Reprinted from:
ADVANCES IN APPLIED MECHANICS, VOL. 16
© 1976
ACADEMIC PRESS, INC.
New York San Francisco London

25

Generation of Magnetic Fields by Fluid Motion

H. K. MOFFATT

Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Cambridge, England

I. Introduction

The existence of the magnetic field of the Earth, and its variation with time, presents a profound challenge to geophysics. This field, though influenced slightly by electric currents in the ionosphere, is predominantly of internal origin and is associated with a large-scale azimuthal current distribution in the liquid core of the Earth. It is well known (see, e.g., Hide and Roberts, 1961) that the temperature of the core is far above the critical value (the "Curie point") at which permanent magnetization can persist. Moreover, in the absence of any regenerative action, the electric currents in the core would decay through ordinary resistive ("ohmic") dissipation in a time of order 10^4–10^5 years. Geomagnetic studies indicate, however, that the Earth's field has existed in one form or another for at least 10^8 years and is probably as old as the Earth itself, and further that, although the main dipole field exhibits random rapid reversals, a phenomenon reviewed by Bullard (1968), it remains at least quasi-steady between reversals for periods up to order 10^6 years, i.e., one or two orders of magnitude greater than the natural decay time. It is now generally agreed that this persistence of the Earth's field can only be explained in terms of electromagnetic induction, whereby the electric currents that provide the field are generated by motion of the fluid in the core across the self-same field, which permeates the core region as well as the nonconducting exterior.

The characteristic feature of such dynamo action is that the field is maintained exclusively by the action of the fluid velocity and without the help of any external source of field. This type of behavior is most simply illustrated in terms of the simple disk dynamo illustrated in Fig. 1. The electrically conducting disk rotates about its axis with angular velocity Ω, and a conducting wire makes sliding contact with the rim of the disk and with its axis, as shown; the wire is twisted into a circle in its passage from the rim to the

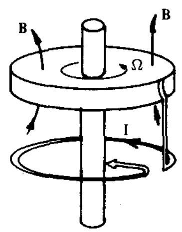

FIG. 1. The self-exciting disk dynamo; note particularly the concentrated shear at the sliding contact and the lack of reflectional symmetry of the system.

axis in such a way that any current flowing in the wire gives rise to a magnetic field with nonzero flux Φ across the disk. The rotation of the disk in the presence of this flux generates a radial electromotive force, and since a closed circuit is available, a net current $I(t)$ flows along the wire. The flux Φ then equals MI, where M is the mutual inductance between the wire and the rim of the disk, and I is given simply by

$$L\, dI/dt + RI = M\Omega I, \tag{1.1}$$

where L and R are, respectively, the self-inductance and resistance of the complete current circuit. Clearly, if $R < M\Omega$, the current grows, and if Ω is maintained at a constant value, this growth is exponential. The state with $I = 0$ is then unstable to the growth of small electromagnetic disturbances.

Such growth cannot, of course, continue indefinitely. The Lorentz force $\mathbf{j} \wedge \mathbf{B}$ (where $\mathbf{j}$ is the current distribution in the disk and $\mathbf{B}$ the magnetic field) provides a net resisting torque MI^2, and (1.1) must be coupled with the equation of angular motion of the disk

$$C\, d\Omega/dt = G - MI^2, \tag{1.2}$$

where C is the moment of inertia of the disk about its axis, and G the applied torque. If G (rather than Ω) is kept constant, then as I increases, Ω decreases until an equilibrium is reached in which

$$G = MI^2, \qquad \Omega = R/M. \tag{1.3}$$

Note that the angular velocity of the disk in this equilibrium state does not depend on the applied torque!

This simple dynamo relies for its success on the carefully contrived path that the current is forced to follow. The conductor (disk + wire) occupies a region of space that is not simply connected, a feature that is of course not shared by the conducting core of the Earth. There are, however, two features of the disk dynamo configuration that deserve particular emphasis, because these features do recur in the fluid context and are both intimately associated with successful dynamo action. First, there is a region of concentrated shear at the sliding contact on the rim of the disk; the counterpart of this in the fluid context is *differential rotation*, which plays an important part in generating toroidal magnetic field from poloidal magnetic field (see Section III,D). Second, the configuration *lacks reflectional symmetry* in that the sense of twist of the wire in Fig. 1 bears a very definite relation to the sense of the angular velocity of the disk: if the twist is reversed, then $M\Omega I$ in (1.1) is replaced by $-M\Omega I$, and rather than dynamo action we have accelerated decay of any transient current in the wire. This lack of reflectional symmetry also has its counterpart in homogeneous fluid systems. The simplest measure of the lack of reflectional symmetry of a localized fluid motion $\mathbf{u}(\mathbf{x})$ is its *helicity*

$$H = \int \mathbf{u} \cdot (\nabla \wedge \mathbf{u}) \, d^3\mathbf{x}, \tag{1.4}$$

a quantity that admits interpretation in terms of the degree of linkage (or "knottedness") of its constituent vortex lines (Moffatt, 1969). We shall describe in the following sections (particularly Section V) how the presence of helicity is of vital importance for the process of regeneration of poloidal field from toroidal field, i.e., for the closing of the dynamo cycle that makes field regeneration a reality.

The Earth is, of course, not the only celestial body that exhibits a significant large-scale magnetic field. Among the planets, Jupiter, Mars, and Mercury are now known to have this property also; the radii, rotation rates, and dipole moments of these planets in comparison with the Earth are displayed in Table 1 (from Dolginov, 1975). A theory that successfully explains the Earth's field may clearly have relevance in the context of these

TABLE 1

COMPARATIVE FIGURES FOR THE EARTH, JUPITER,[a] MARS,[b] AND MERCURY[c]

	Radius R (km)	Dipole moment μ (G km^3)	μ/R^3 (G)	Rotation period (days)
Earth	6371	8.05×10^{10}	3.11×10^{-1}	1
Jupiter	71,351	1.31×10^{15}	3.61	0.415
Mars	3386	2.47×10^{7}	6.36×10^{-4}	1
Mercury	2439	4.8×10^{7}	3.31×10^{-3}	58

[a] Warwick (1963); Smith *et al.* (1974).
[b] Dolginov *et al.* (1973).
[c] Ness *et al.* (1974).

other planetary fields. The internal constitution of Mercury, Mars, and Jupiter is, of course, largely a matter of speculation at present; it may be that detailed observation of the surface magnetic fields of these planets will in the long run provide (via theoretical argument) information about their interiors. In the case of Jupiter, it has been argued (see, e.g., Hide, 1974) that the core consists of a liquid alloy of metallic hydrogen and helium under high pressure and that this core provides the seat of magnetohydrodynamic dynamo action.

The magnetic field of the Sun (which is believed to be typical of "cool" stars with convective outer envelopes) is much more complex in its structure and behavior than that of the Earth, and it too is widely (though not universally) believed to be the result of dynamo action involving the two features, differential rotation and motions lacking reflectional symmetry, described above. Paradoxically, although the Sun is certainly remote as compared with the Earth's liquid core, we have much more detailed information about its magnetic field, simply because it may be detected and measured at its visible surface, i.e., at the surface of the convective region where, from a magnetohydrodynamic point of view, all the interesting action takes place. In the case of the Earth, we have no prospect or hope of any direct measurement of the magnetic field, or indeed of any other quantity, either in the liquid core or on its surface, and we must make do with what we know of the field on the surface of the solid Earth, a pale shadow of the field of the deep interior, and a dim and distant reflection of the fluid turmoil in that deep interior which is at the heart of our problem.

The Sun's field is characterized by *active regions* and by a *weak general field* near the north and south poles of its axis of rotation. Active regions are associated with strong upwelling from the convective envelope with an associated vertical stretching of any magnetic field lines that are convected upward. When this stretching is particularly localized and intense, the strong vertical field that is created (of the order of thousands of gauss) can locally suppress thermal convection; the resulting decrease in heat transport leads to a local cooling of the surface; radiation from this local region (of the order of hundreds of kilometers in horizontal extent) is largely suppressed, and in consequence it is seen from the Earth as a dark spot on the surface of the Sun. Such sunspots occur in pairs, roughly along a line of latitude, but with the leading spot (i.e., that to the East) slightly nearer the equatorial plane. Sunspot activity has been followed for more than 300 years and is known to follow a roughly periodic cycle with half-period of about 11 years. At the beginning of a sunspot cycle, pairs of spots appear within the band of latitudes about $\pm 30°$ from the equatorial plane, with statistical symmetry about this plane, first at the higher latitudes only, then gradually over a wider band of latitudes that drifts, as the cycle proceeds, toward the equatorial plane. In any pair of sunspots, the vertical magnetic field is positive in one and negative in the other; if positive in the westerly spot, the pair has positive polarity, otherwise negative. In any half-cycle of 11 years, all sunspot pairs in the northern hemisphere have the same polarity, and all in the southern hemisphere have the opposite polarity. In the following half-cycle, these polarities are reversed.

The weak polar field of the Sun also follows a somewhat irregular periodic evolution with approximately the same period as that of the sunspot cycle. The field was first measured by direct magnetograph measurements in 1952 (Babcock and Babcock, 1955) and it has been followed closely since that date. The field around the north pole reversed in 1958 and again in 1971; the field around the south pole reversed in 1957 and again in 1972! At each reversal, for about one year, the fields at north and south poles therefore had quadrupole rather than dipole symmetry about the equatorial plane. The reversals apparently occur during that part of the sunspot cycle when sunspot activity is at its maximum.

These observations are compatible with the following qualitative picture (essentially conceived by Parker, 1955a): the global magnetic field of the Sun includes poloidal and toroidal ingredients that are not steady but vary periodically in time, with period approximately 22 years. The poloidal field can be observed and has the polar reversal behavior described above; the toroidal field is contained in some way beneath the visible surface of the sun and cannot be directly detected. This toroidal field is coupled with the poloidal field and in a typical half-period drifts like a wave from polar regions toward equatorial regions, intensifying as it progresses. When this field reaches a certain critical level of intensity, local upwelling instabilities may develop in which ropes of toroidal flux are stretched vertically upward, breaking through the visible surface of the sun and forming sunspots as described above. As the buoyant fluid rises through several scale-heights, it expands due to the decreasing ambient pressure; as a result of the tendency to conserve angular momentum, the rising blob develops a rotation (the sense of rotation being such that it has negative helicity in the northern hemisphere, positive in the southern): hence the twist of the sunspot pair from the original line of latitude of the underlying toroidal field. As the periodic evolution proceeds, the toroidal fields of opposite signs from the two hemispheres interpenetrate and annihilate each other in the equatorial zone and the sunspot activity in consequence dies away. The process then repeats itself, the toroidal field again growing from polar to equatorial regions (but with a complete change of polarity from one half-cycle to the next).

What part does the weak poloidal field play in this process? It just must be present, for otherwise the dynamo cycle cannot proceed. On the one hand, all the little eruptions over the solar surface generate a field with a radial component (i.e., poloidal field); since the surface eruptions are most intense in equatorial latitudes, one might expect the poloidal field to be most evident in these regions also. However, poloidal field can be redistributed by large-scale meridional circulation in the convective zone and this must presumably play a part in sweeping poloidal field back to the polar regions. Meridional circulation also tends to generate differential rotation (conservation of angular momentum again) and this differential rotation is the means by which the toroidal field is regenerated from the poloidal.

These complicated interactions may seem far removed from the simplicity of the disk dynamo described at the outset; yet the two features—differential rotation and lack of reflexional symmetry—appear in the solar context as vital ingredients in its periodic behavior; the lack of reflexional symmetry appears in the rising, twisting blobs, described by Parker (1955a) as "cyclonic events," and directly responsible for sunspot formation.

The above description is, of course, purely qualitative and suggestive. In the sections that follow, we shall endeavor to show how the various physical ideas implicit in the description may be placed on a secure mathematical foundation, and to relate the various approaches to dynamo theory that have made progress in this direction over the last 20 years. The reader who wishes further background material in the terrestrial and solar contexts may consult a number of review articles that have appeared in recent years (Parker, 1970a; Roberts, 1971; Weiss, 1971, 1974; Roberts and Soward, 1972; Vainshtein and Zel'dovich, 1972; Moffatt, 1973; Gubbins, 1974) and the very extensive detailed references that these articles contain.

