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DYNAMO INSTABILITY AND FEEDBACK IN A STOCHASTICALLY DRIVEN SYSTEM

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ABSTRACT

In §1, a method of treatment of the equation

$$\frac{\partial y_i}{\partial t} = \frac{\partial}{\partial x_j}\left(a_{ijk}(\underset{\sim}{x},t)\,y_k\right) + \lambda\nabla^2 y_i\,,$$

where $a_{ijk}(\underset{\sim}{x},t)$ is a random tensor field of known statistical properties is reviewed, with particular reference (i) to the magnetohydrodynamic turbulent dynamo problem, and (ii) to the problem of diffusion of a passive scalar field by turbulent motion. The result of Steenbeck, Krause and Radler (1966), that in the former context dynamo action can occur (i.e. the ensemble average of $\underset{\sim}{y}$ can grow without limit) provided the statistics of the turbulent field lack reflexional symmetry, is discussed within the framework of the above general equation.

In §§ 2 and 3, the feedback mechanism in the magnetohydrodynamic context is considered. It is supposed that a velocity field lacking reflexional symmetry is generated in an electrically conducting fluid by a random body force of known statistical properties. Conditions are then conducive to the growth of large scale magnetic field perturbations. The growth is limited by the fact that the growing Lorentz force progressively modifies the statistical structure of the velocity field, until ultimately a statistical equilibrium is achieved. It is shown that in this equilibrium the magnetic energy density may exceed the kinetic energy density by a factor $O(L/\ell) \gg 1$, where L is the scale of the magnetic field, and ℓ the scale of the turbulence.

1. INSTABILITIES OF THE LINEAR DIFFUSION EQUATION WITH A RANDOM CONVECTIVE TERM

Let $a_{ijk}(\underset{\sim}{x},t)$ be a random tensor field, statistically homogeneous in $\underset{\sim}{x}$ and stationary in t, with $\langle a_{ijk}\rangle = 0$, the angular brackets representing an ensemble average. We shall consider the evolution of the random vector field $y_i(\underset{\sim}{x},t)$ satisfying the equation

$$\frac{\partial y_i}{\partial t} = \frac{\partial}{\partial x_j}\left(a_{ijk}\,y_k\right) + \lambda\nabla^2 y_i \tag{1.1}$$

and the initial condition $\underset{\sim}{y}(\underset{\sim}{x},0) = \underset{\sim}{y}_0(\underset{\sim}{x})$ where $\int \underset{\sim}{y}_0^2\,d\underset{\sim}{x} < \infty$.

Two particular choices of a_{ijk} correspond to well-known problems in the turbulence context. First, if

$$a_{ijk} = \epsilon_{ij\ell}\,\epsilon_{\ell mk}\,u_m(\underset{\sim}{x},t)\,, \tag{1.2}$$

where $\underset{\sim}{u}$ is a (turbulent) velocity field, then with $\underset{\sim}{y} = \underset{\sim}{B}$, (1.1) becomes the induction equation of magnetohydrodynamics,

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \wedge (\underset{\sim}{u} \wedge \underset{\sim}{B}) + \lambda \nabla^2 \underset{\sim}{B} . \tag{1.3}$$

In this context, it has been established by Steenbeck, Krause and Radler (1966) (hereafter SKR) that a sufficient condition for unlimited growth of $\langle \underset{\sim}{B}^2 \rangle$ (i.e. for turbulent dynamo action) is that the statistical properties of the $\underset{\sim}{u}$-field should lack reflexional symmetry, i.e. should not be invariant under a change from a left-handed to a right-handed frame of reference. This situation is likely to occur whenever the external conditions lack reflexional symmetry, e.g. in a rotating fluid with a mean density gradient or a mean energy flux parallel to the rotation vector. The theory of SKR has crucial importance in geomagnetism and in cosmical electrodynamics, and a range of applications and elaborations have already been worked out. Two recent reviews of dynamo theory (Roberts 1971, Weiss 1971) between them give a comprehensive list of references, particularly to developments since 1966, which are largely the outcome of the SKR breakthrough.

