(13)

LARGE SCALE MOTIONS IN A TURBULENT BOUNDARY LAYER; WAVES VERSUS EDDIES

by H. K. Moffatt

Abstract

The dynamics of those fluctuating motions in a turbulent boundary layer which have a scale comparable with the scale of the mean flow is considered. Motions on this scale, 'large eddies', can arise through interactions of Fourier components of the velocity field having a much smaller scale. The large eddies are convected and distorted by the mean flow and they lose energy to small scale turbulence by a process that may (for want of better) be represented by an eddy viscosity. The equilibrium of these processes determines (in principle) the statistical structure of the large eddies. Two methods of analysis are considered, the first of which highlights the wave-nature of possible modes of motion, and the second of which places the emphasis on the distortion and development of anisotropy in certain other modes of motion. The methods are to some extent complementary, but it is argued that the second method is physically more appropriate in treating disturbances that are localised in the y direction.

Some of the features of the correlation curves obtained by Grant (1958) are interpreted in terms of initially isotropic turbulence that has been subjected to a uniform mean shear. The shear induces an anisotropy that admits comparison with experiment, and the indications are that this is an approach that merits further study.

1. Introduction

A great variety of experimental results are now available concerning the statistical structure of the velocity fluctuations $\underset{\sim}{u}(\underset{\sim}{x}, t)$ in a turbulent boundary layer. For example, the two-point normalised correlation tensor (at zero time delay),

$$R_{ij}(\underset{\sim}{r}, y) = \frac{\overline{u_i(\underset{\sim}{x})\, u_j(\underset{\sim}{x}+\underset{\sim}{r})}}{\left[\overline{u_i^2(\underset{\sim}{x})}\ \overline{u_j^2(\underset{\sim}{x}+\underset{\sim}{r})}\right]^{1/2}}, \tag{1.1}$$

has been plotted by Grant (1958) for $i = j = 1, 2, 3,$ and for separations r in the three principal directions, and by Tritton (1967), particularly for $ij = 12, 21$. Space-time correlations for $i = j = 1$ have been extensively studied by Favre, Gaviglio and Dumas (1957, 1958). Several attempts have been made to explain the curves in terms of the dominance of particular structures at various depths in the boundary layer (see particularly Townsend 1956), but as yet no coherent theory is available to explain why any one structure is intrinsically more likely to appear than any others.

Some of the curves obtained by Grant are reproduced in figure 1(a) and (b). In (a), the fixed probe is in the outer part of the boundary layer (at $y/\delta_0 = 0.66$, the intermittency factor is approximately 0.9), while in (b) it is well within the constant stress region. The curves in (a) bear some qualitative resemblance to corresponding correlation curves for isotropic turbulence [Batchelor 1953, figure 3.2] in that the three curves $R_{11}(r, 0, 0)$, $R_{22}(0, r, 0)$ and $R_{33}(0, 0, r)$ are everywhere positive, and the other six all go negative for large r (a requirement of mass conservation in isotropic turbulence). In other respects however, the curves are indicative of pronounced anisotropy. Notice (i) the large magnitude of $R_{11}(r, 0, 0)$ for large r compared with $R_{22}(0, r, 0)$ and $R_{33}(0, 0, r)$, (Tritton (1967) has extended the measurements of $R_{11}(r, 0, 0)$ out to $r/\delta_0 = 2.5$, and finds that $R_{11}(r, 0, 0)$ is still about 0.1 at this large separation), (ii) the long tail of $R_{33}(r, 0, 0)$ compared with that of $R_{33}(0, r, 0)$ - curves that would be identical in isotropic turbulence, (iii) the contracted scale of $R_{11}(0,0,r)$ relative to $R_{11}(0, r, 0)$.

The curves in (b) do not even bear a qualitative resemblance to the curves of isotropic turbulence. $R_{33}(0, 0, r)$ now goes negative for $r/\delta_0 \gtrsim 0.3$; and $R_{11}(0, r, 0)$, $R_{22}(r, 0, 0)$, $R_{22}(0, 0, r)$ and $R_{33}(r, 0, 0)$ remain positive for all r. The most noticeable single feature of the curves is the very pronounced tail on $R_{11}(r, 0, 0)$. The rapid fall-off of $R_{22}(0, 0, r)$ is also remarkable.