II. Magnetokinematic Preliminaries

A. Idealization of the Kinematic Dynamo Problem

Suppose that fluid of uniform electrical conductivity σ is confined to a simply connected region of space V inside a closed surface S, and suppose that the region $\hat{V}$ exterior to S (extending to infinity) is nonconducting. Let $\mathbf{u}(\mathbf{x}, t)$ be the velocity field in V, satisfying

$$\mathbf{n} \cdot \mathbf{u} = 0 \qquad \text{on} \quad S, \tag{2.1}$$

and let $\rho(\mathbf{x}, t)$ be the density field satisfying the equation of mass conservation

$$\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{2.2}$$

For many purposes it will be sufficient to restrict attention to incompressible fluids of uniform density for which

$$\rho = \rho_0, \qquad \nabla \cdot \mathbf{u} = 0. \tag{2.3}$$

Let $\mathbf{j}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$, and $\mathbf{E}(\mathbf{x}, t)$ denote electric current, magnetic field, and electric field, respectively. Neglecting displacement current (certainly valid for phenomena on the long time scales considered), the equations relating $\mathbf{j}$, $\mathbf{B}$, and $\mathbf{E}$ in V are

$$\nabla \cdot \mathbf{B} = 0, \tag{2.4}$$

$$\mu_0 \mathbf{j} = \nabla \wedge \mathbf{B} = \mu_0 \sigma(\mathbf{E} + \mathbf{u} \wedge \mathbf{B}), \tag{2.5}$$

$$\partial \mathbf{B}/\partial t = -\nabla \wedge \mathbf{E}, \tag{2.6}$$

where $\mu_0 = 4\pi \times 10^{-7}$ S.I. units. In $\hat{V}$, where $\mathbf{j} = 0$, $\mathbf{B}$ is determined by

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \wedge \mathbf{B} = 0. \tag{2.7}$$

Moreover, both normal and tangential components of $\mathbf{B}$ must be continuous across S, i.e.,

$$[\mathbf{n} \cdot \mathbf{B}]^+_- = 0, \qquad [\mathbf{n} \wedge \mathbf{B}]^+_- = 0 \quad \text{on} \quad S, \tag{2.8}$$

where $\mathbf{n}$ is the unit outward normal on S. Finally, we require that $\mathbf{B}$ be without singularities in V and in $\hat{V}$, and that there should be no sources of magnetic field at infinity; this means that $\mathbf{B}$ must be at most dipole, i.e., $O(r^{-3})$ as $r = |\mathbf{x}| \to \infty$.

Elimination of $\mathbf{j}$ and $\mathbf{E}$ from (2.4)–(2.6) gives the well-known induction equation (which holds in V),

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) + \lambda \nabla^2 \mathbf{B}, \tag{2.9}$$

where $\lambda = (\mu_0 \sigma)^{-1}$ is the *magnetic diffusivity* of the fluid. For given $\mathbf{u}$, we wish to explore the evolution of the field $\mathbf{B}$ as determined by (2.7)–(2.9) and the subsidiary conditions mentioned. A simple measure of the field level is the total magnetic energy

$$M(t) = (2\mu_0)^{-1} \int_{V + \hat{V}} \mathbf{B}^2 \, d^3\mathbf{x}. \tag{2.10}$$

If, for given $\mathbf{u}$, $M(t) \to 0$ as $t \to \infty$, then the motion $\mathbf{u}$ does not act as a dynamo. If $M(t) \not\to 0$ as $t \to \infty$, then the motion does act as a dynamo, there being ultimately sufficient rate of generation of magnetic energy by fluid motion to counteract the natural decay of magnetic energy due to ohmic dissipation associated with the finite conductivity of the fluid.

In a complete theory that takes account of the dynamics of the fluid motion, $\mathbf{u}$ is, of course, constrained to satisfy the Navier–Stokes equation (with Coriolis forces, Lorentz forces, buoyancy forces, etc., included if the context so requires). In a purely kinematic approach at the outset, it proves useful to widen the scope of the investigation and to imagine that **u** is any kinematically possible velocity field (without dynamical restriction); the influence of dynamic constraints, which will be considered in Section VIII, is more easily comprehended after close investigation of the kinematic problem.

B. Magnetic Field Representations

1. *Poloidal and Toroidal Decomposition*

Any solenoidal field **B** may be expressed as the sum of a poloidal ingredient $\mathbf{B}_P$ and a toroidal ingredient $\mathbf{B}_T$, where

$$\mathbf{B}_P = \nabla \wedge \nabla \wedge (\mathbf{x}S(\mathbf{x})), \qquad \mathbf{B}_T = \nabla \wedge (\mathbf{x}T(\mathbf{x})); \tag{2.11}$$

S and T are the *defining scalars* for these fields. Note that

$$\nabla \wedge \mathbf{B}_T = \nabla \wedge \nabla \wedge (\mathbf{x}T) \tag{2.12}$$

is a poloidal field with defining scalar T, while

$$\nabla \wedge \mathbf{B}_P = -\nabla^2(\nabla \wedge \mathbf{x}S) = -\nabla \wedge (\mathbf{x}\nabla^2 S) \tag{2.13}$$

is a toroidal field with defining scalar $-\nabla^2 S$. Note further that $\mathbf{x} \cdot \mathbf{B}_T = 0$, i.e., the lines of force of the $\mathbf{B}_T$ field lie on spheres $r =$ const, as do the lines of force of $\nabla \wedge \mathbf{B}_P$, a property that makes the representation particularly useful when problems with spherical boundaries are considered.

2. *Axisymmetric Fields*

A field **B** is axisymmetric about an axis $0z$ when its defining scalars S and T are independent of the azimuth angle ϕ about $0z$. If $S = S(r, \theta)$, $T = T(r, \theta)$, where θ is measured from $0z$, then

$$\mathbf{B}_P = \left(\frac{1}{r^2 \sin\theta}\frac{\partial\chi}{\partial\theta}, \frac{-1}{r\sin\theta}\frac{\partial\chi}{\partial r}, 0\right), \qquad \mathbf{B}_T = (0, 0, B_\phi), \tag{2.14}$$

where

$$\chi = -r\sin\theta\, \partial S/\partial\theta, \qquad B_\phi = -\partial T/\partial\theta. \tag{2.15}$$

χ, the *flux function*, is the analog of the Stokes stream function for solenoidal velocity fields, and the lines of force of the $\mathbf{B}_P$ field are given by $\chi =$ const. The $\mathbf{B}_P$ field may also be expressed in the form $\mathbf{B}_P = \nabla \wedge (A\mathbf{i}_\phi)$, where $A = \chi/r\sin\theta$, and $\mathbf{i}_\phi$ is a unit vector in the ϕ direction.

3. *Two-Dimensional Fields*

It is frequently illuminating to consider configurations in which **B** depends only on two cartesian coordinates, say x and y. In this case, the representation analogous to the above is $\mathbf{B} = \mathbf{B}_P + \mathbf{B}_T$, where now

$$\mathbf{B}_P = (\partial A/\partial y,\ -\partial A/\partial x,\ 0), \qquad \mathbf{B}_T = (0,\ 0,\ B_z(x,\ y)), \tag{2.16}$$

and the $\mathbf{B}_P$ lines are given by $A = \text{const}$.

C. Alfvén's Theorem and Woltjer's Invariant

It is an immediate consequent of (2.4)–(2.6) that, if $\Phi(t)$ is the flux of **B** across any surface spanning a closed curve $C(t)$ that moves with the fluid, then

$$d\Phi/dt = -\oint_{C(t)} (\mathbf{E} + \mathbf{u} \wedge \mathbf{B}) \cdot d\mathbf{x} = -\oint_{C(t)} \sigma^{-1}\mathbf{j} \cdot d\mathbf{x}, \tag{2.17}$$

so that, in the perfect conductivity limit ($\sigma = \infty$), Φ is constant for any material curve $C(t)$. It follows that in this limit **B** lines are frozen in the fluid, and in an incompressible flow, stretching of the **B** lines leads to proportionate intensification of the **B** field.

Closely associated with the "frozen-field" concept is the invariance of the integral

$$H_M = \int_V \mathbf{A} \cdot \mathbf{B}\, d^3\mathbf{x}, \tag{2.18}$$

(Woltjer, 1958). Here **A** is the vector potential of **B**, i.e., $\mathbf{B} = \nabla \wedge \mathbf{A}$, and it is supposed that **B** is a localized field so that the integral exists. The volume V in (2.18) may be any volume bounded by a material surface S on which $\mathbf{B} \cdot \mathbf{n} = 0$. The interpretation of H_M is identical with that for the helicity integral (1.4) (which is likewise constant whenever circumstances are such that vortex lines move with the fluid), i.e., H_M is a measure of the degree of topological complexity of the field **B** within the surface S—and this measure cannot, of course, change when the **B** lines are frozen in the fluid.

We may note in passing that the solution of (2.9) in the limit $\sigma = \infty$ (i.e., $\lambda = 0$) may be expressed in Lagrangian variables in the form (due to Cauchy)

$$B_i(\mathbf{x},\ t) = B_j(\mathbf{a},\ 0)\ \partial x_i/\partial a_j\,, \tag{2.19}$$

where $\mathbf{x}(\mathbf{a},\ t)$ is the position at time t of the fluid particle that was at position $\mathbf{a}$ at time $t = 0$.

D. Natural Decay Modes and Force-Free Fields

If **u** is steady, i.e., $\mathbf{u} = \mathbf{u}(\mathbf{x})$, then the problem (2.7)–(2.9) admits solutions proportional to $\exp(-pt)$, where possible values of p are determinate as eigenvalues of the problem. If, for all these values, $\operatorname{Re} p > 0$, then **B** inevitably decays with time, while if for any eigenvalue $\operatorname{Re} p < 0$, the corresponding field structure (the eigenfunction) grows exponentially in time until Lorentz forces modify the velocity field (cf. the rotating-disk situation discussed in the introduction). If $p = p_r + ip_i$ then the condition $p_r = 0$ is critical in that it determines the onset of dynamo action for the corresponding field structure $\mathbf{B}_p(\mathbf{x})$. If, when $p_r = 0$, it also happens that $p_i = 0$, then the resulting mode is steady under critical conditions; this is the sort of behavior that we look for in the context of the Earth's magnetic field, which is steady (with weak fluctuations) over very long periods. If, on the other hand, $p_i \neq 0$ when $p_r = 0$, then the resulting mode is oscillating under critical conditions; this type of behavior would be relevant in the solar context.

Of course, when $\mathbf{u} \equiv 0$, all the p's are real and positive; in the important case when the volume V is spherical with radius R, the eigenvalues are given by

$$p_{nq} = \lambda R^{-2} x_{nq}^2 , \tag{2.20}$$

where x_{nq} is the qth zero of the Bessel function $J_{n+1/2}(x)$. The structure of the corresponding fields for $r < R$ are given, in the notation of Bullard and Gellman (1954), by

$$\mathbf{S}_n^{mc} + i\mathbf{S}_n^{ms} = \nabla \wedge \nabla \wedge [\mathbf{x} r^{-1/2} J_{n+1/2}(\lambda^{-1} p_{nq} r) e^{im\phi}], \tag{2.21}$$

$$\mathbf{T}_{n-1}^{mc} + i\mathbf{T}_{n-1}^{ms} = \nabla \wedge [\mathbf{x} r^{-1/2} J_{n-1/2}(\lambda^{-1} p_{nq} r) e^{im\phi}]. \tag{2.22}$$

The field $\mathbf{S}_1^{oc}$ matches to a dipole field in the exterior region $r > R$, the field $\mathbf{S}_2^{oc}$ matches to an axisymmetric quadrupole field, and so on. From (2.20) the time scale of decay of these modes of simple structure is $O(R^2 \lambda^{-1})$ (as can, of course, be anticipated from dimensional analysis).

The natural decay modes are closely related to field structures that are *force-free*, i.e., for which the Lorentz force $\mathbf{j} \wedge \mathbf{B}$ everywhere vanishes. Such fields arise naturally in the context of the kinematic dynamo problem, and it will be useful to set out some of their properties here. First, for such fields there exists a scalar field $K(\mathbf{x})$ such that

$$\mu_0^{-1} \mathbf{j} = \nabla \wedge \mathbf{B} = K(\mathbf{x}) \mathbf{B}, \tag{2.23}$$

and, since $\nabla \cdot \mathbf{j} = \nabla \cdot \mathbf{B} = 0$, it follows that

$$\mathbf{B} \cdot \nabla K = 0 \quad \text{and} \quad \mathbf{j} \cdot \nabla K = 0, \tag{2.24}$$

i.e., **B** lines and **j** lines lie on a surface $K = \text{const.}$

The simplest example of a force-free field, with K constant everywhere, is (in cartesians)

$$\mathbf{B} = B_0(\sin Kz, \cos Kz, 0); \tag{2.25}$$

the property $\nabla \wedge \mathbf{B} = K\mathbf{B}$ is trivially verified. The $\mathbf{B}$ lines lie in planes $z = \text{const}$ and rotate in a left-handed sense with increasing z. The vector potential of $\mathbf{B}$ is just $\mathbf{A} = K^{-1}\mathbf{B}$, so that

$$\mathbf{A} \cdot \mathbf{B} = K^{-1}\mathbf{B}^2 = K^{-1}B_0^2 . \tag{2.26}$$

The magnetic helicity density $\mathbf{A} \cdot \mathbf{B}$ is thus uniform; the field structure (2.25) has "maximal helicity" (Kraichnan, 1973).