Secondly, if

$$a_{ijk} = -\delta_{ij}\, u_k(\underset{\sim}{x}, t), \tag{1.4}$$

then, with $\underset{\sim}{y} = \underset{\sim}{G} = \nabla\theta$, (1.1) becomes the gradient of the equation

$$\frac{\partial \theta}{\partial t} = -\underset{\sim}{u} . \nabla\theta + \lambda \nabla^2 \theta , \tag{1.5}$$

describing the convection and diffusion of a passive scalar contaminant. In this context, the effect of the turbulence on a localised θ-field is equivalent to that of an eddy diffusivity; the expression for this diffusivity was obtained by Taylor (1921) for $\lambda = 0$, and the first correction (for small λ) by Saffman (1962). These results may be recovered by the method described below, and in addition a simple expression for the limiting form of the eddy diffusivity for large λ may be obtained.

The method of treatment of (1.1) is as follows:

Let $\underset{\sim}{Y}(\underset{\sim}{x}, t) = \langle \underset{\sim}{y}(\underset{\sim}{x}, t) \rangle$ and let $\underset{\sim}{y}' = \underset{\sim}{y} - \underset{\sim}{Y}$; then from (1.1),

$$\frac{\partial Y_i}{\partial t} = \frac{\partial}{\partial x_j} E_{ij} + \lambda \nabla^2 Y_i , \tag{1.6}$$

where $E_{ij} = \langle a_{ijk} y_k' \rangle$. Subtracting this from (1.1),

$$\frac{\partial y_i'}{\partial t} = \frac{\partial}{\partial x_j}\left(a_{ijk} Y_k\right) + \frac{\partial}{\partial x_j}\left(a_{ijk} y_k' - \langle a_{ijk} y_k' \rangle\right) + \lambda \nabla^2 y_i' . \tag{1.7}$$

Equation (1.7), with the initial condition $y_i' = 0$, establishes a linear relation between y_i' and Y_k, say

$$y_i'(\underset{\sim}{x}, t) = \int K_{ij}(\underset{\sim}{x}, t; \underset{\sim}{x}', t') \, Y_j(\underset{\sim}{x}', t') \, d^3\underset{\sim}{x}' \, dt' , \tag{1.8}$$

where K_{ij} is a functional of a_{ijk} and of the parameter λ. The range of integration is over all $\underset{\sim}{x}'$ and over $0 \leq t' \leq t$.

We can now construct the tensor E_{ij} needed in (1.6), viz.,

$$E_{ij}(\underset{\sim}{x}, t) = \int V_{ijk}(\underset{\sim}{\xi}, \tau) \, Y_k(\underset{\sim}{x}', t') \, d^3\underset{\sim}{\xi} \, d\tau \tag{1.9}$$

where $\underset{\sim}{\xi} = \underset{\sim}{x}' - \underset{\sim}{x}$, $\tau = t' - t$, and

$$V_{ijk}(\underset{\sim}{\xi}, \tau) = \langle a_{ijm}(\underset{\sim}{x}, t) \, K_{mk}(\underset{\sim}{x}, t; \underset{\sim}{x}' t') \rangle . \tag{1.10}$$

V_{ijk} depends only on $\underset{\sim}{\xi}$ and τ because of the assumed homogeneity and stationarity of a_{ijk}; it also depends implicitly on λ. We may certainly suppose that

$$V_{ijk}(\underset{\sim}{\xi}, \tau) \rightarrow 0 \quad \text{as} \quad |\underset{\sim}{\xi}| \rightarrow \infty \quad \text{and as} \quad \tau \rightarrow -\infty , \tag{1.11}$$

but the scales ℓ, t_o characteristic of this tensor (which could for example be defined in terms of its moments) may depend on λ as well as on the statistical properties of a_{ijk}.

Let us now consider solutions $\underset{\sim}{Y}(\underset{\sim}{x}, t)$ of (1.6) having scales L, T satisfying

$$L \gg \ell \quad , \quad T \gg t_o .$$

Then in (1.9) we may expand $Y_k(\underset{\sim}{x}', t')$ in Taylor series about $(\underset{\sim}{x}, t)$, and integrate term by term, obtaining

$$E_{ij} = A^{(0)}_{ijk} Y_k + A^{(1)}_{ijkl} \frac{\partial Y_k}{\partial x_l} + \cdots + B^{(1)}_{ijk} \frac{\partial Y_k}{\partial t} + \cdots, \qquad (1.12)$$

where

$$A^{(0)}_{ijk} = \int V_{ijk}\, d^3\xi\, d\tau, \quad A^{(1)}_{ijkl} = \int \xi_l V_{ijk}\, d^3\xi\, d\tau, \quad B^{(1)}_{ijk} = \int \tau V_{ijk}\, d^3\xi\, d\tau, \text{ etc.} \qquad (1.13)$$

The range of integration in these moments is over all ξ, and over $-t \leqslant \tau \leqslant 0$; for $t \gg t_0$, the time range is effectively $-\infty < \tau \leqslant 0$, and the tensors defined by (1.13) are then simply constant tensors, determined solely by the statistical properties of a_{ijk} and by λ.