It is possible to infer from these curves correlation length scales for the turbulence, which may be interpreted as scales (in the three principal directions) of the energy containing eddies of the turbulence. If we define

$$\ell_x = \int_0^\infty R_{11}(r,0,0)\,dr, \quad \ell_y = \int_0^\infty R_{22}(0,r,0)\,dr, \quad \ell_z = \int_0^\infty R_{33}(0,0,r)\,dr,$$

then, from inspection of the curves, very approximately, in the outer layer,

$$\ell_x \approx \tfrac{1}{3}\delta_0 , \quad \ell_y \approx \tfrac{1}{5}\delta_0 , \quad \ell_z \approx \tfrac{1}{4}\delta_0 ,$$

and in the constant stress layer,

$$\ell_x \approx \tfrac{1}{3}\delta_0 , \quad \ell_y \approx \tfrac{1}{8}\delta_0 , \quad \ell_z \approx \tfrac{1}{10}\delta_0 .$$

The estimates of l_x are if anything too small, due to the uncertainty of contributions from large r. The simplest measures of the degree of anisotropy of the turbulence are the scale ratios l_x/l_y , l_x/l_z and these increase from values of order $(\frac{5}{3}, \frac{4}{3})$ to values of order $(\frac{8}{3}, \frac{10}{3})$ as y/δ_0 decreases from $\frac{2}{3}$ to about $\frac{1}{20}$.

The existence of motions elongated in the x-direction has been shown by Bakewell (1966) to extend right down into the viscous sublayer as far as $y^+ = 1.25$. The film by Kline (1964) and the photographs of Kline et al (1967) provide striking visual evidence of the growing importance of motions having little variation in the x-direction as the wall is approached.

The term 'large eddies' (Townsend 1956) is usually used to denote those motions having a scale comparable with the scale of the mean velocity profile. In the case of the turbulent wake, the large eddies are reasonably distinct (inscale) from the energy containing eddies, but in the case of the boundary layer the distinction is really too fine to be meaningful. Townsend has estimated that about 20 per cent of the total turbulent energy is contained in motions having a scale (in the y and z directions) greater than $0.4\,\delta_0$; the correlations lengths would suggest that about 50 percent of the turbulent energy is contained in motions having a scale $(l_y{}^2 + l_z{}^2)^{\frac{1}{2}}$ greater than about $0.1\,\delta_0$; and there is no discernible gap in the spectrum between large eddies (scale $> 0.4\,\delta_0$) and energy containing eddies (scale, say ≈ 0.1 or

$0.2\,\delta_0$).

In these circumstances, it might be thought that any attempt to decompose the turbulence into 'large eddies' $\underset{\sim}{v}(\underset{\sim}{x}, t)$ on the one hand and 'small-scale turbulence $\underset{\sim}{w}(\underset{\sim}{x}, t)$ on the other must be doomed at the outset. The great merit of such a decomposition, if it is possible, is that since in a sense the large eddies are weak relative to the mean flow (though by no means infinitesimal) it should be legitimate to obtain equations that are linear in $\underset{\sim}{v}$, the non-linear turbulent effects being conveniently summarised in the small-scale turbulence w. The role of the small-scale turbulence is two-fold; firstly, interactions of Fourier components of wave-numbers $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$ (satisfying $|\underset{\sim}{k}_1\delta_0|, |\underset{\sim}{k}_2\delta_0| \gg 1$) continuously generate new Fourier components of the velocity field with wave-vectors $(\underset{\sim}{k}_1 \pm \underset{\sim}{k}_2)$; and if it so happens that, say, $|\underset{\sim}{k}_1 - \underset{\sim}{k}_2|\delta_0 = O(1)$ then we have a source of energy for large eddy motion. On the other hand, there is a transfer of energy back from the large eddies to the small eddies, and if it is assumed (unjustifiably!) that the scales of $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are quite distinct, this effect may reasonably be represented by an eddy viscosity $\nu_T(y)$. The picture then is that large eddies are initiated by random interactions of nearly equal Fourier components of the velocity field, distorted by interaction with the mean flow, and attenuated by something like an eddy viscosity mechanism. The equilibrium of these processes will determine a steady statistical structure.

The hope for such a description of large eddy dynamics lies to some extent in the qualitative nature of the correlation curves of figure 1. There can be little doubt that the anisotropy is due to the persistent effect of the mean shear, and the obvious thing to do is to see what happens to isotropic turbulence when it is sheared, and when the non-linear self-modulation of the velocity field is represented by an eddy viscosity. A calculation of the response of turbulence to uniform shear $(\alpha y, 0, 0)$ neglecting viscous and non-linear effects (Moffatt 1965) shows that the dominant contribution to the turbulent energy ultimately (i.e. for $\alpha t \gg 1$) comes from those Fourier components whose wave vectors are nearly in the y - z plane, i.e. from eddies with negligible variation in the x-direction; moreover this energy is concentrated in the x-component of velocity. If viscosity is included in this calculation (and this may include the eddy viscosity of small scale turbulence) then the turbulence ultimately decays (a result recorded by Pearson 1959); if the Reynolds number R based on α^{1} and on the initial length scale of the turbulence is large however, then a high degree of anisotropy develops before the ultimate decay. A stationary random isotropic driving force generates a stationary random, but anisotropic, turbulent velocity field; the greater the value of R the greater the degree of anisotropy. The hope is that for suitable choice of R, the correlation tensor of the sheared turbulence will compare favourably with the measured correlation tensor at a given value of y/δ_0 in the boundary layer. Preliminary numerical calculations (Townsend, private communication) suggest that this hope is well-founded. [J. Fluid Mech. 41 (1970), 13]