This and other similar examples have the property that the $\mathbf{j}$ field extends to infinity. There are in fact *no* force-free fields for which $\mathbf{j}$ is confined to a finite volume and $\mathbf{B}$ is everywhere continuous and $O(r^{-3})$ at infinity (see, e.g., Roberts, 1967, p. 109). It is, however, possible (Chandrasekhar, 1956) to construct force-free fields in a sphere V that match smoothly onto current-free fields in the exterior region $\hat{V}$ that do not vanish at infinity: let

$$S(r, \theta) = Ar^{-1/2}J_{3/2}(Kr) \cos \theta, \tag{2.27}$$

and

$$\mathbf{B} = \nabla \wedge (\mathbf{x}S) + K^{-1}\nabla \wedge \nabla \wedge (\mathbf{x}S) \qquad \text{for} \quad r < R. \tag{2.28}$$

Then it may be readily verified that, since S satisfies the Helmholtz equation $(\nabla^2 + K^2)S = 0$, the field (2.28) does satisfy $\nabla \wedge \mathbf{B} = K\mathbf{B}$. However, since $\mathbf{j} \cdot \mathbf{n} = 0$ on $r = R$, and since $\mathbf{B}$ is parallel to $\mathbf{j}$, we must also satisfy $\mathbf{B} \cdot \mathbf{n} = 0$ on $r = R$ (a condition that the decay modes do not satisfy) and this requires that $J_{3/2}(KR) = 0$. The exterior field in $\hat{V}$ is purely poloidal and is given by $\mathbf{B} = K^{-1}\nabla \wedge \nabla \wedge (\mathbf{x}S)$, where

$$S = -B_0(r - R^3/r^2) \cos \theta, \qquad B_0 = \tfrac{1}{3}AR^{-1/2}\, dJ_{3/2}(KR)/dR, \tag{2.29}$$

the latter condition ensuring smoothness across $r = R$.

III. Convection, Distortion, and Diffusion of B Lines

In this section, we consider certain particular solutions of the induction equation (2.9) when $\mathbf{u}(\mathbf{x})$ is prescribed and steady. The behavior is strongly dependent on the order of magnitude of the magnetic Reynolds number $R_m = u_0 l/\lambda$ [where u_0 and l are, respectively, velocity and length scales characteristic of $\mathbf{u}(\mathbf{x})$]. Moreover, great care must be exercised when the two limiting processes $R_m \to \infty$ and $t \to \infty$ are considered—the solution may depend in a most sensitive manner on the order in which these limits are taken.

A. Balance of Stretching and Diffusion in a Magnetic Flux Rope

Equation (2.9) represents the evolution of **B** under the joint action of the stretching of magnetic field lines and of their diffusion relative to the fluid. A well-known steady solution of the equation in which these effects exactly balance exists when **u** is the uniform extensional straining motion given by

$$\mathbf{u} = (-\alpha x, \ -\alpha y, \ 2\alpha z), \qquad \alpha > 0. \tag{3.1}$$

The steady solution of (2.9) is then

$$\mathbf{B} = (0, 0, B_0 \exp(-\alpha/\lambda)(x^2 + y^2)), \tag{3.2}$$

representing a flux rope of gaussian structure aligned with the z axis. The field (3.2), in fact, provides the asymptotic solution of (2.9) as $t \to \infty$ (with λ fixed and nonzero). The flux in the rope is $\pi B_0 \lambda/\alpha$, so that for given flux, B_0 becomes very large when λ is very small, under this type of persistent stretching. This type of motion may be expected (locally) to generate the strong vertical magnetic fields observed in sunspots, as described in the introduction.

B. Flux Expulsion by Flows with Closed Streamlines

The phenomenon of flux expulsion was first explicitly considered by Parker (1963) and Weiss (1966). Suppose we have a steady two-dimensional incompressible flow with stream function $\psi(x, y)$ and suppose that the fluid is permeated at time $t = 0$ by a magnetic field that is uniform and in the plane of the flow. For $t > 0$ the flow distorts the field and diffusion, of course, also influences its behavior. If $R_m \ll 1$, diffusion dominates and the field perturbations remain small [in fact, $O(R_m)$ relative to the initial field]. If $R_m \gg 1$, the picture is very much more complicated. During an initial phase whose duration is $O(R_m^{1/2})l/u_0$ (l and u_0 being the scales introduced above), diffusion is negligible and the field perturbations grow in intensity to $O(R_m^{1/2})$ times the initial field. There is then an intermediate stage, which has been studied in a particular case by Parker (1966), and whose duration is $O(R_m^{3/2})l/u_0$, during which diffusion causes the breaking off of closed flux loops in regions of closed streamlines; these flux loops slowly decay and disappear and there is an associated net reduction in the total flux threading the region of closed streamlines. Finally, the field settles down to a steady state, in which the flux across any region of closed streamlines is exponentially small. The difference between $\lim_{t\to\infty}\lim_{\lambda\to 0}$ and $\lim_{\lambda\to 0}\lim_{t\to\infty}$ is quite striking in this context: the former procedure gives a field whose energy density increases without limit (as t^2); the latter procedure gives a steady field whose energy density is generally *less* than that of the uniform field that we started with!

The following simple proof that the flux across any region of closed streamlines must vanish (under the second limiting procedure) appears to be new. Using the representation (2.16a) for the magnetic field, (2.5) may be expressed in the simple form

$$DA/Dt \equiv \partial A/\partial t + \mathbf{u}\cdot\nabla A = \lambda\nabla^2 A. \tag{3.3}$$

It is supposed here that there is no applied electric field, so that $E_z = -\partial A/\partial t$. Under steady conditions then,

$$\nabla\cdot(\mathbf{u}A) = \mathbf{u}\cdot\nabla A = \lambda\nabla^2 A, \tag{3.4}$$

and in the limit $\lambda\to 0$, $\mathbf{u}\cdot\nabla A = 0$, so that A is constant on streamlines, or equivalently $A = A(\psi)$. We now adapt the argument of Batchelor (1956) (as applied to the vorticity equation) to the present context: let C be any closed streamline, and integrate (3.4) over the area enclosed by C. Since $\mathbf{n}\cdot\mathbf{u} = 0$ on C (where $\mathbf{n}$ is normal to C), the left-hand side integrates to zero. This focuses attention on the effects of diffusion when λ is small but not quite zero. The right-hand side, on integration, gives

$$0 = \oint_C \lambda\mathbf{n}\cdot\nabla A\, ds = \lambda\, dA/d\psi \oint_C \partial\psi/\partial n\, ds = \lambda K_C\, dA/d\psi, \tag{3.5}$$

where K_C is the circulation around C. It follows that in the steady state (which may, of course, take a long time to attain) $dA/d\psi = 0$, i.e., $A = \text{const}$, i.e., $\mathbf{B} = 0$ throughout the region of closed streamlines.

A net flux of field across, say, a periodic array of eddies with closed streamlines cannot, of course, be simply eliminated by this mechanism. What happens, as is clearly demonstrated in the numerical solutions of Weiss (1966), is that the flux is concentrated into sheets of thickness $O(R_m^{-1/2})$ at the boundaries between adjacent eddies. A horizontal row of eddies will concentrate vertical flux in this way at the vertical cell boundaries, whereas any horizontal flux will be expelled to the regions above and below the eddies.

The above proof can be simply adapted to cover the corresponding axisymmetric situation when steady meridional circulation acts on an axisymmetric poloidal field. Again, if the relevant magnetic Reynolds number is large, the field is ultimately excluded from any region of closed streamlines. This result has important implications for dynamo theory: meridional circulation that is weak can be conducive to efficient dynamo action, but meridional circulation that is too strong merely expels poloidal field from regions of closed streamlines (i.e., from the whole fluid region for an enclosed flow) and this effect is bound to be counterproductive, as indeed demonstrated in the numerical studies of P. H. Roberts (1972).

C. Topological Pumping of Magnetic Flux

A rather fundamental variant of the flux expulsion mechanism has recently been discovered by Drobyshevski and Yuferev (1974). This study was motivated by the observation that in steady Bénard convection between horizontal planes, fluid rises at the center of the convection cells and falls at the periphery; the regions of rising fluid are therefore separated from each other, whereas the regions of falling fluid are all connected. If the fluid is permeated by a horizontal magnetic field, then a field line near the upper plane will be distorted to lie entirely in a region of falling fluid and will therefore tend to be convected downward, a tendency that may be resisted to some extent by diffusion. A field line near the lower plate *cannot* be distorted to lie everywhere in the disconnected regions of rising fluid and cannot therefore be convected upward. One would therefore expect a net tendency for flux to be transported toward the lower plate; the Bénard layer should act as a valve, allowing horizontal flux to pass downward but not upward.

The particular velocity field chosen by Drobyshevski and Yuferev to demonstrate this effect is (in dimensionless form)

$$\mathbf{u} = (-\sin x(1 + \tfrac{1}{2}\cos y)\cos z,\; -(1 + \tfrac{1}{2}\cos x)\sin y\cos z,\; (\cos x + \cos y + \cos x\cos y)\sin z), \tag{3.6}$$

for which the cell boundaries are square; the more realistic choice of hexagonal cell boundaries would complicate the analysis without greatly adding to the insight provided. The magnetic field in the absence of fluid motion (or equivalently at zero magnetic Reynolds number) is taken to be $(B_0, 0, 0)$, i.e., uniform in the x direction, and it is supposed that the boundaries $z = 0, \pi$ are perfectly conducting so that the flux πB_0 is trapped in the gap between them. The steady solution of (2.9) may be obtained as a power series in R_m when $R_m \ll 1$; what is of interest is the average of this field over the horizontal plane, which turns out to have the expansion

$$B(z) = B_0\left(1 + \frac{7R_m^2}{48\pi^2}\cos 2z + \frac{R_m^3}{240\pi^3}(28\cos z - 3\cos 3z) + O(R_m^4)\right). \tag{3.7}$$

The asymmetry about $z = \pi/2$ appears at the $O(R_m^3)$ level, and this term indeed shows the expected increase in flux in the lower half of the gap.

This phenomenon is of potential interest in the solar context. Convection in the Sun's outer layers is turbulent, but there are also fairly stable and persistent large-scale structures, reminiscent of Bénard cells, that survive even in the presence of this turbulence. These cells do show a preferred tendency for fluid to rise in discrete regions and to fall in connected regions, and any toroidal flux permeating this region will in consequence tend to be transported downward. The relevant magnetic Reynolds number is $R_{me} = u_s l_s / \lambda_e$ where u_s and l_s are scales characteristic of the large-scale structures and λ_e is an effective eddy diffusivity associated with the small-scale turbulence (see Section V); R_{me} will probably be of order unity (even though the magnetic Reynolds number based on molecular diffusivity is very large).

The phenomenon of flux pumping has been further investigated by Proctor (1975), who has demonstrated that when $R_m \ll 1$, pumping can occur even when the topological distinction between upward- and downward-moving fluid is absent. Proctor analyzes the effect of two-dimensional motions in detail and shows that a lack of geometrical symmetry about the midplane is sufficient to lead to a net transport of flux either up or down; e.g., if $\langle w^3 \rangle \neq 0$, where w is the vertical velocity at the midplane and the angular brackets denote a horizontal average, then there will be a net transport, which Proctor describes as *geometrical* (as opposed to topological) *pumping*. When $R_m \gg 1$, however, he demonstrates that this type of two-dimensional geometrical pumping is almost nonexistent, whereas in this limit the Drobyshevski and Yuferev mechanism may be expected to be most effective (although no detailed analysis has yet been carried out).

D. Generation of Toroidal Field by Differential Rotation

We conclude this section with a brief discussion of the process by which toroidal field is generated from poloidal field by differential rotation. Physically it is clear that differential rotation about an axis will tend to distort the field lines of an initially poloidal axisymmetric field $\mathbf{B}_P$ if the angular velocity ω varies along the field lines. In fact, if $\mathbf{u} = \omega(s, z)\mathbf{k} \wedge \mathbf{x}$, where (s, ϕ, z) are cylindrical polar coordinates and $\mathbf{k}$ a unit vector along $0z$, and if $\mathbf{B} = \mathbf{B}_P(s, z) + B_\phi(s, z, t)\mathbf{i}_\phi$, then the ϕ component of (2.9) is

$$\partial B_\phi / \partial t = s(\mathbf{B}_P \cdot \nabla)\omega + \lambda(\nabla^2 - s^{-2})B_\phi . \tag{3.8}$$

For the moment we shall suppose that $\mathbf{B}_P$ is maintained steadily by some unspecified mechanism. As expected, it is the gradient of ω along $\mathbf{B}_P$ lines that gives rise to generation of toroidal field. If λ is small, and if initially

$B_\phi = 0$, then B_ϕ grows linearly with time until $|B_\phi| = O(R_m)|\mathbf{B}_P|$, at which stage a steady state is established. There is no question of flux expulsion in this case since $\mathbf{B}_P$ is axisymmetric (if $\mathbf{B}_P$ included any nonaxisymmetric ingredients then these *would* be expelled).

The ultimate steady solution of (3.8) may be easily obtained if $\mathbf{B}_P$ and ω are prescribed. For example, if $\mathbf{B}_P = B_0\mathbf{k}$ where B_0 is uniform, and if $\omega = \omega(r)$ where $r^2 = s^2 + z^2$, then the steady solution of (3.8) is

$$B_\phi = -\tfrac{1}{3}B_0(\lambda r^3)^{-1} \sin\theta \cos\theta \int_0^r r_1^4 \omega(r_1)\, dr_1. \tag{3.9}$$

Note that B_ϕ as given by this solution is antisymmetric about the "equatorial" plane $\theta = \pi/2$, and that if $\omega(r_1)$ decreases sufficiently rapidly as $r_1 \to \infty$ for the integral in (3.9) to converge, then $B_\phi = O(r^{-3})$ as $r \to \infty$.