The expansion (1.12) may be rearranged, using (1.6) repeatedly, so that only space-derivatives appear; in this way we obtain

$$E_{ij} = A^{(0)}_{ijk} Y_k + A^{(1)}_{ijkl} \frac{\partial Y_k}{\partial x_l} + \cdots, \qquad (1.14)$$

where, in fact, $A^{(1)}_{ijkl} = \tilde{A}^{(1)}_{ijkl} + B^{(1)}_{ijm} A^{(0)}_{mlk}$, and higher coefficients may be similarly obtained. Equation (1.6) then becomes an equation with constant coefficients

$$\frac{\partial Y_i}{\partial t} = A^{(0)}_{ijk} \frac{\partial Y_k}{\partial x_j} + A^{(1)}_{ijkl} \frac{\partial^2 Y_k}{\partial x_j \partial x_l} + \cdots + \lambda \nabla^2 Y_i. \qquad (1.15)$$

The term involving $A^{(1)}_{ijkl}$ may be recognized as describing an 'eddy diffusion' effect which certainly appears in both the particular cases given by (1.2) and (1.4). The term involving $A^{(0)}_{ijk}$ is less familiar. It appears in the magnetohydrodynamic case (1.2), and is described in that context (for notational reasons) as the 'α-effect' by Steenbeck _et al_ (1966)*; (would 'SKR-effect' or 'helicity effect' - see Moffatt 1970a - not be more apt?). It does not appear in the second situation described by (1.4), the reason being essentially that the kernel function V_{ijk} in (1.10) is in this case expressible as a gradient and so the

*Footnote: E.N. Parker anticipated the SKR-effect through inspired physical reasoning (Parker 1955); he used the symbol Γ where SKR use α !

expression for $A^{(0)}_{ijk}$ given by (1.13) vanishes.

<u>The case when $a_{ijk}(\underset{\sim}{x},t)$ is statistically isotropic</u>

Suppose now that $a_{ijk}(\underset{\sim}{x},t)$ is statistically isotropic, i.e. that its statistical properties are invariant under rotations of the frame of reference. Then, in particular, the tensors $A^{(m)}_{ijk---}$ are isotropic, i.e.

$$A^{(0)}_{ijk} = \alpha\, \epsilon_{ijk}\,, \tag{1.16}$$

$$A^{(1)}_{ijkl} = \beta_1 \delta_{ij}\delta_{kl} + \beta_2 \delta_{ik}\delta_{jl} + \beta_3 \delta_{il}\delta_{jk}\,, \tag{1.17}$$

etc.

If the statistical properties of $a_{ijk}(\underset{\sim}{x},t)$ are invariant under reflexions of the frame of reference as well as rotations, then $A^{(m)}_{ijk---} = 0$ when m is even, and in particular $\alpha = 0$. Only if the random field lacks reflexional symmetry can we have $\alpha \neq 0$. Note that α is a pseudo-scalar*, while β_1 , β_2 and β_3 are pure scalars.

When $\alpha \neq 0$, the first term on the right of (1.15) clearly dominates the development of $\underset{\sim}{Y}(\underset{\sim}{x},t)$ provided the scale L is sufficiently large, and we then have simply

$$\frac{\partial \underset{\sim}{Y}}{\partial t} = \alpha\, \nabla \wedge \underset{\sim}{Y} \quad + \text{ negligible terms.} \tag{1.18}$$

Any mode for which

$$\nabla \wedge \underset{\sim}{Y} = (\operatorname{sgn}\alpha)\, K\, \underset{\sim}{Y} \tag{1.19}$$

then grows exponentially according to

$$\underset{\sim}{Y}(\underset{\sim}{x},t) = \underset{\sim}{Y}_0(\underset{\sim}{x})\, e^{|\alpha| K t}\,, \tag{1.20}$$

if K is sufficiently small. For example, the Fourier components

$$\underset{\sim}{Y}_0(\underset{\sim}{x}) = \tilde{Y}_0(K)\left(\cos Kz,\ \mp \sin Kz,\ 0\right) \tag{1.21}$$

satisfy $\nabla \wedge \underset{\sim}{Y}_0 = \pm K\underset{\sim}{Y}_0$, and one of these Fourier components will grow exponentially if it is represented in the Fourier transform of the initial field $\underset{\sim}{y}_0(\underset{\sim}{x})$.