The above type of approach puts the emphasis on the change of structure of a disturbance as it is sheared by the mean flow. An alternative, and to some extent, complementary approach, which likewise involves a linearised treatment of the velocity fluctuations, has been developed by Landahl (1967) and (in the context of the sublayer) by Schubert and Corcos (1967). In Landahl's theory, no distinction is made between large scale and small scale turbulence, but the distinction is to some extent implicit in that the non-linear terms in the equation of motion are treated as forcing terms, independent of the response. In the theory of Schubert and Corcos, the non-linear terms are neglected, with slender justification, and the velocity fluctuations are driven by the pressure field which is conceived as being imposed at the outer edge of the sublayer. Both theories give rise to an Orr-Sommerfeld equation involving the mean profile $U(y)$ and a forcing term. The free modes of the equation are all damped waves, and a stationary random forcing term therefore again gives rise to a stationary random response. A notable feature of the numerical results of Schubert and Corcos is the marked anisotropy of the response; fluctuations in the sublayer for which l_x/l_z is large are more easily excited than those for which l_x/l_z is $O(1)$.

The development of an arbitrary localised disturbance on a scale of say $\frac{1}{5}\delta_0$, and imbedded in the outer layer (say $y/\delta_0 \approx \frac{1}{2}$), may be followed by two methods which should give equivalent results if suitably interpreted. Method I is to treat the disturbance as a sum of eigenfunctions of the Orr-Sommerfeld equation, in the spirit of Landahl (1967), the coefficients in the sum being determined by the initial structure of the disturbance. This approach puts the emphasis on those aspects of the disturbance which do <u>not</u> change with time; and the eigenfunction with the weakest damping rate will ultimately dominate suggesting that a wavy structure may ultimately emerge. Method II is to treat the mean shear as locally uniform (an approximation that of course improves as the scale l_y of the disturbance decreases) and to follow the distortion of the disturbance in the manner adopted by Moffatt (1965). This method puts the emphasis on the changing structure of the disturbance, and there is no suggestion of a wavy structure developing at any stage.

In order to illustrate the merits and limitations of the two methods, a particularly simple example will be worked out (in section 3) by both methods (and the results shown to be the same). The calculation suggests that, under circumstances where method II is applicable, it gives results which are physically more illuminating than method I. However, method II has certain severe limitations (for example, it cannot cope with the effects of curvature of the mean velocity profile)*, and then method I must be used; but the inference is that results derived by this method must then be interpreted with extreme caution.

*But see Lighthill, paper presented at same Symposium.

2. The equations describing large eddy development

With a mean velocity field $\underset{\sim}{U} = (U(y), 0, 0)$, the equation for the turbulent fluctuation $u_i(\underset{\sim}{x}, t)$ may be written

$$\frac{Du_i}{Dt} + u_2 U'(y)\delta_{i1} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + f_i , \tag{2.1}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x} , \tag{2.2}$$

and

$$f_i = -\frac{\partial}{\partial x_j}\left(u_i u_j - \overline{u_i u_j}\right) + \nu\nabla^2 u_i , \tag{2.3}$$

the overbar denoting the usual time average. Let $\langle \ldots \rangle$ denote an average over some scale $\epsilon\delta_0$, where $\epsilon \ll 1$, and let $\underset{\sim}{v} = \langle \underset{\sim}{u} \rangle$, $\underset{\sim}{w} = \underset{\sim}{u} - \underset{\sim}{v}$. If (2.1) is averaged $\langle \ldots \rangle$ and linearised in $\underset{\sim}{v}$, we obtain

$$\frac{Dv_i}{Dt} + v_2 U'(y)\delta_{i1} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\langle p \rangle + \langle f_i \rangle , \tag{2.4}$$

where

$$\langle f_i \rangle = -\frac{\partial}{\partial x_j}\left(\langle w_i w_j \rangle - \overline{w_i w_j}\right) + \nu\nabla^2 v_i . \tag{2.5}$$

Physical considerations, as outlined in §1, suggest now that we write

$$\langle f_i \rangle = T_i(\underset{\sim}{x}, t) + \frac{\partial}{\partial x_j}\nu_T(y)\frac{\partial v_i}{\partial x_j} , \tag{2.6}$$

where T_i represents the source term for the $\underset{\sim}{v}$-field, and the eddy viscosity $\nu_T(y)$ includes the kinematic viscosity ν.