Contrast the situation when $\mathbf{B}_P$ is an irrotational field with uniform gradient, i.e.,

$$\mathbf{B}_P = C(-2s\mathbf{i}_s + z\mathbf{k}), \tag{3.10}$$

where $\mathbf{i}_s$ is a unit vector in the s direction. The steady solution for B_ϕ then has two ingredients proportional to $\sin\theta$ and $\partial P_3(\cos\theta)/\partial\theta$, but the former ingredient dominates for large r, and again provided $\omega(r_1)$ decreases sufficiently rapidly as $r_1 \to \infty$, the asymptotic behavior of B_ϕ for large r is

$$B_\phi \sim (2C \sin\theta/3\lambda r^2) \int_0^\infty r_1^4 \omega(r_1)\, dr_1. \tag{3.11}$$

This field is symmetric about $\theta = \pi/2$, and $O(r^{-2})$ at infinity. In general, therefore, if a poloidal field $\mathbf{B}_P$ is weakly varying in a region of differential rotation, then it is the local gradient of the field (rather than the local field itself) that determines the toroidal field generated at remote points when conditions are steady.

IV. Some Basic Results

A. Cowling's Theorem and Related Results

The impossibility of steady maintenance of an axisymmetric magnetic field by motions axisymmetric about the same axis (Cowling, 1934) is so well known as hardly to require comment here. The modifications and extensions of the theorem are numerous, but all reflect the basic fact that axisymmetric meridional circulation can redistribute poloidal flux but cannot systematically regenerate it. The equation for the flux function $\chi(r, \theta)$ under axisymmetric conditions [analogous to (3.35)] is

$$\partial\chi/\partial t + \mathbf{u}_P \cdot \nabla\chi = \lambda D^2\chi, \tag{4.1}$$

where $\mathbf{u}_P$ is the meridional velocity (assumed axisymmetric) and D^2 the Stokes operator. The structure of this parabolic equation essentially ensures (Braginskii, 1964a) that $\nabla\chi$ everywhere tends to zero. Even if λ is nonuniform but satisfies merely the natural condition $\mathbf{u}_P \cdot \nabla\lambda = 0$, standard manipulation of (4.1) and the relevant boundary conditions leads to this same conclusion (yet another minor extension of Cowling's celebrated result!). Of course, when $|\nabla\chi|$ and so $|\mathbf{B}_P|$ have reached a negligibly weak level, the toroidal field B_ϕ must likewise decay, again essentially because of the parabolic structure of (3.8).

An interesting variation on Cowling's theorem has been claimed by Pichakhchi (1966). This is that steady dynamo action is impossible if the electric field $\mathbf{E}$ vanishes everywhere. (In the axisymmetric case with $\mathbf{B}_T = 0$, this is just a rewording of Cowling's theorem since $\mathbf{E} \equiv 0$ in this situation under steady conditions.) If a steady dynamo with $\mathbf{E} \equiv 0$ *were* possible, then the field $\mathbf{B}$ would [from (2.5)] satisfy $\mathbf{B} \cdot (\nabla \wedge \mathbf{B}) = 0$, i.e., it would have zero helicity everywhere and a correspondingly simple topological structure. The fact that this is apparently not possible is of course significant.

There is also a counterpart of Cowling's theorem for two-dimensional velocity and magnetic fields (Zel'dovich, 1957; Lortz, 1968). In this case, (3.3) is the governing equation, and by standard manipulations, provided $A = o(r^{-1})$ at infinity,

$$(d/dt) \iint A^2 \, dx \, dy = -2\lambda \iint (\nabla A)^2 \, dx \, dy. \tag{4.2}$$

It follows that a steady state is possible only if $\nabla A = 0$, i.e., only if $B_x = B_y = 0$. Again, this removes the source of any possible regeneration of B_z, which must also therefore vanish in a steady state.

If $\mathbf{u}$ and $\mathbf{B}$ are stationary random functions (with zero mean) of x and y, then (4.2) must be replaced by

$$d\langle A^2\rangle/dt = -2\lambda\langle(\nabla A)^2\rangle, \tag{4.3}$$

where $\langle \ \rangle$ indicates averaging over the x-y plane. No matter how small λ may be, this again implies ultimate decay of the magnetic energy density. In the limit $\lambda \to 0$, $\langle A^2\rangle$ apparently remains constant; however, this holds only so long as $\langle(\nabla A)^2\rangle$ remains finite; in fact, $\langle(\nabla A)^2\rangle$ increases (just as in the particular situation described in Section III,B) until it is $O(\lambda^{-1})$; at this stage the length scale of the field is $O(\lambda^{1/2})$ and the inexorable decay of the field then sets in.

The impossibility of dynamo action under purely toroidal motion (Bullard and Gellman, 1954; Backus, 1958) is also closely related to Cowling's theorem, in that it follows from the equation

$$D(\mathbf{x} \cdot \mathbf{B})/Dt = (\mathbf{B} \cdot \nabla)(\mathbf{x} \cdot \mathbf{u}) + \lambda \nabla^2(\mathbf{x} \cdot \mathbf{B}), \tag{4.4}$$

which may be derived from (2.9) together with $\nabla \cdot \mathbf{u} = 0$. If $\mathbf{u}$ is purely toroidal, then $\mathbf{x} \cdot \mathbf{u} = 0$ and so $\mathbf{x} \cdot \mathbf{B}$ inevitably decays everywhere to zero. The decay of $\mathbf{B}_P$ and $\mathbf{B}_T$ is almost an immediate consequence. Busse (1975) has on the basis of Eq. (4.4) obtained a necessary condition (in terms of a lower bound on the poloidal velocity) that must be satisfied for successful dynamo action.

B. Rotor Dynamos

In view of the various antidynamo theorems described above, it was of course of crucial importance that the possibility of steady dynamo action in a fluid of uniform conductivity occupying a simply connected domain be unambiguously established for at least one kinematically possible velocity field (no matter how artificial from a dynamical point of view). Until this was done (Herzenberg, 1958) it was by no means clear that a master theorem proving the absolute impossibility of such dynamo action might not at some stage be proved. Herzenberg's dynamo consisted of two spherical rotors (i.e., quasi-eddies) embedded in a fluid sphere, the conductivity being uniform throughout. In fact, it is easier to comprehend the three-rotor problem (Fig. 2) considered† by Gibson (1968). Suppose that three spheres S_1, S_2, S_3 each of radius a, with centers located at the points $(d, 0, 0)$, $(0, d, 0)$, and $(0, 0, d)$, where $d \gg a$, rotate with angular velocities $(0, 0, -\omega)$, $(-\omega, 0, 0)$, and $(0, -\omega, 0)$, respectively, where $\omega > 0$; the conductivity σ throughout the whole space is assumed uniform. Note immediately the lack of reflectional symmetry in this configuration. The principle of the dynamo is roughly as follows: suppose that there are nearly uniform fields of the form $\mathbf{B}_1 \approx (0, 0, B)$, $\mathbf{B}_2 \approx (B, 0, 0)$, and $\mathbf{B}_3 \approx (0, B, 0)$ in the neighborhoods of S_1, S_2, and S_3, respectively. Then we have the possibility of a cyclic generation in which the toroidal field generated by rotation of S_α $(\alpha = 1, 2)$ acts as the "applied" field $\mathbf{B}_{\alpha+1}$ in the neighborhood of $S_{\alpha+1}$, and the toroidal field generated by rotation of S_3 acts as the applied field $\mathbf{B}_1$ in the neighborhood of S_1. The subtlety of the problem derives from the fact that, as mentioned at the end of Section III,D, the fields generated by differential rotation in each case are determined to an important degree by the local field gradient as well as by the local field itself, and this has to be taken into account in the

† The particular configuration considered here was also discussed by Venezian (1967).

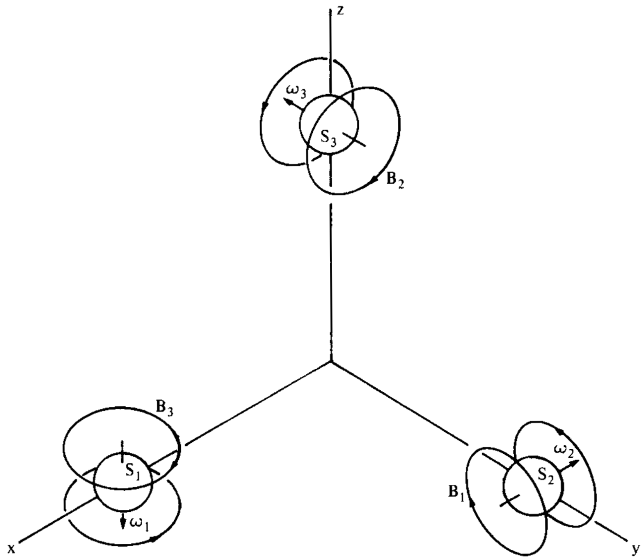

FIG. 2. The three-rotor dynamo of Gibson (1968). The spheres rotate as indicated and generate toroidal fields $\mathbf{B}_\alpha$, which act as the applied poloidal fields for S_α ($\alpha = 1, 2, 3$).

detailed calculation. The condition obtained by Gibson for steady dynamo action (adapted to this particular configuration) is

$$R_m = \omega a^2/\lambda = 10\sqrt{2}\,(d/a)^3, \tag{4.5}$$

correct to leading order in the small parameter a/d.

A working dynamo based on the interaction of two rotors has been constructed in the laboratory by Lowes and Wilkinson (1963, 1968). The rotors are cylinders inclined to each other and embedded in a block of material of the same conductivity, electrical communication between the rotors and the block being provided by a lubricating film of mercury. Not only was dynamo action demonstrated with this model [through observation of a sudden large increase in the local magnetic field when the angular velocities of the cylinders were increased beyond a certain critical value—analogous to that given by (4.5)], but also reversals of the field were observed when the dynamo was functioning in the fully nonlinear regime—an observation of the greatest interest in view of the known random reversals of the Earth's magnetic field mentioned in the introduction.

A further ingenious example of dynamo action associated with a pair of rotors has been analyzed by Gailitis (1970). In this case, the rotors are toroidal rather than spherical, and the velocity field is axisymmetric about the common axis of the two toruses. The magnetic field that is maintained is, however, nonaxisymmetric, and there is no conflict with Cowling's theorem.

V. The Mean Electromotive Force Generated by a Random Velocity Field

A. The Two-Scale Approach

Motion in the convective zone of the Sun is certainly turbulent, and any dynamo theory that fails to take account of this fact is hardly realistic. Likewise, motion in the core of the Earth almost certainly consists of a mean and a random ingredient. It is not clear whether the random ingredient is turbulence in the normal sense of the word, or rather a random field of waves influenced by Lorentz, Coriolis, and buoyancy forces; from a purely kinematic point of view, this distinction is not crucial, and we merely assume throughout this section that $\mathbf{u}(\mathbf{x}, t)$ is a stationary random function of both $\mathbf{x}$ and t with zero mean—effects of nonzero mean velocity will be considered in Section VI.

We are primarily concerned with the evolution of the mean magnetic field, which may be supposed to have a characteristic length scale L large compared with the scale l that characterizes the "background" velocity field $\mathbf{u}$; in the case of turbulence, l will be the scale of the energy-containing eddies (Batchelor, 1953), while in the case of random waves, l will be, say, the wavelength of the most energetic modes represented in the spectrum of $\mathbf{u}$. Either way, we can average the induction equation (2.9) over a spatial scale intermediate between l and L to obtain

$$\partial \mathbf{B}_0/\partial t = \nabla \wedge \mathscr{E} + \lambda \nabla^2 \mathbf{B}_0 , \tag{5.1}$$

where $\mathbf{B}_0 = \langle \mathbf{B} \rangle, \mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, and $\mathscr{E} = \langle \mathbf{u} \wedge \mathbf{b} \rangle$. This two-scale approach was introduced by Steenbeck *et al.* (1966) and has since had a revolutionary impact on the subject†. A series of papers by these authors, developing their approach to "mean field electrodynamics" has been collected together in English translation by Roberts and Stix (1971). The main problem, of course, analogous to the closure problem of turbulence dynamics, is to find a means of expressing $\mathscr{E}$ in terms of $\mathbf{B}_0$ so that (5.1) may be integrated. The task is, however, easier because the basic equation for $\mathbf{B}$ is linear, albeit with a random coefficient.