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* Cf the mean helicity $\langle \underset{\sim}{u} \cdot (\nabla \wedge \underset{\sim}{u}) \rangle$.

In the reflexionally symmetric case, $\alpha = 0$, and (1.15) becomes

$$\frac{\partial \underset{\sim}{Y}}{\partial t} = (\beta_1 + \beta_3)\nabla\nabla\cdot\underset{\sim}{Y} + (\beta_2 + \lambda)\nabla^2\underset{\sim}{Y} \quad \text{+ negligible terms.} \tag{1.22}$$

In the first example, with $\underset{\sim}{Y} = \langle \underset{\sim}{B} \rangle$, $\nabla\cdot\underset{\sim}{Y} = 0$, and we simply have an 'eddy diffusivity'

$$\lambda_t^{(1)} = \beta_2 . \tag{1.23}$$

In the second example, with $\underset{\sim}{Y} = \langle \nabla\theta \rangle$, $\nabla\wedge(\nabla\wedge\underset{\sim}{Y}) = 0$ so that $\nabla\nabla\cdot\underset{\sim}{Y} = \nabla^2\underset{\sim}{Y}$, and we have an eddy diffusivity

$$\lambda_t^{(2)} = \beta_1 + \beta_2 + \beta_3 . \tag{1.24}$$

One would of course expect these diffusivities to turn out to be positive in real physical situations, and they do turn out to be positive in every limiting case amenable to analysis. However no general proof that they must invariably be positive appears to be available, and there remains the intriguing possibility that in certain circumstances the (net) diffusivity may be negative with interesting inverse diffusion consequencies.

<u>The case when $a_{ijk}(\underset{\sim}{x},t)$ is statistically anisotropic but reflexionally symmetric</u>

In these circumstances again, $A_{ijk}^{(0)}$ need not vanish. We might for example have a situation where

$$A_{ijk}^{(0)} = \lambda_i\,\delta_{jk} + \mu_j\,\delta_{ki} + \nu_k\,\delta_{ij} , \tag{1.25}$$

where $\underset{\sim}{\lambda}$, $\underset{\sim}{\mu}$ and $\underset{\sim}{\nu}$ are any three vectors. Putting

$$\underset{\sim}{Y}(\underset{\sim}{x},t) = \tilde{\underset{\sim}{Y}}(\underset{\sim}{K})\,e^{i\underset{\sim}{K}\cdot\underset{\sim}{x} + mt} , \tag{1.26}$$

in (1.15), and retaining only the O(K) term on the right-hand side, we then have

$$m\tilde{Y}_i = i\left(\lambda_i K_k + (\underset{\sim}{\mu}\cdot\underset{\sim}{K})\,\delta_{ki} + \nu_k K_i\right)\tilde{Y}_k . \tag{1.27}$$

The determinantal condition for a non-trivial solution gives a cubic equation for m with roots

$$m_1 = i\underset{\sim}{K}\cdot\underset{\sim}{\mu} \quad , \qquad m_{2,3} = -i\underset{\sim}{K}\cdot\underset{\sim}{\mu} \pm \sqrt{(\underset{\sim}{K}\cdot\underset{\sim}{\lambda})(\underset{\sim}{K}\cdot\underset{\sim}{\nu}) - K^2\underset{\sim}{\lambda}\cdot\underset{\sim}{\nu}}\,. \tag{1.28}$$

Hence we have an instability when K is in a direction satisfying

$$(\underset{\sim}{K}\cdot\underset{\sim}{\lambda})(\underset{\sim}{K}\cdot\underset{\sim}{\nu}) > K^2\underset{\sim}{\lambda}\cdot\underset{\sim}{\nu}\,. \tag{1.29}$$

[This situation cannot arise in the magnetohydrodynamic context, in which, because of the particular structure (1.2) of a_{ijk}, A_{ijk} is anti-symmetric in i and j; the particular form (1.25) would then require $\underset{\sim}{\nu} = 0$ and $\underset{\sim}{\mu} = -\underset{\sim}{\lambda}$; the three roots $m_{1,2,3}$ are imaginary and there can be no question of instability.]