Equations (2.4) and (2.6), together with

$$\frac{\partial v_i}{\partial x_i} = 0, \tag{2.7}$$

are linear equations which determine (in principle) the response v_i if T_i and $\nu_T(y)$ are known. In practice, of course, only the statistical properties of the field T_i will be accessible, and only the statistical properties of the field $v_i(\underset{\sim}{x}, t)$ may be inferred.

If now, for simplicity of notation, we replace (v_1, v_2, v_3) by (u, v, w), and eliminate u, w and <p> from (2.4) and (2.7), we obtain (cf Landahl 1967) the inhomogeneous Orr-Sommerfeld equation

$$\frac{D}{Dt}\nabla^2 v - U''(y)\frac{\partial v}{\partial x} - \nabla^2 \frac{\partial}{\partial x_i}\nu_T(y)\frac{\partial v}{\partial x_i} = q, \tag{2.8}$$

where
$$q = \nabla^2 T_2 - \frac{\partial}{\partial y}\nabla . \underset{\sim}{T} . \tag{2.9}$$

Defining the Fourier transform

$$\hat{q}(k_1, y, k_3, t) = \iint_{-\infty}^{\infty} q(x, y, z, t)\, e^{i(k_1 x + k_3 z)} dk_1 dk_3, \tag{2.10}$$

and similarly for $\hat{v}$, (2.8) becomes

$$\frac{\hat{D}}{Dt}\hat{\nabla}^2\hat{v} - ik_1 U''(y)\hat{v} - \hat{\nabla}^2\left[\nu_T \hat{\nabla}^2 \hat{v} + \nu_T' \frac{\partial \hat{v}}{\partial y}\right] = \hat{q}, \tag{2.11}$$

where now

$$\frac{\hat{D}}{Dt} = \frac{\partial}{\partial t} + ik_1 U(y), \quad \hat{\nabla}^2 = \frac{\partial^2}{\partial y^2} - \kappa^2, \quad \kappa^2 = k_1^2 + k_3^2. \tag{2.12}$$

For given $\hat{q}$, this should be solved subject to the boundary conditions

$$\hat{v} = \frac{\partial \hat{v}}{\partial y} = 0 \quad at \quad y = 0, \infty. \tag{2.13}$$

3. Waves or Eddies? A simple example

It is natural as a preliminary, to study the free modes associated with the homogeneous form of (2.11), with $\hat{q} \equiv 0$. To simplify the discussion, and to throw some light on the relation between 'wave-modes' of the type discussed by Landahl (1967) and the distorting eddies discussed by Townsend (1956), we shall suppose that the scale of the disturbance $\hat{v}(y, t)$* in the y-direction is small compared with δ_0 (its scale in the x- and z-directions being unrestricted), i.e. that

$$\left| U(y) \frac{\partial^2 \hat{v}}{\partial y^2} \right| \gg \left| U''(y)\, \hat{v} \right| . \tag{3.1}$$

If the disturbance is localised near the level $y = y_0$, we may then make a linear approximation to $U(y)$:

$$U(y) \approx U(y_0) + \alpha (y - y_0), \tag{3.2}$$

where $\alpha = (dU/dy)_{y=y_0}$. The disturbance is convected at velocity $U(y_0)$ and distorted by the uniform shear $\alpha(y - y_0)$. If we take axes moving in the x-direction with velocity $U(y_0)$, the effect of the convection disappears, and only the distortion remains.

The variation of ν_T with y may also be ignored if the scale of the disturbance in the y-direction is sufficiently limited; writing $Y = y - y_0$, equation (2.11) (with $\hat{q} = 0$) becomes

$$\left(\frac{\partial}{\partial t} + ik_1 \alpha Y \right) \left(\frac{\partial^2}{\partial Y^2} - \kappa^2 \right) \hat{v} - \nu_T \left(\frac{\partial^2}{\partial Y^2} - \kappa^2 \right)^2 \hat{v} = 0 . \tag{3.3}$$

The analysis of the development of $\hat{v}(Y, t)$ from a given initial distribution,

$$\hat{v}(Y, 0) = \hat{v}_0(Y), \tag{3.4}$$

is particularly simple and <u>illuminating in the inviscid limit $\nu_T = 0$</u>; the

* The dependence on k_1 and k_3 will be omitted unless explicitly required.