The equation for $\mathbf{b}$, obtained by subtracting (5.1) from (2.9), is

$$\partial \mathbf{b}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}_0) + \nabla \wedge (\mathbf{u} \wedge \mathbf{b} - \langle \mathbf{u} \wedge \mathbf{b} \rangle) + \lambda \nabla^2 \mathbf{b}, \tag{5.2}$$

and if we suppose that $\mathbf{b} = 0$ at same initial instant $t = 0$, this establishes a linear relationship between $\mathbf{b}$ and $\mathbf{B}_0$, and so between $\mathscr{E}$ and $\mathbf{B}_0$; since the

† The concepts of a mean electromotive force and of an associated eddy conductivity were already present in earlier work (e.g., Kovasznay, 1960).

scale L of $\mathbf{B}_0$ is very large, such a relationship may presumably be developed as a series

$$\mathscr{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk}\, \partial B_{0j}/\partial x_k + \gamma_{ijkl}\, \partial^2 B_{0j}/\partial x_k\, \partial x_l + \cdots, \tag{5.3}$$

the coefficients α_{ij}, β_{ijk}, ..., being pseudotensors determined in principle by the statistical properties of the $\mathbf{u}$ field and the parameter λ (which, of course, plays an important part in the solution of (5.2) (*pseudo* because $\mathscr{E}$ is a polar vector, whereas $\mathbf{B}_0$ is an axial vector). It is important to note that these pseudotensors do not depend on $\mathbf{B}_0$; hence α_{ij} may be evaluated on the simplifying assumption that $\mathbf{B}_0$ is uniform; then β_{ijk} may be evaluated on the assumption that $\partial B_{0j}/\partial x_k$ is uniform, and so on. Attention in most investigations has been focused on the first two terms of (5.3) on the grounds that it is essentially a series in ascending powers of l/L, and subsequent terms are likely to have negligible effect when L is large. There are, however, considerable subtleties here, particularly when λ is very small, and the possible influence of subsequent terms perhaps deserves investigation. For the present, however, we truncate the series (5.3) after the second term and investigate some of the consequences.

First, suppose that the $\mathbf{u}$ field exhibits no preferred direction in its statistical properties; since α_{ij}, β_{ijk}, *are* essentially statistical properties of the turbulence, they must then be invariant under rotations of the frame of reference and must therefore take the form

$$\alpha_{ij} = \alpha\delta_{ij}, \qquad \beta_{ijk} = \beta\varepsilon_{ijk}, \tag{5.4}$$

where α is a pseudoscalar and β a pure scalar. Here the crucial role played by "lack of reflectional symmetry" comes into evidence. If the $\mathbf{u}$ field is reflectionally (as well as rotationally) symmetric in its statistical properties, then α (being a pseudoscalar that is not invariant under change from a right- to a left-handed frame of reference) must vanish. No such conclusion applies to the β term. If the turbulence lacks reflectional symmetry, then the α term will in general be nonzero, and the relation between $\mathscr{E}$ and $\mathbf{B}_0$ becomes

$$\mathscr{E} = \alpha\mathbf{B}_0 - \beta\nabla \wedge \mathbf{B}_0. \tag{5.5}$$

In view of the assumed statistical homogeneity of the $\mathbf{u}$ field, α and β are constants, and (5.1) becomes

$$\partial\mathbf{B}_0/\partial t = \alpha\nabla \wedge \mathbf{B}_0 + (\lambda + \beta)\nabla^2\mathbf{B}_0. \tag{5.6}$$

Hence β plays the role of a turbulent diffusivity; it is to be expected that $\beta > 0$, although no general proof of this appears yet to be available. The α term, on the other hand, has quite a novel structure (from the point of view of conventional electrodynamics) and is in fact of crucial importance for dynamo theory.

The average of Ohm's law (2.5), incorporating (5.5), becomes

$$\mathbf{J}_0 = \sigma(\mathbf{E}_0 + \alpha\mathbf{B}_0 - \beta\nabla \wedge \mathbf{B}_0), \tag{5.7}$$

where $\mathbf{J}_0 = \langle \mathbf{j} \rangle$, $\mathbf{E}_0 = \langle \mathbf{E} \rangle$, or equivalently,

$$\mathbf{J}_0 = \sigma_e(\mathbf{E}_0 + \alpha\mathbf{B}_0), \qquad \sigma_e = \sigma(1 + \beta\sigma\mu_0)^{-1}. \tag{5.8}$$

The α term therefore tends to drive mean current along the lines of mean magnetic field. In a spherical geometry, this effect in the presence of a toroidal field will generate a toroidal current, which acts as the source of a poloidal field. This therefore is the key to the means by which poloidal field may be regenerated from toroidal field by nonaxisymmetric random motions.

The explosive character of Eq. (5.6) may be recognized very quickly if we suppose for the moment that our fluid fills all space and if we consider a magnetic field having an initial structure satisfying $\nabla \wedge \mathbf{B}_0 = K\mathbf{B}_0$ (i.e., one of the force-free fields of Section II,D). For such a field, $\nabla^2\mathbf{B}_0 = -K^2\mathbf{B}_0$, and so according to (5.6) the field will retain its spatial structure and develop exponentially like exp ωt, where

$$\omega = \alpha K - (\lambda + \beta)K^2. \tag{5.9}$$

Clearly we have exponential growth (i.e., dynamo action) if

$$\alpha K > (\lambda + \beta)K^2 \tag{5.10}$$

(and clearly we must choose K to have the same sign as α in order to ensure this possibility). Provided $|K|$ is sufficiently small (i.e., provided the scale of $\mathbf{B}_0$ is sufficiently large), condition (5.10) is satisfied. Dynamo growth is then assured for force-free modes of sufficiently large length scale.

It is, of course, important to obtain an explicit representation of α_{ij} and β_{ijk} in terms of the statistical properties of the $\mathbf{u}$ field; this stage of the problem is analogous to the statistical mechanics problem of obtaining expressions for the various transport coefficients in terms of the statistical substructure of the medium considered. In the present context the substructure is provided by the background random velocity field. Unfortunately, determination of α_{ij} and β_{ijk} is possible only in certain limiting situations; however, these limiting situations are in themselves of particular interest and will be considered in the following sections.

B. The Strong Diffusion Limit

If $R_m = u_0 l/\lambda \ll 1$, where $u_0 = \langle \mathbf{u}^2 \rangle^{1/2}$, then the diffusion term $\lambda\nabla^2\mathbf{b}$ in (5.2) clearly dominates over the "interaction" term

$$\mathbf{G} = \nabla \wedge (\mathbf{u} \wedge \mathbf{b} - \langle \mathbf{u} \wedge \mathbf{b} \rangle), \tag{5.11}$$

which may be neglected to lowest order. The resulting equation

$$\partial \mathbf{b}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}_0) + \lambda \nabla^2 \mathbf{b} \tag{5.12}$$

is soluble by straightforward means, as recognized by Liepmann (1952) in a discussion of the spectrum of field fluctuations generated by turbulence in the presence of a uniform magnetic field.

It is important now to distinguish between conventional turbulence whose characteristic time scale (the turnover time of the energy-containing eddies) is of order l/u_0, and a random wave field whose time scale (the inverse of the dominant frequency) is determined by the relevant dispersion relation and is quite independent of l/u_0. In the former case, $|\partial \mathbf{b}/\partial t|$ is of the same order of magnitude as $|\mathbf{G}|$ and should, for consistency, be dropped also.

Let us first determine α_{ij} in this situation. As remarked earlier, to do this we may suppose $\mathbf{B}_0$ uniform, and (5.12) becomes simply

$$\lambda \nabla^2 \mathbf{b} = -(\mathbf{B}_0 \cdot \nabla)\mathbf{u}. \tag{5.13}$$

Let $\mathbf{p}(\mathbf{k}, t)$ and $\mathbf{q}(\mathbf{k}, t)$ be the space Fourier transforms of $\mathbf{u}$ and $\mathbf{b}$, respectively. [The fact that $\mathbf{p}$ and $\mathbf{q}$ must be generalized functions (Lighthill, 1959) does not invalidate any of the operations that follow.] Then from (5.13),

$$-\lambda k^2 \mathbf{q} = -i(\mathbf{B}_0 \cdot \mathbf{k})\mathbf{p}. \tag{5.14}$$

The spectrum tensor $\Phi_{ij}(\mathbf{k})$ is related to $\mathbf{p}(\mathbf{k}, t)$ by

$$\langle p_i(\mathbf{k}, t) p_j^*(\mathbf{k}', t)\rangle = \Phi_{ij}(\mathbf{k})\, \delta(\mathbf{k} - \mathbf{k}'), \tag{5.15}$$

where an asterisk here and subsequently denotes a complex conjugate, and similarly for the spectrum tensor $\Gamma_{ij}(\mathbf{k})$ of the $\mathbf{b}$ field. The relation between these two spectrum tensors may be derived immediately from (5.14) (Golitsyn, 1960) in the form

$$\Gamma_{ij}(\mathbf{k}) = (\lambda k^2)^{-2} (\mathbf{B}_0 \cdot \mathbf{k})^2 \Phi_{ij}(\mathbf{k}). \tag{5.16}$$

More interestingly, the vector $\mathscr{E}$ may be obtained from (5.14) in the form $\mathscr{E}_i = \alpha_{ij} B_{0j}$, where

$$\alpha_{ij} = i\varepsilon_{ikl} \lambda^{-1} \int k^{-2} k_j \Phi_{kl}(\mathbf{k})\, d^3\mathbf{k}. \tag{5.17}$$

The Hermitian symmetry of $\Phi_{kl}(\mathbf{k})$ (Batchelor, 1953) guarantees that this expression for α_{ij} is real; moreover, the incompressibility condition in the form $k_i \Phi_{ij}(\mathbf{k}) = 0$ may be used to show that α_{ij} [as given by (5.17)] is also symmetric (Moffatt, 1970a).

For turbulence that is statistically invariant under rotations (i.e., showing no preferred direction), $\Phi_{kl}(\mathbf{k})$ takes the form

$$\Phi_{kl}(\mathbf{k}) = \frac{E(k)}{4\pi k^4}(k^2\,\delta_{kl} - k_k k_l) + \frac{iF(k)}{8\pi k^4}\varepsilon_{klm}k_m\,, \tag{5.18}$$

where $E(k)$ (≥ 0 for all k) is the energy spectrum function, and $F(k)$ the helicity spectrum function, satisfying

$$\tfrac{1}{2}\langle \mathbf{u}^2\rangle = \int_0^\infty E(k)\,dk, \qquad \langle \mathbf{u}\cdot\nabla\wedge\mathbf{u}\rangle = \int_0^\infty F(k)\,dk. \tag{5.19}$$

The Schwarz inequality

$$\langle \mathbf{p}\cdot\mathbf{k}\wedge\mathbf{p}^*\rangle^2 \leq \langle |\mathbf{p}|^2\rangle\langle |\mathbf{k}\wedge\mathbf{p}|^2\rangle \tag{5.20}$$

may be translated into spectral terms to show that, for all k,

$$|F(k)| \leq 2kE(k). \tag{5.21}$$

It is clear that only the antisymmetric part of Φ_{kl} contributes to (5.17), and substitution of (5.18) in fact gives $\alpha_{ij} = \alpha\delta_{ij}$, where

$$\alpha = -(1/3\lambda)\int_0^\infty k^{-2}F(k)\,dk. \tag{5.22}$$

Thus α is expressed as a weighted integral of the helicity spectrum function. If $F(k)$ is either positive for all k or negative for all k, then α and $\langle \mathbf{u}\cdot\nabla\wedge\mathbf{u}\rangle$ have opposite signs; in this case an order of magnitude estimate of α is given from (5.19b) and (5.22) in the form

$$\alpha \approx -\tfrac{1}{3}\lambda^{-1}l^2\langle \mathbf{u}\cdot\boldsymbol{\omega}\rangle, \tag{5.23}$$

where $\boldsymbol{\omega} = \nabla\wedge\mathbf{u}$.

The pseudotensor β_{ijk} can be obtained by the same method and is likewise proportional to λ^{-1}. It follows on dimensional grounds that all components of β_{ijk} have order of magnitude at most $\lambda^{-1}u_0^2 l^2 = R_\mathrm{m}^2\lambda$; since $R_\mathrm{m} \ll 1$, effects associated with β_{ijk} are in this limit swamped by the molecular diffusion term $\lambda\nabla^2\mathbf{B}_0$ in (5.1) and may reasonably be ignored.

The conclusion that α_{ij} is symmetric does not persist if the calculation leading to (5.17) is extended to higher powers in R_m. The result (5.17) may be regarded as the leading term of an expansion of the form

$$\alpha_{ij} = u_0\sum_{n=1}^{\infty} R_\mathrm{m}^n\alpha_{ij}^{(n)}, \tag{5.24}$$

where the $\alpha_{ij}^{(n)}$ are dimensionless pseudotensors that may in principle be determined by iteration based on (5.2); $\alpha_{ij}^{(n)}$ can by this means be expressed as an n-fold weighted integral of an $(n + 1)$th-order spectrum tensor. Krause (1968) has considered the question of convergence of this type of expansion and concludes that it does converge for all finite R_m, although the indications are that the convergence may be extremely slow for large R_m. There are further indications from parallel work by G. O. Roberts (1970, 1972) on spatially periodic dynamos that $\alpha_{ij}^{(n)}$ is symmetric or antisymmetric according as n is odd or even; this conjecture requires further investigation in the turbulence context.