2. THE ESTABLISHMENT OF A NON-LINEAR EQUILIBRIUM

It has been shown in §1 that when the statistical properties of the a_{ijk}-field lack reflexional symmetry, equation (1.1) leads to exponential growth of Fourier components of the $\underset{\sim}{Y}$-field of sufficiently small wave-number. In a real physical situation, the growth must ultimately be limited by some back-reaction which modifies the statistical properties of the a_{ijk}-field. In the magnetohydrodynamic context, this back reaction is provided by the Lorentz force; the effect has been analysed when the $\underset{\sim}{u}$-field consists of a random field of decaying inertial waves in a rotating fluid (Moffatt 1970b). It has been further discussed in general terms by Roberts (1971). We shall here examine the magnetohydrodynamic situation, on the assumption that a body force $\underset{\sim}{f}(\underset{\sim}{x},t)$ of known statistical properties is applied to the fluid in order to generate the velocity field from which the magnetic field may be nourished and sustained. We assume that $\langle \underset{\sim}{f} \rangle = 0$. The equations governing the velocity field $\underset{\sim}{u}(\underset{\sim}{x},t)$ and the Alfvén velocity field $\underset{\sim}{v}(\underset{\sim}{x},t) = (\mu\rho)^{\frac{1}{2}}\ \underset{\sim}{B}(\underset{\sim}{x},t)$ are

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = -\nabla\chi + \underset{\sim}{v}\cdot\nabla\underset{\sim}{v} + \nu\nabla^2\underset{\sim}{u} + \underset{\sim}{f}\,, \tag{2.1}$$

$$\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{v} = \underset{\sim}{v}\cdot\nabla\underset{\sim}{u} + \lambda\nabla^2\underset{\sim}{v}, \tag{2.2}$$

and

$$\nabla.\underset{\sim}{u} = \nabla.\underset{\sim}{v} = 0. \tag{2.3}$$

As before, let $\underset{\sim}{V}(\underset{\sim}{x},t) = \langle \underset{\sim}{v} \rangle$ and suppose that the scale L of $\underset{\sim}{V}$ is large compared with the scale ℓ of any relevant statistical property of the $\underset{\sim}{u}$-field generated. The equation satisfied by $\underset{\sim}{V}$ is

$$\frac{\partial \underset{\sim}{V}}{\partial t} = \nabla_\wedge \langle \underset{\sim}{u}\wedge\underset{\sim}{h}\rangle + \lambda\nabla^2\underset{\sim}{V}, \tag{2.4}$$

where $\underset{\sim}{h} = \underset{\sim}{v} - \underset{\sim}{V}$, and the main objective is to obtain an expression for $\langle \underset{\sim}{u}\wedge\underset{\sim}{h}\rangle$ in terms of $\underset{\sim}{V}$ so that (2.4) may be integrated (see the expressions (2.6) and (2.13) below).

Assuming that $\langle \underset{\sim}{u}\rangle = 0$ (and this is something that will require further consideration in retrospect), the equations for $\underset{\sim}{u}$ and $\underset{\sim}{h}$ are

$$\left.\begin{aligned}\frac{\partial \underset{\sim}{u}}{\partial t} &= -\nabla\chi + \underset{\sim}{V}.\nabla\underset{\sim}{h} + \underset{\sim}{h}.\nabla\underset{\sim}{V} + \nu\nabla^2\underset{\sim}{u} + \underset{\sim}{f} + N.L.\\ \frac{\partial \underset{\sim}{h}}{\partial t} &= \underset{\sim}{V}.\nabla\underset{\sim}{u} - \underset{\sim}{u}.\nabla\underset{\sim}{V} + \lambda\nabla^2\underset{\sim}{h} + N.L.\end{aligned}\right\} \tag{2.5}$$

where N.L. indicates terms quadratic in the fluctuating quantities $\underset{\sim}{u}$ and $\underset{\sim}{h}$. We shall suppose that the forcing field $\underset{\sim}{f}$ is sufficiently weak and the field $\underset{\sim}{V}$ (which grows exponentially on the linear analysis of §1) sufficiently strong, for these non-linear terms to be negligible. We may also neglect the terms $\underset{\sim}{h}.\nabla\underset{\sim}{V}$ and $\underset{\sim}{u}.\nabla\underset{\sim}{V}$ on account of the slow variation of the field $\underset{\sim}{V}$. We are then left with two coupled linear equations describing forced Alfvén waves in an effectively uniform field $\underset{\sim}{V}$.