two methods mentioned in §1 will be illustrated in detail for this case. The effects of eddy viscosity will then be briefly considered. It will be assumed that y_0 is in the outer layer (say $y_0/\delta_0 \approx \frac{1}{2}$) and that the disturbance, localised near $y = y_0$, is negligibly influenced by the wall. Where convenient, it will then be legitimate to replace the conditions (2.13) by the conditions

$$\hat{v} = \frac{\partial \hat{v}}{\partial Y} = 0 \quad \text{at} \quad Y = \pm\infty. \tag{3.5}$$

Inviscid development

Method I

The equation

$$\left(\frac{\partial}{\partial t} + i k_1 \alpha Y\right)\left(\frac{\partial^2}{\partial Y^2} - k^2\right)\hat{v} = 0 \tag{3.5}$$

admits solutions of the form

$$\tilde{v}(Y)\, e^{-i\kappa c t}, \tag{3.6}$$

representing wave modes travelling with velocity c in the direction $(k_1, 0, k_3)$. The function $\tilde{v}(Y)$ must be a linear combination of $e^{\kappa Y}$ and $e^{-\kappa Y}$ except possibly at the 'critical' point where

$$Y = \frac{\kappa c}{\alpha k_1} = \eta, \text{ say.} \tag{3.7}$$

Since $\hat{v}$ must certainly be continuous across $Y = \eta$, the solution satisfying the conditions (3.5) is

$$\tilde{v}(Y) = \left.\begin{matrix} V(\eta)\, e^{-\kappa(Y-\eta)} & (Y>\eta), \\ V(\eta)\, e^{\kappa(Y-\eta)} & (Y<\eta), \end{matrix}\right\} \tag{3.8}$$

where $V(\eta)$ is the amplitude of the mode centred on $Y = \eta$. For varying η we have a continuous spectrum of modes with wave speed given by (3.7).

The general solution of (3.5) satisfying $|\hat{v}| \to 0$ as $|Y| \to \infty$ is a superposition of such modes, viz.

$$\hat{v}(Y,t) = \int_{-\infty}^{\infty} \tilde{v}(Y)\, e^{-i\kappa c t}\, d\eta$$

$$= \int_{-\infty}^{Y} V(\eta)\, e^{-\kappa(Y-\eta)}\, e^{-i\alpha k_1 \eta t} + \int_{Y}^{\infty} V(\eta)\, e^{\kappa(Y-\eta)}\, e^{-i\alpha k_1 \eta t}\, d\eta. \quad (3.9)$$

The initial condition gives

$$\hat{v}_0(Y) = \int_{-\infty}^{Y} V(\eta)\, e^{-\kappa(Y-\eta)}\, d\eta + \int_{Y}^{\infty} V(\eta)\, e^{\kappa(Y-\eta)}\, d\eta, \quad (3.10)$$

an integral equation for $V(\eta)$ with solution

$$V(\eta) = \tfrac{1}{2}\kappa\, \hat{v}_0(\eta) - \frac{1}{2\kappa}\, \hat{v}_0''(\eta). \quad (3.11)$$

Hence (3.9) gives $\hat{v}(Y, t)$ explicitly in terms of its initial distribution.

<u>Method II</u>

Equation (3.5) also admits solutions of the form

$$\hat{v}_1(Y,t) = A(t)\, e^{i k_2(t) Y}. \quad (3.12)$$

Substitution in (3.5) gives

$$\left(\frac{\partial}{\partial t} + i k_1 \alpha Y\right)\left[A(t)\left(k_2^2(t) + \kappa^2\right) e^{i k_2(t) Y}\right] = 0,$$

and this can be satisfied (for all Y) only if

$$k_2'(t) = -\alpha k_1, \quad \text{i.e.} \quad k_2(t) = k_{20} - \alpha k_1 t, \quad (3.13)$$

where $k_{20} = k_2(0)$, and

$$A(t)\left(k_2^2(t) + \kappa^2\right) = \text{const.} = A(0)\left(k_{20}^2 + \kappa^2\right). \quad (3.14)$$

Hence, if the initial disturbance is represented in terms of its Fourier transform,

$$\hat{v}_0(Y) = \int_{-\infty}^{\infty} A(k_{20})\, e^{ik_{20}Y}\, dk_{20}\,, \tag{3.15}$$

then

$$\hat{v}(Y,t) = \int_{-\infty}^{\infty} A(k_{20}) \frac{k_{20}^2 + \kappa^2}{(k_{20} - \alpha k_1 t)^2 + \kappa^2}\, e^{i(k_{20} - \alpha k_1 t)Y}\, dk_{20}\,. \tag{3.16}$$

This must of course be equivalent to (3.9); the two expressions are shown to be the same in the Appendix.