In general, however, it is clear that α_{ij} may be decomposed into its symmetric and antisymmetric parts:

$$\alpha_{ij} = \alpha_{ij}^{(S)} - \varepsilon_{ijk} V_k \,. \tag{5.25}$$

The effect of the antisymmetric part in (5.1) is to provide a term $\nabla \wedge (\mathbf{V} \wedge \mathbf{B}_0)$ on the right-hand side, where $\mathbf{V}$ is a velocity (uniform in so far as the turbulence is homogeneous) that merely convects the large-scale field pattern. In conjunction with the α effect, the force-free modes considered in Section V,A would then propagate relative to the fluid with phase velocity $\mathbf{V}$. If there is also a mean fluid velocity $\mathbf{U}$, then the *effective* mean velocity as far as the magnetic field is concerned is $\mathbf{U} + \mathbf{V}$; this concept of an "effective velocity" is a crucial ingredient of Braginskii's (1964a) theory of nearly axisymmetric configurations (see Section VI).

The appearance of an electromotive force with a component parallel to the ambient magnetic field [Eq. (5.5)] bears a simple physical interpretation, which was in fact given by Parker (1955a), who on the basis of inspired physical reasoning and heuristic argument anticipated the main lines of development of the subject by more than ten years! Consider the effect of a localized motion with positive helicity (a "cyclonic event" in Parker's terminology). Such a motion tends to generate an Ω-shaped loop in a line of force of an ambient magnetic field $\mathbf{B}_0$ (Fig. 3a), and the loop is twisted so that its normal has a nonzero component in the original field direction.

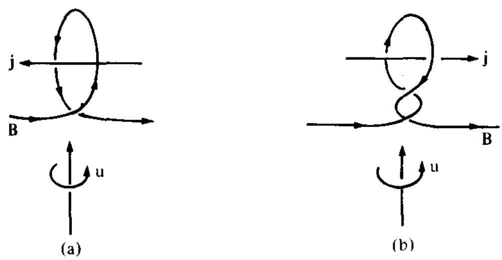

FIG. 3. Possible distortion of a line of force by a cyclonic event.

When the twist is of limited amount (as it certainly will be in the strongly diffusive situation considered above) the loop may be conceived as being due to a current antiparallel to $\mathbf{B}_0$; a random superposition of such motions may then be expected to generate a mean current $\mathbf{J}_0$ antiparallel to $\mathbf{B}_0$, as implied by the result $\mathscr{E} = \alpha \mathbf{B}_0$ with $\alpha < 0$.

This argument seems reliable both in the strong diffusion limit considered here and in the alternative situation considered by Parker when diffusion is weak but the events are so short-lived that the "limited twist" picture of Fig. 3a is applicable. In this weak diffusion limit, however, if the events are *not* short-lived, then a twist through $3\pi/2$ (Fig. 3b) will give just the opposite effect from a twist through $\pi/2$, and this introduces some uncertainty into the sign of the effective value for α. Since the sign of α is of crucial importance for some of the dynamos described in Sections VII,C and D it becomes important to have a reliable expression for α in the weak diffusion limit also (see Section V,D).

C. Evaluation of α_{ij} and β_{ijk} for a Random Wave Field

For a random wave field, it is natural to use a double Fourier transform in both space and time variables for $\mathbf{u}(\mathbf{x}, t)$:

$$\mathbf{u}(\mathbf{x}, t) = \iint \tilde{\mathbf{u}}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \, d^3\mathbf{k} \, d\omega, \tag{5.26}$$

and similarly for $\mathbf{b}(\mathbf{x}, t)$. The field has a random wave character (rather than a "turbulent" character) if the velocity amplitudes in each constituent wave are small compared with their phase velocities, i.e., provided $|k^3 \omega \tilde{\mathbf{u}}(\mathbf{k}, \omega)| \ll \omega/k$. In this situation the interaction term $\mathbf{G}$ is again negligible in (5.2) and so $\mathbf{b}$ is determined by (5.12). Now, however, there is no reason to drop $\partial \mathbf{b}/\partial t$; moreover, (5.12) should be valid for all values of λ, both small and large, the case when λ is small being now of greater potential interest. Note that in the limit of infinitesimal wave amplitudes, the constituent waves in (5.26) are noninteracting, with a dispersion relationship (that will be determined by dynamic considerations) $\omega = \omega(\mathbf{k})$. For finite amplitude waves, however, nonlinear interactions will provide some forcing of waves at nonresonant frequencies, and ω need not then be restricted by a dispersion relation; there may of course also be some extraneous forcing of waves.

Taking the Fourier transform of (5.12), with $\mathbf{B}_0(\mathbf{x})$ a field whose second and higher space derivatives vanish, and constructing $\langle \mathbf{u} \wedge \mathbf{b} \rangle$ as before leads to the following expressions for α_{ij} and β_{ijk}:

$$\alpha_{ij} = i\lambda\varepsilon_{ikl} \iint (\omega^2 + \lambda^2 k^4)^{-1} k^2 k_j \Phi_{kl}(\mathbf{k}, \omega)\, d\mathbf{k}\, d\omega, \tag{5.27}$$

$$\beta_{ijk} = \operatorname{Re} \varepsilon_{iml} \iint (\omega^2 + \lambda^2 k^4)^{-1} (i\omega + \lambda k^2) \times \left\{\frac{\partial}{\partial k_j} k_k \Phi_{lm}(\mathbf{k}, \omega) - \Phi_{jm}(\mathbf{k}, \omega)\delta_{kl}\right\} d\mathbf{k}\, d\omega, \tag{5.28}$$

where now $\Phi_{lm}(\mathbf{k}, \omega)$ is the double Fourier transform of the space–time velocity correlation tensor. Note that results for the strong diffusion limit (Section V,B) may be recovered by replacement of $\Phi_{ij}(\mathbf{k}, \omega)$ by $\Phi_{ij}(\mathbf{k})\, \delta(\omega)$; in this limit, magnetic adjustment to velocity fluctuations becomes instantaneous.

In the case when the wave amplitudes are isotropically distributed, (5.24) becomes $\alpha_{ij} = \alpha\, \delta_{ij}$, where

$$\alpha = \tfrac{1}{3}\alpha_{ii} = -\tfrac{1}{3}\lambda \int_{-\infty}^{\infty} d\omega \int_0^{\infty} dk(\omega^2 + \lambda^2 k^4)^{-1} k^2 F(k, \omega), \tag{5.29}$$

where $F(k, \omega)$ bears the same relation to $\Phi_{kl}(\mathbf{k}, \omega)$ as $F(k)$ bears to $\Phi_{kl}(\mathbf{k})$ in (5.18). Similarly, (5.28) becomes $\beta_{ijk} = \beta\varepsilon_{ijk}$, where

$$\beta = \tfrac{1}{6}\beta_{ijk}\varepsilon_{ijk} = \tfrac{2}{3}\lambda \int_{-\infty}^{\infty} d\omega \int_0^{\infty} dk(\omega^2 + \lambda^2 k^4)^{-1} k^2 E(k, \omega) \tag{5.30}$$

(Krause and Roberts, 1973; Roberts and Soward, 1975). Evidently β is positive and again α has the opposite sign from $F(k, \omega)$ if this function is single-signed.

It is noteworthy that both expressions (5.29) and (5.30) vanish in the perfect conductivity limit $\lambda \to 0$ [unless there is a divergence of the integrals in the neighborhood of $\omega = 0$, a possibility that can certainly be discounted if $\Phi_{ij}(\mathbf{k}, \omega) = O(\omega^2)$ as $\omega \to 0$]. The reason is essentially that in this limit $\mathbf{b}$ and $\mathbf{u}$ are *in phase* in each constituent wave, so that $\tilde{\mathbf{u}} \wedge \tilde{\mathbf{b}}^*$ makes no contribution to $\langle \mathbf{u} \wedge \mathbf{b} \rangle$. Dillon (1975) has shown that, as far as α is concerned, this result holds even when effects of the interaction terms $\mathbf{G}$ are included: by expanding in powers of the amplitude of the constituent waves, Dillon shows by induction that, to all orders, $\alpha_{ii} = 0$ when $\lambda = 0$!

The vanishing of α and β in the limit $\lambda \to 0$ does, however, depend critically on the assumption of a well-established wave field, with no transient effects present, and no zero frequency ingredients. If we adopt the alternative point of view and solve (5.12) with $\lambda = 0$, $\mathbf{B}_0$ uniform, and subject to the initial condition $\mathbf{b}(\mathbf{x}, 0) = 0$, we obtain

$$\mathbf{b}(\mathbf{x}, t) = \mathbf{B}_0 \cdot \int_0^t \nabla \mathbf{u}(\mathbf{x}, \tau)\, d\tau, \tag{5.31}$$

and so

$$\langle \mathbf{u} \wedge \mathbf{b} \rangle_i = \varepsilon_{ijp} \int_0^t \langle u_j(\mathbf{x}, t)\, \partial u_p(\mathbf{x}, \tau)/\partial x_k \rangle\, d\tau\, B_{0k}, \tag{5.32}$$

and the relevant value of α when the field $\mathbf{u}$ is isotropic and t large is evidently

$$\alpha = -\frac{1}{3}\int_0^\infty \langle \mathbf{u}(\mathbf{x}, t) \cdot \nabla \wedge \mathbf{u}(\mathbf{x}, t - \tau)\rangle\, d\tau = -\frac{2\pi}{3}\int_0^\infty F(k, 0)\, dk, \tag{5.33}$$

which need not vanish.

So which expression is "correct", (5.29) or (5.33)? If $F(k, 0) \neq 0$, then (5.33) is appropriate [and may in fact be obtained directly from (5.29) by replacing $F(k, \omega)$ by $F(k, 0)$ and integrating with respect to ω]; if $F(k, 0) = 0$ however (i.e., if the wave spectral density vanishes as $\omega \to 0$), then the expression (5.29) is appropriate with the implication [consistent with (5.33)] that $\alpha \to 0$ as $\lambda \to 0$.

This distinction is clearly important since a factor λ (or equivalently R_m^{-1}) in order of magnitude estimates of α makes a big difference when (as in the turbulent convection zone of the sun) $R_m \gg 1$. The approach of Parker (1955a, 1970b) (see Section V,D) gives a result akin to (5.33) and corresponds to the limiting procedure $\lim_{t\to\infty} \lim_{\lambda\to 0}$; the approach of Braginskii (1964a,b) gives a result more akin to (5.29) and corresponds to the inverse procedure $\lim_{\lambda\to 0} \lim_{t\to\infty}$.

D. Effect of Turbulence in the Weak Diffusion Limit, $\lambda \to 0$

For turbulence with typical time scale $O(l/u_0)$, there is no justification for the neglect of the interaction term $\mathbf{G}$ when λ is small, and we must return to the exact Eq. (5.2). If we fix $t > 0$ and let $\lambda \to 0$, then the Cauchy solution (2.19) is valid and we may construct

$$\mathscr{E}_i = \langle \mathbf{u} \wedge \mathbf{b} \rangle_i = \langle \mathbf{u} \wedge \mathbf{B} \rangle_i = \varepsilon_{ijk} \langle v_j(\mathbf{a}, t) B_l(\mathbf{a}, 0)\, \partial x_k/\partial a_l \rangle, \tag{5.34}$$

where $\mathbf{v}(\mathbf{a}, t) = \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t)$. If further we adopt the initial condition $\mathbf{b}(\mathbf{x}, 0) = 0$, then $\mathbf{B}_l(\mathbf{a}, 0) = \mathbf{B}_{0l}(\mathbf{a}, 0)$; and with the usual artifice that α_{ij} may be calculated by assuming $\mathbf{B}_0$ uniform, we obtain immediately

$$\alpha_{ij}(t) = \varepsilon_{imk}\langle v_m(\mathbf{a}, t)\, \partial x_k/\partial a_j\rangle = \varepsilon_{imk} \int_0^t \langle v_m(\mathbf{a}, t)\, \partial v_k(\mathbf{a}, \tau)/\partial a_j\rangle\, d\tau. \quad (5.35)$$

This tensor has a superficial resemblance with the diffusion tensor $D_{ij}(t)$ for a scalar field (Taylor, 1922):

$$D_{ij}(t) = \int_0^t \langle v_i(\mathbf{a}, t) v_j(\mathbf{a}, \tau)\rangle\, d\tau. \quad (5.36)$$

There are, however, several difficulties in the interpretation of (5.35), which may be catalogued as follows:

(i) If the turbulence is statistically stationary in space and time, then the integrand in (5.36) is a function of $t - \tau$ only and the integral certainly converges as $t \to \infty$. The integrand of (5.35) may, however, depend on t and $t - \tau$ independently (since $\partial v_k/\partial a_l$ is not a stationary random function of τ—see Lumley, 1962) and there is no obvious guarantee that the integral converges as $t \to \infty$.

(ii) If the integral (5.35) diverges or oscillates as $t \to \infty$ (as is not implausible bearing in mind the discussion about loop twisting at the end of Section V,B), then the effects of weak molecular diffusion must in some way be reincorporated to force convergence in a time of order l^2/λ. This is extremely difficult, although possibly the approach of Saffman (1960) for the corresponding scalar problem might succeed.