We can now Fourier transform these equations and construct the convolution $\langle \underset{\sim}{u}\wedge\underset{\sim}{h}\rangle$ in terms of the spectrum tensor $F_{ij}(\underset{\sim}{k},\omega)$ of $\underset{\sim}{f}$; omitting the details, which are straightforward, we obtain

$$\langle \underset{\sim}{u}\wedge\underset{\sim}{h}\rangle_i = -\int \frac{\lambda k^2(\underset{\sim}{k}.\underset{\sim}{V})[i\,\epsilon_{ijk}F_{jk}(\underset{\sim}{k},\omega)}{(\lambda^2k^4+\omega^2)\,|\sigma|^2}\,d^3\underset{\sim}{k}\,d\omega, \tag{2.6}$$

where

$$\sigma = -(\omega + i\nu k^2) + (\underset{\sim}{k}.\underset{\sim}{V})^2(\omega + i\lambda k^2)^{-1}. \tag{2.7}$$

The quantity in square brackets under the integral sign is real in view of the Hermitian symmetry condition $F^*_{jk}(\underset{\sim}{k}, \omega) = F_{kj}(\underset{\sim}{k}, \omega)$ for the spectrum tensor of a real vector field. It is no surprise that the expression is non zero only if $F_{jk}(\underset{\sim}{k}, \omega)$ lacks reflexional symmetry. The expression (2.6) is evidently non-linear in $\underset{\sim}{V}$ due to the dependence of σ on $\underset{\sim}{V}$ in the denominator. As $\underset{\sim}{V}$ grows in strength, $|\langle \underset{\sim}{u} \wedge \underset{\sim}{h} \rangle|$ ultimately decreases, and the mechanism responsible for the growth of $\underset{\sim}{V}$ is correspondingly controlled.

Evaluation of the integral (2.6) for given $F_{jk}(\underset{\sim}{k}, \omega)$ is in general a very difficult (if not impossible) matter. It may be sufficient for this lecture to consider a choice of $F_{jk}(\underset{\sim}{k}, \omega)$ which is very special, but which nevertheless serves to illustrate the type of effect that may be expected in general. Suppose that, locally,

$$\underset{\sim}{V} = (V, 0, 0), \tag{2.8}$$

and suppose that

$$\underset{\sim}{f}(\underset{\sim}{x}, t) = Re\, \underset{\sim}{\hat{f}}\, e^{ik_0(x - Vt)}, \quad \underset{\sim}{\hat{f}} = \frac{1}{\sqrt{2}} f_0(0, 1, -i). \tag{2.9}$$

This is a forcing wave at the 'resonance' frequency $\omega = k_0 V$, with a helical structure. Note that

$$\underset{\sim}{\tilde{f}} \wedge \underset{\sim}{\tilde{f}}^* = f_0^2(i, 0, 0), \tag{2.10}$$

and correspondingly,

$$i\epsilon_{ijk} F_{jk}(\underset{\sim}{k}, \omega) = -\langle f_0^2 \rangle \delta_{i1}\, \delta(\underset{\sim}{k} - \underset{\sim}{k_0})\, \delta(\omega - k_0 V), \tag{2.11}$$

where $\underset{\sim}{k_0} = (k_0, 0, 0)$. We put angular brackets round f_0^2 to indicate that this amplitude may be a random variable with an ensemble average. Suppose further that the dissipative effects are weak, viz. that

$$\nu k_0 \ll V \ , \quad \lambda k_0 \ll V, \tag{2.12}$$

so that, to lowest order, $\sigma = -i(\nu+\lambda)k_0^2$; then, from (2.6),

$$\langle \underset{\sim}{u} \wedge \underset{\sim}{h} \rangle = \frac{\lambda \langle f_0^2 \rangle \underset{\sim}{V}}{k_0^3 (\lambda+\nu)^2 V^2} \ . \tag{2.13}$$