The important thing to notice about (3.16) is that for $\alpha t \gg 1$, $\hat{v}(k_1, Y, k_3, t)$ will be $O(\alpha t)^{-2}$ except for the small range of wave numbers for which $k_1/k_{20} = O(\alpha t)^{-1}$, for which $\hat{v}$ remains $O(1)$. This simply reflects the fact that the disturbance is drawn out in the x-direction, its scale increasing asymptotically as αt. After sufficient shearing action, the disturbance is dominated by those Fourier components which have little or no variation in the x-direction.

The effect of eddy viscosity

Both of the above methods may be extended to the case when $\nu_T \neq 0$. For method I, the eigenfunctions of equation (3.3) with the boundary conditions (2.13) may be obtained in terms of Airy functions (see, for example, Reid 1965); the wave speeds are slightly changed (for large Reynolds number) and each mode is weakly damped. The discontinuity in the $\tilde{u}(Y)$ corresponding to the eigenfunction (3.8) at $Y = \eta$ is smoothed out through a critical layer through the influence of viscosity. The simplified boundary conditions (3.5) cannot be used for this method, since there are no non-trivial solutions of (3.3) satisfying (3.5). This may be regarded as an indication of the inappropriateness of the method, since it is scarcely conceivable that the wall can have any significant effect on a disturbance that is localised near $y = y_0$ in the outer layer; moreover it is hard to see why the complicated structure of an eigenmode across a critical layer at some level $y = y_c$ (where y_c is very different from y_0) can be relevant to the development of a disturbance which is vanishingly small outside some close neighbourhood of y_0.

Method II, by contrast, is modified in a relatively simple way when account is taken of eddy viscosity; all that is required is the inclusion of a factor

$$\exp - \nu_T \int_0^t \left(k_2^2(t) + \kappa^2\right) dt = \exp - \nu_T \left(k_0^2 t - k_1 k_{02} \alpha t^2 + \tfrac{1}{3} k_1^2 \alpha^2 t^3\right) \quad (3.17)$$

(where $k_0{}^2 = \kappa^2 + k_{02}{}^2$) in the expression (3.14) for $A(t)$. A mode for which $k_1 \alpha t = O(k_{02})$ (the only kind that contributes to $\hat{v}(Y, t)$ for $\alpha t \gg 1$) is damped out after a time of order $(\nu_T k_0^2)^{-1}$.
A localised disturbance of initial scale ℓ is damped out after a time of order $t_v = \ell^2/\nu_T$; at this stage, its degree of anisotropy q (i.e. the ratio of its scale in the x-direction to its scale in the y- or z-directions) is of order $\alpha t_v = \alpha \ell^2/\nu_T$ and this is simply the turbulent Reynolds number based on the scale of the disturbance in (say) the y-direction and the local mean velocity gradient. The turbulent Reynolds number $R_s = \alpha \delta_0^2/\nu_T$ (with $\alpha = \tau_w^{1/2}/K\delta_0$) can be estimated from the mean velocity profile (Townsend 1956, § 10.8) and is in the neighbourhood of 130. With $\ell \approx \frac{1}{5}\delta_0$ (the ratio of the two terms in (3.1) being then approx. 0.04), we get a degree of anisotropy

$$q \approx \frac{130}{25} = 5.2,$$

corresponding to eddies of extent $\sim \delta_0$ in the x-direction. This is a little greater than the estimates given in § 1 of the degree of anisotropy based on correlation measurements; this may be partly due to the general crudity of the estimates, but a possible further reason is that the neglected terms non-linear in $\underset{\sim}{v}$ (i.e. that part of the non-linearity of the turbulencethat cannot be represented by an eddy viscosity) have the general effect of resisting shear, and therefore of decreasing the actual degree of anisotropy of the turbulence under statistically steady conditions.

4. The generation of Reynolds stress by streamwise vortices

Since both experimental observations and the considerations of the preceding section suggest that the large eddies depend only weakly on the x-coordinate, it is natural to look in particular at motions which are quite independent of x . The simplest form of (2.4), linearised, and with $f_i = 0$, $\nu_T = 0$, is then

$$\frac{\partial v_i}{\partial t} + v_2 U'(y)\,\delta_{i1} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\langle p\rangle . \quad (4.1)$$