(iii) Even if the integral (5.35) does converge, there is no absolute guarantee that it gives a good approximation to the relevant value of α when λ is small but nonzero. We have seen in Section V,C that expression (5.33) (which is certainly a convergent integral) may be quite misleading when R_m is large but finite and $t \to \infty$. The same may be true in the present context in which Lagrangian (rather than Eulerian) correlation tensors make a natural appearance. In the linearized context of Section V,C, transients certainly decay as $\lambda t \to \infty$; it is not known whether the same is true when the interaction term $\mathbf{G}$ is retained. The question may be rephrased as follows: suppose $\mathbf{u}(\mathbf{x}, t)$ is a stationary random function of $\mathbf{x}$ and t, and $\mathbf{b}(\mathbf{x}, 0)$ a stationary random function of $\mathbf{x}$, both with zero mean, but with $\langle \mathbf{u} \wedge \mathbf{b}\rangle \neq 0$ at $t = 0$, and let $\mathbf{b}(\mathbf{x}, t)$ be determined by the exact induction equation with $\lambda \neq 0$. Does $\langle \mathbf{u} \wedge \mathbf{b}\rangle$ tend to zero or to infinity, or to something between as $t \to \infty$? [The related question of what happens to $\langle \mathbf{b}^2\rangle$ as $t \to \infty$ is an old one (Batchelor, 1950), which has been studied closely from many points of view (Schlüter and Biermann, 1950; Moffatt, 1961, 1963; Saffman, 1963; Kraichnan and Nagarajan, 1967) but to which no clear-cut answer has yet emerged.]

The expression (5.35) hears a close relationship with the expression obtained by Parker (1970b) on the basis of his cyclonic events model, namely,

$$\alpha_{ij} = \tfrac{1}{2} n \varepsilon_{imk} \left\langle \int X_j(\mathbf{a}) \, \partial X_m / \partial a_k \, d^3\mathbf{a} \right\rangle, \tag{5.37}$$

where $\mathbf{X}(\mathbf{a})$ is the displacement of the particle that starts at position $\mathbf{a}$ in an event, the angular brackets represent an average over events, and n is the number of events per unit volume per unit time. Expression (5.35) is preferable to (5.37) simply because (5.35) is defined (and therefore meaningful) for any turbulent velocity field, whereas (5.37) is defined only for velocity fields that can be represented in terms of random cyclonic events; although such a representation is physically illuminating, it is unlikely that an arbitrary turbulent velocity field will in general admit such a representation.

In the isotropic ("no preferred direction") situation, (5.35) becomes $\alpha_{ij} = \alpha \, \delta_{ij}$, where

$$\alpha(t) = -\tfrac{1}{3} \int_0^t \langle \mathbf{v}(\mathbf{a}, t) \cdot \nabla_{\mathbf{a}} \wedge \mathbf{v}(\mathbf{a}, \tau) \rangle \, d\tau, \tag{5.38}$$

and again the appearance of a helicity-type correlation (this time in terms of Lagrangian variables) is to be noted. It would be of great interest to be able to reexpress (5.38) in terms of standard Eulerian correlations (and undoubtedly an infinite series of these would be needed)—but the relationship between Lagrangian and Eulerian statistical quantities presents one of the notoriously difficult central unsolved problems of the theory of turbulence.

The tensor $\beta_{ijk}(t)$ has an integral representation similar to (5.35) (Moffatt, 1974). In the isotropic situation this reduces to the form $\beta_{ijk} = \beta(t)\varepsilon_{ijk}$, where

$$\begin{aligned} \beta(t) = D(t) - \tfrac{1}{6} \int_0^t \int_0^t \{ & \langle \mathbf{v}(\mathbf{a}, t) \cdot \nabla_{\mathbf{a}} \mathbf{v}(\mathbf{a}, \tau_1) \cdot \mathbf{v}(\mathbf{a}, \tau_2) \rangle \\ & - \langle \mathbf{v}(\mathbf{a}, t) \cdot \mathbf{v}(\mathbf{a}, \tau_2) \nabla_{\mathbf{a}} \cdot \mathbf{v}(\mathbf{a}, \tau_1) \rangle \} \, d\tau_1 \, d\tau_2 \\ & + \tfrac{1}{6} \alpha(t) \int_0^t \alpha(\tau) \, d\tau, \end{aligned} \tag{5.39}$$

where $D(t) = \frac{1}{3} D_{ii}$, and D_{ij} and $\alpha(t)$ are given by (5.36) and (5.38), respectively. The last term of this expression ensures that if $\alpha(t)$ tends to a constant value as $t \to \infty$, then on this diffusionless theory $\beta(t)$ is unbounded as $t \to \infty$.†

† *Footnote added in proof*: The functions $\alpha(t)$ and $\beta(t)$ given by (5.38) and (5.39) have recently been computed by R. H. Kraichnan (private communication) for gaussian turbulence of maximal helicity; the results indicate that both $\alpha(t)$ and $\beta(t)$ tend to constant nonzero values of order u_0 and $u_0 l$, respectively. This means that the divergence in the final term of (5.39) is compensated by an equal divergence in the double integral term. This interesting result may be peculiar to the assumed gaussian statistics.

Hence it is *essential* to retain the effects of molecular diffusivity in order to determine the appropriate form of β in the weak diffusion limit when the turbulence lacks reflectional symmetry. This calculation (if it could be done) would inevitably lead to an expression of the form $\beta = u_0 l f(R_m)$, where $f(R_m) \to \infty$ as $R_m \to \infty$. (It may be noted that situations such as this are not unprecedented: in the problem of longitudinal diffusion in turbulent pipe flow, for example, the eddy diffusion coefficient is inversely proportional to the molecular diffusivity.)

E. THE FORMS OF α_{ij} AND β_{ijk} IN AXISYMMETRIC TURBULENCE

Suppose now that the turbulence (or random wave field) is not invariant under all rotations, but only under those about the direction defined by the unit vector **e**. This situation may well arise when the fluid is in a state of mean rotation $\mathbf{\Omega}$ and when the turbulence is in consequence influenced by Coriolis forces [a situation recently realized and studied in the laboratory by Ibbetson and Tritton (1975)]; in this case, $\mathbf{e} = \mathbf{\Omega}/\Omega$ and the spectrum tensor $\Phi_{ij}(\mathbf{k}, \omega)$ is axisymmetric about the direction **e**. The most general form that Φ_{ij} can take in such circumstances, consistent with Hermitian symmetry and the incompressibility condition $k_i \Phi_{ij}(\mathbf{k}, \omega) = 0$, is

$$\begin{aligned}\Phi_{ij}(\mathbf{k}, \omega) = {} & \phi_1(k^2\delta_{ij} - k_i k_j) + \phi_2(k e_i e_j + k\mu^2\delta_{ij} - \mu k_i e_j - \mu k_j e_i) \\ & + i\phi_3 \varepsilon_{ijk} k_k + i\phi_4 \varepsilon_{ijk} e_k \\ & + \phi_5(\mathbf{k} \wedge \mathbf{e})_i k_j + \phi_5^*(\mathbf{k} \wedge \mathbf{e})_j k_i \\ & + \phi_6(\mathbf{k} \wedge \mathbf{e})_i e_j + \phi_6^*(\mathbf{k} \wedge \mathbf{e})_j e_i,\end{aligned} \tag{5.40}$$

where

$$i\phi_4 + k^2\phi_5 + \mu k \phi_6 = 0. \tag{5.41}$$

Here, $\phi_1, \ldots, \phi_6$ are functions of $\mathbf{k}$, $\mathbf{k} \cdot \mathbf{e} = k\mu$, and ω, $\phi_1, \ldots, \phi_4$ being real, and ϕ_5 and ϕ_6 complex. The terms involving ϕ_1 and ϕ_2 (which are scalar functions) are reflectionally symmetric, while those involving $\phi_3, \ldots, \phi_6$ (pseudoscalar functions) are not. The energy spectrum function $E(k, \mu, \omega)$ and helicity spectrum function $F(k, \mu, \omega)$ are given by

$$E(k, \mu, \omega) = 2\pi k^2 \Phi_{ii} = 4\pi k^4 \phi_1 + k(1 + \mu^2)\phi_2, \tag{5.42}$$

$$F(k, \mu, \omega) = -4\pi i k^2 k_m \varepsilon_{klm} \Phi_{kl} = 8\pi k^4\{\phi_3 + \tfrac{1}{3}(\mu^2 - 1)\ \mathrm{Im}\ \phi_6\}. \tag{5.43}$$

The situation is evidently much more complex than in the isotropic case!

The pseudotensors α_{ij} and β_{ijk} may still, however, be calculated from formulas (5.27) and (5.28), and symmetry considerations imply that these must now take the form

$$\alpha_{ij} = \alpha_1 \delta_{ij} + \alpha_2 e_i e_j, \tag{5.44}$$

$$\begin{aligned}\beta_{ijk} = \beta_1 \varepsilon_{ijk} + \beta_2 \varepsilon_{jkm} e_m e_i + \beta_3 \varepsilon_{kim} e_m e_j + \beta_4 \varepsilon_{ijm} e_m e_k \\ + \beta_5 \delta_{jk} e_i + \beta_6 \delta_{ki} e_j + \beta_7 \delta_{ij} e_k + \beta_8 e_i e_j e_k,\end{aligned} \tag{5.45}$$

where α_1, α_2, β_5, β_6, β_7, and β_8 (being pseudoscalars) may be expressed as weighted integrals of the functions ϕ_3, ..., ϕ_6 and their derivatives with respect to k and $\mathbf{k} \cdot \mathbf{e}$, while β_1, β_2, β_3, and β_4 may be expressed similarly in terms of ϕ_1 and ϕ_2. The details of the calculation are not of great interest; it is enough to note that in general β_1, β_2, β_3, and β_4 have the same order of magnitude, as do β_5, β_6, β_7, and β_8.

The relation (5.3) between $\mathscr{E}$ and $\mathbf{B}_0$ (considering only the first two terms) then reduces to the form

$$\begin{aligned}\mathscr{E} = \alpha_1 \mathbf{B}_0 + \alpha_2(\mathbf{e} \cdot \mathbf{B}_0)\mathbf{B}_0 - \beta_1 \nabla \wedge \mathbf{B}_0 - \beta_2 \mathbf{e} \cdot (\nabla \wedge \mathbf{B}_0)\mathbf{e} + \beta_3(\mathbf{e} \wedge \nabla)(\mathbf{B}_0 \cdot \mathbf{e}) \\ - \beta_4(\mathbf{e} \cdot \nabla)(\mathbf{e} \wedge \mathbf{B}_0) + \beta_6 \nabla(\mathbf{B}_0 \cdot \mathbf{e}) + \beta_7(\mathbf{e} \cdot \nabla)\mathbf{B}_0 + \beta_8(\mathbf{e} \cdot \nabla)(\mathbf{B}_0 \cdot \mathbf{e})\mathbf{e}.\end{aligned} \tag{5.46}$$

The α effect, represented by the first two terms, is now anisotropic but has not suffered any fundamental modification. A nonisotropic α effect of this form still gives rise to amplification of field modes proportional to exp $i\mathbf{K} \cdot \mathbf{x}$ provided (Moffatt, 1970a)

$$\alpha_1^2 K^2 + \alpha_1 \alpha_2 (K^2 - (\mathbf{K} \cdot \mathbf{e})^2) > \lambda^4 K^4. \tag{5.47}$$

Similarly, the eddy diffusivity effect is now anisotropic: the terms involving β_2, β_3, and β_4 (which occur even in reflectionally symmetric conditions) are all (presumably) of a dissipative nature like the term involving β_1. The term involving β_6 is of little interest (unless β_6 is allowed to vary slowly as it would in an inhomogeneous field of turbulence) since only the curl of $\mathscr{E}$ contributes to (5.1). Similarly the term involving β_7 may be replaced by $-\beta_7 \mathbf{e} \wedge (\nabla \wedge \mathbf{B}_0)$ (the difference being irrotational). This effect was discovered by Rädler (1969a) and was expressed in the form $\gamma_1 \mathbf{\Omega} \wedge \mathbf{J}_0$, where $\gamma_1 = -\beta_7 \mu_0/\Omega$, and is now generally known as the $\mathbf{\Omega} \wedge \mathbf{J}$ effect. The coefficient γ_1 is a pure scalar, and it is tempting to conclude (as did Rädler) that the $\mathbf{\Omega} \wedge \mathbf{J}$ effect can therefore occur when the random motion is reflectionally symmetric; this conclusion does not, however, appear to be justified, since if $\Phi_{ij}(\mathbf{k}, \omega)$ is reflectionally symmetric, then β_7 (and so γ_1) certainly vanishes. This point is of more than academic interest, because it means that if the $\mathbf{\Omega} \wedge \mathbf{J}$ effect is present, then in general an anisotropic α effect will be present also. Incorporation of the $\boldsymbol{\Omega} \wedge \mathbf{J}$ effect in a model and simultaneous exclusion of the α effect (Rädler, 1969b; P. H. Roberts, 1972, Sect. 6) is therefore perhaps somewhat artificial. Nevertheless, it must be said that these models do show that the $\boldsymbol{\Omega} \wedge \mathbf{J}$ effect, in conjunction with differential rotation can provide dynamo action, the reason being that the electromotive force $\gamma_1 \boldsymbol{\Omega} \wedge \mathbf{J}_0$ tends to generate toroidal current from poloidal and vice versa; the effect therefore provides a possible means whereby the dynamo cycle may be closed.

