} = \frac{\phi_0}{B_0\nu}, \tag{2.24}$$

where ν is the kinematic viscosity of the fluid, and it seems likely that the flow in the layer, with its jet-like character, will become unstable when R exceeds some number of order unity.

When the electrode geometry is three dimensional (e.g. as in Fig. 1), the discontinuity surfaces acquire a curvature of order a^{-1}, [assuming $a/b = 0(1)$ in Fig. 1]. Inertia forces in the steady flow are then not zero but of order of magnitude w^2/a. These are negligible compared with viscous and magnetic forces if

$$w^2/a \ll w/\nu\delta^2 \text{ or } RM^{-1/2} \ll 1. \tag{2.25}$$

When M is large, if ϕ_0 is gradually increased from zero, it seems likely that the first manifestation of inertia forces will be in the development of instabilities rather than in a steady distortion of the steady flow pattern.

3. Electrode surface parallel to applied field

An entirely different situation arises when current is forced to flow *across* an applied field. Consider for example the geometry of Fig. 4 in which $\boldsymbol{B}_0$ is parallel to two parallel electrodes on which conditions vary discontinuously. The structure of the governing equation,

$$\nabla^4\phi = \frac{M^2}{a^2}\frac{\partial^2\phi}{\partial x^2}, \tag{3.1}$$

is such that it is easy to find a solution if $\partial\phi/\partial y$ rather than ϕ is prescribed on $y = 0,\ a$. This corresponds to the prescription of $\boldsymbol{j}\cdot\boldsymbol{n}$ on $y = 0, a$, and although artificial, the solution indicates the type of behaviour to expect in general. We therefore adopt the boundary conditions

$$\begin{aligned} &\phi_y = -\sigma^{-1}j_0,\ w = 0 \text{ on } y = 0,\ a,\ (x < 0),\\ &\phi_y = 0,\ w = 0 \text{ on } y = 0,\ a,\ (x > 0). \end{aligned} \tag{3.2}$$

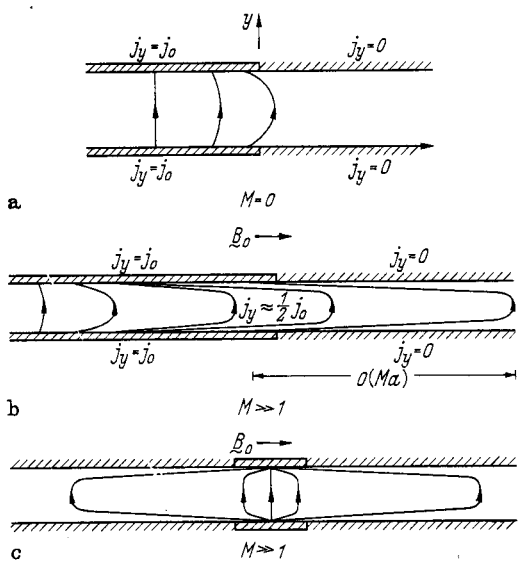

Fig. 4 a—c. Electrodes parallel to applied field. The double line with dashes between represents a "laminated electrode" on which $\boldsymbol{j}\cdot\boldsymbol{n}$ can be prescribed

Again we shall suppose that there are no applied pressure gradients. The portions $x > 0$ of the plates $y = 0,\ a$ are insulators. In the absence of a magnetic field a fringing effect on the current flow lines occurs in the region within a distance $0(a)$ of $x = 0$, (Fig. 4a). Outside this region, $j_y = j_0$ for $x < 0$, and $j_y = 0$ for $x > 0$.

In the presence of the field $\boldsymbol{B}_0$, as $x \to -\infty$, $j_y \to j_0$ and there is a uniform body force $j_0 B_0$ which drives the z-component of velocity

$$w \sim \frac{B_0 j_0}{2\eta}\, y\,(y - a) = w_0(y), \text{ say, as } x \to -\infty. \tag{3.3}$$