Moreover, since $\nabla \cdot \underset{\sim}{v} = 0$, $\nabla^2 \langle p \rangle = 0$, and for a localised disturbance within the boundary layer, this requires $\langle p \rangle = 0$. Hence, with $v_i = (u, v, w)$,

$$v = \text{const.} = v_0(y, z), \tag{4.2}$$

$$w = \text{const.} = w_0(y, z), \tag{4.3}$$

and
$$u(y, z, t) = u_0(y, z) - v_0(y, z)\, U'(y)\, t. \tag{4.4}$$

Equation (4.4) expresses the simple Prandtl mechanism whereby fluctuations in the x-component of velocity are generated through the transport of mean momentum by fluctuations normal to the wall. If $u_0(y, z) = C\, v_0(y, z)$ (so that initially the planes of circulation within the eddies are at an angle $\gamma = \tan^{-1} C$ to the x-axis) then

$$u(y, z, t) = \left[C - t\, U'(y)\right] v_0(y, z), \tag{4.5}$$

so that at time t the streamlines of the disturbance lie in the surfaces

$$x - Cy + t\, U(y) = \text{const.} \tag{4.6}$$

These surfaces are tilted progressively backwards (relative to the direction of the shear) contrary to the description of the process given in Townsend (1956) §6.1). If, in the neighbourhood of $y = y_0$, we write $U = U_0 + \alpha(y - y_0)$, the surfaces (4.6) are the planes

$$x - (C - \alpha t)(y - y_0) = \text{const.}, \tag{4.7}$$

i.e. the plane of circulation in a typical eddy rotates anticlockwise, and asymptotes to the plane $y = y_0$ (figure 2).

For a random superposition of eddies of the above kind, the initial planes of circulation being randomly oriented, the shear will cause all the planes of circulation to tilt backwards, as indicated in figure 2. Eddy viscosity causes an ultimate decay of each eddy, and its magnitude will determine the orientation of the plane of circulation, when the energy of the eddy is a maximum. Clearly, if C is randomly distributed with $\overline{C} = 0$, the average plane of circulation will be tilted backwards as in figure 3, although not necessarily at the angle 45° to the x-axis suggested by Townsend.

It is easy to see how eddies of this kind contribute to the Reynolds stress. From (4.4),

$$-uv = -u_0 v_0 + v_0^2 U'(y) t. \tag{4.8}$$

The linear growth for a single eddy lasts only so long as the eddy viscosity is negligible (or as long as the linearised description is valid); after a time of order $t_v = \ell^2 \nu_T^{-1}$, the eddy decays. New eddies are continuously generated by the body force T_i. For a random distribution of eddies at different stages of growth and decay, and such that at their initiation u_0 and v_0 are uncorrelated, (4.8) gives

$$-\overline{uv} = A\overline{v^2}U'(y), \tag{4.9}$$

where A is a measure of the mean large eddy lifetime, and this is the contribution to the Reynolds stress from eddies having negligible variation in the streamwise direction.

The contribution to the development of the mean flow in the x-direction from this contribution to the Reynolds stress is given by

$$\frac{DU}{Dt} = U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{\partial}{\partial y} A\overline{v^2}\frac{\partial U}{\partial y}, \tag{4.10}$$

a diffusion equation with diffusivity $A\overline{v^2}$ a function of y. Although the v in this equation refers only to the large eddy velocity, it seems likely that $\overline{v^2}$ has the same qualitative behaviour (as a function of y) as the mean square of the total normal velocity fluctuation $v_2 + w_2$. Equation (4.10) then reflects (i) the trapping of mean flow vorticity near its source at the wall where $A\overline{v^2}$ is very small, and (ii) the effective mixing of mean flow momentum towards the outer edge of the constant stress layer and in the outer layer, particularly around $y/\delta_\varphi \approx 0.25$ where $\overline{(v_2 + w_2)^2}$ has its greatest value.

Appendix: Equivalence of (3.16) and (3.9)

The inverse of (3.15) is

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{v}_o(\eta)\, e^{-ik\eta}\, d\eta, \tag{A1}$$

and substitution in (3.16) gives

$$\hat{v}(Y,t) = \int_{-\infty}^{\infty} \hat{v}_o(\eta)\, F(\eta)\, d\eta, \tag{A2}$$

where

$$F(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{k^2+\kappa^2}{(k-\alpha k_1 t)^2+\kappa^2} - 1\right\} e^{-ik\eta}\, e^{i(k-\alpha k_1 t)Y}\, dk + \frac{1}{2\pi}\, e^{-i\alpha k_1 tY}\int_{-\infty}^{\infty} e^{ik(Y-\eta)}\, dk. \tag{A3}$$

Contour integration gives, for $\eta \leqslant Y$,

$$F(\eta) = i\alpha k_1 t\left(1 - \frac{i\alpha k_1 t}{2\kappa}\right) e^{-i\alpha k_1 t\eta}\, e^{-\kappa(Y-\eta)} + e^{-i\alpha k_1 tY}\,\delta(Y-\eta), \tag{A4}$$

and, for $\eta \geqslant Y$,

$$F(\eta) = -i\alpha k_1 t\left(1 + \frac{i\alpha k_1 t}{2\kappa}\right) e^{-i\alpha k_1 t\eta}\, e^{-\kappa(\eta-Y)} + e^{-i\alpha k_1 tY}\,\delta(Y-\eta) \tag{A5}$$