<u>The implications for dynamo theory</u>

Let us now suppose that $\underset{\sim}{V}$ varies slowly, and that the statistical properties of the $\underset{\sim}{f}$-field are such that the formula (2.13) holds everywhere. This is admittedly unrealistic, in that it requires a forcing helicity wave whose wave-vector $\underset{\sim}{k}_0$ is everywhere locally aligned with the mean field $\underset{\sim}{V}$; but it is the only assumption that leads to a reasonably simple treatment of the feedback phenomenon. Equation (2.4) then becomes

$$\frac{\partial \underset{\sim}{V}}{\partial t} = A \nabla \wedge \left(\frac{\underset{\sim}{V}}{V^2} \right) + \lambda \nabla^2 \underset{\sim}{V} \ , \tag{2.14}$$

where $A = \lambda(\lambda+\nu)^{-2} k_0^{-3} \langle f_0^2 \rangle$. This equation admits solutions of the form

$$\underset{\sim}{V} = V_0(t) \left(\cos Kz \, , \, -\sin Kz \, , \, 0 \right), \tag{2.15}$$

provided $V_0(t)$ satisfies

$$\frac{dV_0}{dt} = \frac{AK}{V_0} - \lambda K^2 V_0 \ . \tag{2.16}$$

For consistency, we require $K \ll k_0$. The particular interest of solutions of this type is that they satisfy the 'force-free' condition (1.19), and are therefore preferentially amplified on the basis of linear theory. According to the non-linear theory of this section, the structure of such modes remains unaltered when the back-reaction is included, but the growth rate is modified.

The solution of (2.16) is given by

$$\frac{1}{2}V_0^2 = \frac{A}{2\lambda} + Ce^{-2\lambda K^2 t}, \tag{2.17}$$

where C is a constant determined by the value of V_0 at some time $t = t_1$ beyond which the approximations (2.12) are satisfied. In a time of order $(\lambda K^2)^{-1} = L^2/\lambda$, the magnetic energy density $M(t) = \frac{1}{2}V_0^2$ (there is in this situation negligible energy in the fluctuating magnetic field) asymptotically attains the value

$$M_{ult} = \frac{A}{2\lambda} = \frac{\langle f_0^2 \rangle}{2(\lambda+\nu)^2 k_0^3 K} . \tag{2.18}$$

The kinetic energy density is given by

$$E_{ult} = \frac{1}{2}\langle \underset{\sim}{u}^2 \rangle = \frac{\langle f_0^2 \rangle}{2|\sigma|^2} = \frac{\langle f_0^2 \rangle}{4(\lambda+\nu)^2 k_0^4} . \tag{2.19}$$

Hence

$$\frac{M_{ult}}{E_{ult}} = \frac{2k_0}{K} = \frac{2L}{l} \gg 1 . \tag{2.20}$$

The magnetic energy in the ultimate steady state is therefore necessarily an order of magnitude <u>greater</u> than the kinetic energy of the background velocity field.

3. DISCUSSION

There are three questions that arise in the course of the above analysis that require special comment: (i) Can we be sure that a mean velocity field $\underset{\sim}{u}_0 = \langle \underset{\sim}{u} \rangle$ does not develop as a result of the non-uniform Reynolds stress distribution implied by the non-uniform large-scale magnetic field? (ii) Is the particular assumption (2.11) concerning the spectrum tensor of the forcing field too special for the result to have any significance? (iii) Is it at all realistic to consider a forcing field whose statistical properties lack reflexional symmetry? Let us take these questions in turn.

(i) Generation of a mean velocity field

The mean momentum equation, from (2.1) is

$$\frac{\partial \underset{\sim}{u}_0}{\partial t} + \underset{\sim}{u}_0 . \nabla \underset{\sim}{u}_0 = -\nabla \chi_0 + \underset{\sim}{V} . \nabla \underset{\sim}{V} - \nabla . \langle \underset{\sim}{u} \underset{\sim}{u} - \underset{\sim}{h} \underset{\sim}{h} \rangle + \nu \nabla^2 \underset{\sim}{u}_0 , \tag{3.1}$$

so that a mean velocity field will develop if either $\underset{\sim}{V} . \nabla \underset{\sim}{V}$ or $\nabla . \langle \underset{\sim}{u}\underset{\sim}{u} - \underset{\sim}{h}\underset{\sim}{h} \rangle$ is rotational; if these terms are irrotational, they can be accommodated by a mean (effective) pressure field $\chi_0(\underset{\sim}{x},t) = \langle \chi \rangle$.