Clearly

$$w \to 0 \quad \text{as} \quad x \to +\infty. \tag{3.4}$$

Hence, from (1.4),

$$\begin{aligned} \phi &\sim \sigma^{-1}j_0\left[y - \frac{M^2}{a^2}\left(\frac{1}{3}y^3 - \frac{1}{2}ay^2\right)\right] = \phi_0(y), \text{ say}\\ &= -\sigma^{-1}j_0\sum_0^\infty D_n \cos k_n y, \text{ as } x \to -\infty, \end{aligned} \tag{3.5}$$

where
$$D_n = -\frac{a}{\pi^2}\left[\frac{2}{n^2} + \frac{4M^2}{\pi^2 n^4}\right](1-(-1)^n),\ (n>1),\ D_0 = \frac{1}{2}a\left(1+\frac{1}{6}M^2\right), \tag{3.6}$$
and
$$\phi \sim \phi_1(cst), \quad \text{as} \quad x \to +\infty. \tag{3.7}$$

As for the previous case, this problem may now be solved by assuming a potential of the form
$$\begin{aligned}\phi &= -\sigma^{-1}j_0 \sum D_n \cos k_n y - \sigma^{-1} j_0 \sum p_n(x) \cos k_n y, \quad (x<0),\\ &= \phi_1 + \sum q_n(x) \cos k_n y, \qquad (x>0),\end{aligned} \tag{3.8}$$
where again $k_n = n\pi/a$. This satisfies the boundary conditions (3.2). The corresponding velocity distribution is given by
$$B_0 \frac{\partial w}{\partial y} = \nabla^2 \phi, \tag{3.9}$$
so that
$$\begin{aligned} w &= B_0 j_0 y(y-a) + \frac{j_0}{B_0\sigma}\sum_1^\infty k_n^{-1}(p_n'' - k_n^2 p_n)\sin k_n y, \quad (x<0),\\ w &= -\frac{j_0}{B_0\sigma}\sum_1^\infty k_n^{-1}(q_n'' - k_n^2 q_n)\sin k_n y, \quad (x>0).\end{aligned} \tag{3.10}$$
This velocity satisfies the condition $w = 0$ on $y = 0, a$ as required.

The procedure for determining p_n and q_n is very similar to that used in Sec. 2 and the result is
$$\begin{aligned} p_n &= \frac{1}{2}D_n \frac{a^2}{M(M^2+4a^2k_n^2)^{1/2}}\left(l_n^{(2)^2} e^{l_n^{(1)}x} - l_n^{(1)^2} e^{l_n^{(2)}x}\right),\\ q_n &= -\frac{1}{2}D_n \frac{a^2}{M(M^2+4a^2k_n^2)^{1/2}}\left(l_n^{(2)^2} e^{-l_n^{(1)}x} - l_n^{(1)^2} e^{-l_n^{(2)}x}\right),\end{aligned} \tag{3.11}$$
where
$$2a l_n^{(1),(2)} = \pm M + (M^2 + 4a^2k_n^2)^{1/2}. \tag{3.12}$$
Since $l_n^{(1)}$ and $l_n^{(2)}$ are real and positive, these expressions do not exhibit any periodicity, and are exponentially damped as $|x| \to \infty$.

As $M \to \infty$,
$$l_n^{(1)} \sim \frac{M}{a} \text{ and } l_n^{(2)} \sim \frac{k_n a}{M}, \tag{3.13}$$
so that
$$p_n \sim -\frac{1}{2}D_n e^{k_n^2 ax/M}, \quad q_n \sim \frac{1}{2}D_n e^{-k_n^2 ax/M}, \tag{3.14}$$
again for all n less than some $N = o(M)$. Hence for $x = o(M/k_N^2 a)$,
$$p_n \sim -\frac{1}{2}D_n, \quad q_n \sim \frac{1}{2}D_n, \tag{3.15}$$
so that
$$\phi \sim \frac{1}{2}\phi_0(y) \text{ for } |x| = o\left(\frac{M}{k_n^2 a}\right) \text{ as } M \to \infty. \tag{3.16}$$
Moreover,
$$w \sim \frac{1}{2}w_0(y) \text{ and } j_y \sim \frac{1}{2}j_0 \text{ for } x = o\left(\frac{M}{k_N^2 a}\right), \tag{3.17}$$
except in boundary layers near both walls $y = 0, a$ of thickness $0(a|x|/M)^{1/2}$. The current flow pattern for $M \gg 1$ is therefore as indicated in Fig. 4b. The fringing effect already evident when $M = 0$ is greatly exaggerated in the presence of a strong field. Fig. 4c illustrates a further case when the source of current is confined to the range $|x| < b$.

Instabilities are again to be expected if the current intensity (and so the velocity in the boundary layers) becomes too large.

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