Substitution in (A2) gives

$$\begin{aligned}\hat{v}(Y,t) = \hat{v}_o(Y)\, e^{-i\alpha k_1 tY} &+ i\alpha k_1 t\left(1 - \frac{i\alpha k_1 t}{2\kappa}\right)\int_{-\infty}^{Y} \hat{v}_o(\eta)\, e^{\kappa(\eta-Y)}\, e^{-i\alpha k_1 t\eta}\, d\eta \\ &- i\alpha k_1 t\left(1 + \frac{i\alpha k_1 t}{2\kappa}\right)\int_{Y}^{\infty} \hat{v}_o(\eta)\, e^{-\kappa(\eta-Y)}\, e^{-i\alpha k_1 t\eta}\, d\eta.\end{aligned} \tag{A6}$$

This expression may equally be obtained from (3.9) on substitution of (3.11) and integration by parts. Hence the equivalence of (3.9) and (3.15) is established.

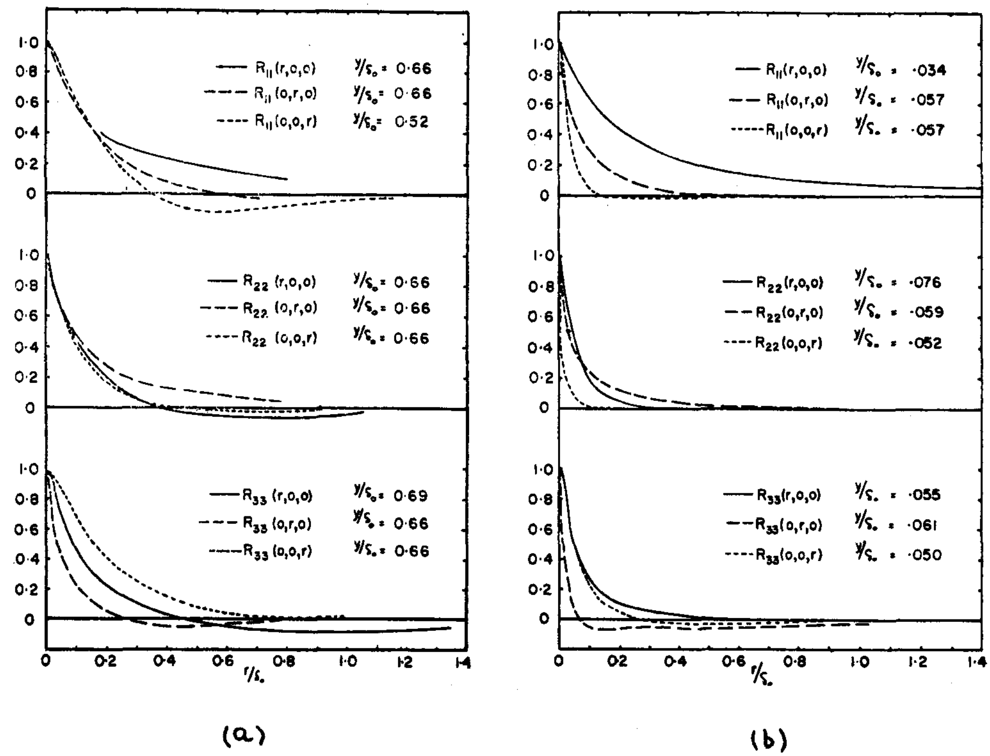

Figure 1. Correlation curves from Grant (1958); in (a) the fixed probe is in the outer layer; in (b) in the constant stress layer

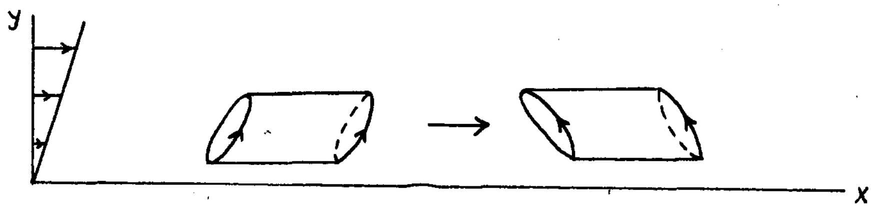

Figure 2. Rotation of the planes of circulation of eddies having no variation in the x-direction by uniform shear.

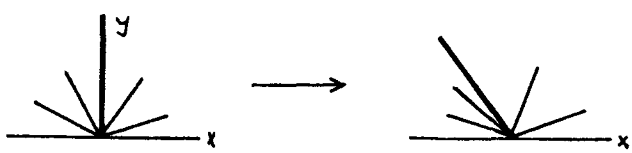

Figure 3. Rotation of planes of circulation of a random superposition of eddies of the type shown in figure 2.