For the particular field (2.15) considered above $\underset{\sim}{V} . \nabla \underset{\sim}{V}$ is certainly irrotational; indeed $\underset{\sim}{V} \wedge (\nabla \wedge \underset{\sim}{V}) = 0$ since the growing field is force-free, so that $\underset{\sim}{V} . \nabla \underset{\sim}{V} = \frac{1}{2} \nabla \underset{\sim}{V}^2$. Furthermore $P_{ij} = \langle u_i u_j - h_i h_j \rangle$ is a symmetric tensor axisymmetric about the direction of $\underset{\sim}{V}$, and depending only on z and t, since $\underset{\sim}{V}$ depends only on z and t. Clearly $P_{13} = P_{23} = 0$, since Oz is a possible principal axis of P_{ij} for all z ; hence

$$\frac{\partial}{\partial x_j} P_{ij}(z,t) = \frac{\partial}{\partial z} P_{i3}(z,t) = \frac{\partial}{\partial z} P_{33}(z,t)\, \delta_{i3} = \frac{\partial}{\partial x_i} P_{33}(z,t) ,$$

and so this term is also irrotational. Hence no mean flow will develop, but a mean pressure field periodic in z will be established.

For a more general field than (2.15) consisting of a superposition of such Fourier components, but having different wave-vectors $\underset{\sim}{K}$, it seems certain that a mean velocity field $\underset{\sim}{u}_0(\underset{\sim}{x},t)$ <u>will</u> develop, but what its structure will be and how it will modify equation (2.16) remains an open question.

(ii) The effect of a forcing field with a continuous spectrum

The assumption (2.11) is indeed too restrictive, and not only because the result (2.13) would then appear to have only local and not global significance. There is some justification for choosing the resonant frequency $\omega = k_0 V$, since the response to a random $\underset{\sim}{f}$-field containing a continuous spectrum of wave-numbers $\underset{\sim}{k}$ and frequencies ω will (for weak dissipative effects) peak around the resonant frequencies of the undamped system. However, in singling out a particular resonant frequency, we overestimate the response, which for a continuous spectrum, is controlled by a narrow band of frequenoies around the resonant frequency. Moreover, we should take account of all the resonant frequencies given by the Alfvén dispersion relation $\omega = \pm \underset{\sim}{k}.\underset{\sim}{V}$. These effects can be taken into account in an asymptotic evaluation of the integral (2.6) for particular choices of continuous spectrum tensors; but it is better first to take some account of the third question (iii) raised above.

(iii) What physical mechanism can generate a lack of reflexional symmetry?

It was recognized by Steenbeck et al (1966) that, as mentioned in the introductory paragraphs, a lack of reflexional symmetry is likely to arise only in a rotating fluid in which a definite direction relative to the rotation vector $\underset{\sim}{\Omega}$ can be distinguished, i.e. only if the 'external conditions' themselves lack reflexional symmetry. A random force field $\underset{\sim}{f}$ is unlikely in itself to lack reflexional symmetry. If however a reflexionally symmetric $\underset{\sim}{f}$-field acts upon a rotating fluid, the resulting $\underset{\sim}{u}$-field will lack reflexional symmetry if the statistical properties of the $\underset{\sim}{f}$-field lack symmetry about planes perpendicular to $\underset{\sim}{\Omega}$, e.g. if (with the convention $\omega > 0$) only waves for which $\underset{\sim}{k}.\underset{\sim}{\Omega} > 0$ are present.

An analysis similar to that described in §2, but with the addition of a Coriolis force $2\underset{\sim}{\Omega} \wedge \underset{\sim}{u}$ in equation (2.1) has been carried out (Moffatt 1971), and full account is taken of the considerations under heading (ii) above. In this calculation, it is supposed that the spectrum of $\underset{\sim}{f}$ is isotropic over the half-space $\underset{\sim}{k}.\underset{\sim}{\Omega} > 0$ and reflexionally symmetric, and characterised by wave-numbers $O(\ell^{-1})$ and frequencies $O(\omega_0)$. Under the assumptions

$$\nu \ll \lambda \ll \Omega \ell^2 , \quad V \gg \lambda/\ell , \quad V \gg \ell\omega_0 , \quad \omega_0 \ll \Omega ,$$

a result analogous to (2.20) is obtained, viz.,

$$\frac{M_{ult}}{E_{ult}} = C\left(\frac{\Omega}{\omega_0}\right)^{\frac{1}{2}}\left(\frac{\nu}{\lambda}\right)^{\frac{1}{2}} \frac{L}{\ell} , \tag{3.3}$$

where C is a dimensionless constant of order unity. Again M_{ult} can be much greater than E_{ult} if the length scale L available for the growth of a mean magnetic field is sufficiently large compared with the length-scale ℓ of the background forcing field.

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