The term involving β_8 was likewise expressed by Rädler in the form

$$\gamma_2(\boldsymbol{\Omega} \cdot \nabla)(\mathbf{B}_0 \cdot \boldsymbol{\Omega})\boldsymbol{\Omega}, \tag{5.48}$$

and he argued that if the rotation $\boldsymbol{\Omega}$, which gives rise to all the nonaxisymmetric contributions to α_{ij} and β_{ijk}, is weak, then the term (5.48) being cubic in $\boldsymbol{\Omega}$ should be negligible. This conclusion must also be questioned; as mentioned above, the coefficients β_7 and β_8 are in general of the same order of magnitude, and it would therefore seem inconsistent to include Rädler's $\boldsymbol{\Omega} \wedge \mathbf{J}$ effect in any model without including also the term of (5.46) involving β_8.

F. Dynamo Equations for Axisymmetric Mean Fields Including Mean Flow Effects

When there is a mean velocity $\mathbf{U}(\mathbf{x})$ in addition to the random fluctuations $\mathbf{u}(\mathbf{x}, t)$, it is no longer realistic to think in terms of homogeneous turbulence; the statistical properties of the turbulence, and in particular the pseudotensors α_{ij}, β_{ijk}, will now themselves be functions of position varying on the same scale as the scale of variation of $\mathbf{U}(\mathbf{x})$. The equation for the mean field, which we shall now simply denote $\mathbf{B}$, becomes

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{U} \wedge \mathbf{B}) + \nabla \wedge \mathscr{E} + \lambda \nabla^2 \mathbf{B}, \tag{5.49}$$

where $\mathscr{E}$ is still given by (5.3), in which for the moment we simply assume that we know the form of $\alpha_{ij}(\mathbf{x})$ and $\beta_{ijk}(\mathbf{x})$.

When $\mathbf{B}$, $\mathbf{U}$, and $\mathscr{E}$ are axisymmetric, it is convenient to separate the toroidal and poloidal ingredients of (5.49). If

$$\mathbf{B} = B\mathbf{i}_\phi + \mathbf{B}_P, \qquad \mathbf{U} = U\mathbf{i}_\phi + \mathbf{U}_P, \tag{5.50}$$

where $U = s\omega(s, z)$ in cylindrical polars (s, ϕ, z), then the toroidal ingredient of (5.49) [cf. (3.8), but note that we have now dropped the subscript ϕ from B_ϕ] is

$$\partial B/\partial t + \mathbf{U}_P \cdot \nabla B = s(\mathbf{B}_P \cdot \nabla)\omega + (\nabla \wedge \mathscr{E})_\phi + \lambda(\nabla^2 - s^{-2})B. \tag{5.51}$$

The poloidal ingredient of (5.49) may be "uncurled," so that if $\mathbf{B}_P = \nabla \wedge (A\mathbf{i}_\phi)$ the equation for A [cf. (4.1), but with $A = \chi/s$] is

$$\partial A/\partial t + \mathbf{U}_P \cdot \nabla A = \mathscr{E}_\phi + \lambda(\nabla^2 - s^{-2})A. \tag{5.52}$$

Note that the ϕ component of electric field $\mathbf{E}$ is given by $E_\phi = -\partial A/\partial t$; there can be no electrostatic contribution under axisymmetric conditions.

Equations (5.51) and (5.52) make it clear that the meridional velocity $\mathbf{U}_P$ merely has a redistributive effect with regard to both fields A and B; in the former case, this leads to expulsion of poloidal flux by the mechanism of Section III,B if λ is small, although now of course this flux may be regenerated by the $\mathscr{E}_\phi$ term. The term $s(\mathbf{B}_P \cdot \nabla)\omega$ represents generation of toroidal field from poloidal field by the differential rotation mechanism (Section III,D). This effect now acts in conjunction with the source term $(\nabla \wedge \mathscr{E})_\phi$, and dynamos are of two types, depending on which of these terms is of dominant importance.

In the simplest situation in which

$$\mathscr{E} = \alpha \mathbf{B} - \beta \nabla \wedge \mathbf{B}, \tag{5.53}$$

the β term leads, as we have seen, to an augmentation of the molecular diffusivity λ, the effective diffusivity being simply $\lambda_e = \beta + \lambda$. The α term gives a contribution $\nabla \wedge (\alpha \mathbf{B}_P)$ in (5.51). The relative magnitude of the two production terms in (5.51) is then given by

$$|s\mathbf{B}_P \cdot \nabla\omega| / |\nabla \wedge (\alpha \mathbf{B}_P)| = O(L^2\omega_0'/\alpha_0), \tag{5.54}$$

where ω_0' and α_0 are typical values of $|\nabla\omega|$ and $|\alpha|$, respectively (such estimates can of course break down locally). If $L^2\omega_0' \ll \alpha_0$, then the differential rotation effect is negligible; the α effect is then solely responsible both for the generation of poloidal field from toroidal *and* vice versa; dynamos that operate in this way are described as α^2 dynamos. The very simple dynamo described at the end of Section V,A is essentially an α^2 dynamo, depending as it does on a two-fold operation of the α effect.

If $L^2\omega_0' \gg \alpha_0$, on the other hand, then differential rotation is responsible for the generation of toroidal field from poloidal, whereas the α effect, through the term αB in (5.52) regenerates poloidal from toroidal field; a dynamo operating in this way is described as an $\alpha\omega$ dynamo ($\alpha\nabla\omega$ would perhaps be more accurate, since it is only the gradient of ω that is relevant here). The equations for the $\alpha\omega$ dynamo in the form

$$\partial B/\partial t + \mathbf{U}_P \cdot \nabla B = s\mathbf{B}_P \cdot \nabla\omega + \lambda(\nabla^2 - s^{-2})B, \tag{5.55}$$

$$\partial A/\partial t + \mathbf{U}_P \cdot \nabla A = \alpha B + \lambda(\nabla^2 - s^{-2})A, \tag{5.56}$$

were obtained originally by Parker (1955a). In this form, the sole effect of the background random motions is the appearance of the term αB in (5.56), and Parker derived this on the basis of his "random cyclonic events" model (see also Parker, 1970b, 1971a,b,c,d,e,f, 1975; Lerche, 1971a,b). Equations having the same structure were derived by systematic perturbation procedures by Braginskii (1964a,b), whose approach will be described in the following section; this latter approach leads, moreover, to the appropriate determination of α in terms of properties of the background fluctuating (nonaxisymmetric) motions.

The approach that we have adopted in this section (following Steenbeck, Krause, and Radler) seems the most general and the most easily comprehended—and ease of comprehension is of crucial importance when attention is turned to the far more difficult dynamic aspects of the problem. It is, of course, reassuring that the various approaches, from rather different standpoints, do converge at the same destination [in the form of Eqs. (5.55) and (5.56)], and this lends confidence to the extensive analytical and numerical studies of these equations that have been carried out, some of which will be described in Section VII.

VI. Braginskii's Theory of Nearly Axisymmetric Fields

A. Lagrangian Transformation of the Induction Equation

An approach to the dynamo problem that is in some respects complementary to that described in Section V was developed by Braginskii (1964a,b) and has since been elucidated and extended by Soward (1972). The approach is based on the idea that, although an axisymmetric field cannot be maintained by axisymmetric motions, weak departures from axisymmetry in both velocity and magnetic fields may be sufficient when λ is small to provide the means of regeneration of the field against ohmic decay. In describing the essence of this theory we follow the approach advocated by Soward.

The starting point is a property of invariance of the induction equation in the frozen field ($\lambda = 0$) limit. For reasons that emerge, it is helpful to modify the notation slightly: let $\tilde{\mathbf{B}}(\tilde{\mathbf{x}}, t)$, $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, t)$ represent field and velocity at $(\tilde{\mathbf{x}}, t)$, and consider a 1–1 continuous mapping $\tilde{\mathbf{x}} \to \mathbf{x}(\tilde{\mathbf{x}}, t)$ satisfying the incompressibility condition that the determinant $\|\partial x_i/\partial \tilde{x}_j\|$ be unity. Then if we define new fields

$$B_k(\mathbf{x}, t) = \tilde{B}_i \,\partial x_k/\partial \tilde{x}_i, \qquad u_k(\mathbf{x}, t) = \partial x_k/\partial t + \tilde{u}_i \,\partial x_k/\partial \tilde{x}_i, \tag{6.1}$$

the invariance property is that the equation

$$\partial \tilde{\mathbf{B}}/\partial t = \tilde{\nabla} \wedge (\tilde{\mathbf{u}} \wedge \tilde{\mathbf{B}}) \tag{6.2}$$

transforms into

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}). \tag{6.3}$$

$\mathbf{B}(\mathbf{x}, t)$ is (physically) the field that would result from $\tilde{\mathbf{B}}(\tilde{\mathbf{x}}, t)$ under the frozen-field distortion $\tilde{\mathbf{x}} \to \mathbf{x}$.

Suppose now that we subject the full induction equation

$$\partial \tilde{\mathbf{B}}/\partial t - \tilde{\nabla} \wedge (\tilde{\mathbf{u}} \wedge \tilde{\mathbf{B}}) = \lambda \tilde{\nabla}^2 \tilde{\mathbf{B}} \tag{6.4}$$

to the same transformation. We obtain, after some elaborate manipulation of the right-hand side,

$$\partial \mathbf{B}/\partial t - \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) = \nabla \wedge \mathscr{E} + \lambda \nabla^2 \mathbf{B}, \tag{6.5}$$

where

$$\mathscr{E}_i = \alpha_{ij} B_j + \varepsilon_{ikp} \beta_{pj}\, \partial B_k/\partial x_j\,, \tag{6.6}$$

and

$$\alpha_{ij} = \lambda \varepsilon_{ikl} \frac{\partial x_k}{\partial \tilde{x}_p} \frac{\partial}{\partial x_j}\left(\frac{\partial x_l}{\partial \tilde{x}_p}\right), \qquad \beta_{pj} = \lambda\left(\frac{\partial x_p}{\partial \tilde{x}_r}\frac{\partial x_j}{\partial \tilde{x}_r} - \delta_{pj}\right). \tag{6.7}$$

Note that α_{ij} (a pseudotensor) and β_{pj} (a tensor) are both quadratic in the displacement $\mathbf{x} - \tilde{\mathbf{x}}$. The similarity between (6.5), (6.6) and (5.1), (5.3) is immediately striking; note, however, that in the present context, the term $\nabla \wedge \mathscr{E}$ is wholly diffusive in origin. Paradoxically, although diffusion is responsible for the natural tendency of a field to decay, it can also be of crucial importance in creating the electromotive force that can counteract this decay.

B. Nearly Axisymmetric Systems

Throughout this section, we shall use angular brackets $\langle\ \rangle$ to denote an average over the azimuth angle ϕ in cylindrical polar coordinates (s, ϕ, z). Thus for any scalar $\psi(s, \phi, z)$,

$$\langle \psi \rangle = (1/2\pi) \int_0^{2\pi} \psi(s, \phi, z)\, d\phi, \tag{6.8}$$

and for any vector $\mathbf{f} = f_s \mathbf{i}_s + f_\phi \mathbf{i}_\phi + f_z \mathbf{i}_z$, we define

$$\langle \mathbf{f} \rangle = \langle f_s \rangle \mathbf{i}_s + \langle f_\phi \rangle \mathbf{i}_\phi + \langle f_z \rangle \mathbf{i}_z\,. \tag{6.9}$$

With this convention, we may talk of the mean toroidal field $\langle f_\phi \rangle \mathbf{i}_\phi$ and the mean poloidal field $\langle f_s \rangle \mathbf{i}_s + \langle f_z \rangle \mathbf{i}_z$, which are by definition axisymmetric.

If the velocity field $\tilde{\mathbf{u}}(\mathbf{x}, t)$ is nearly axisymmetric, then we may express it in the form

$$\tilde{\mathbf{u}} = \langle \tilde{\mathbf{u}} \rangle + \varepsilon \mathbf{u}', \qquad \varepsilon \ll 1. \tag{6.10}$$

A magnetic field $\tilde{\mathbf{B}}(\mathbf{x}, t)$ convected and distorted by such a velocity field must exhibit at least a similar degree of asymmetry, and it is consistent to suppose that

$$\tilde{\mathbf{B}} = \langle \tilde{\mathbf{B}} \rangle + \varepsilon \mathbf{b}'. \tag{6.11}$$

When $\varepsilon = 0$, the field $\langle \tilde{\mathbf{B}} \rangle$ cannot survive, by Cowling's theorem. Braginskii's model is based on the expectation that the mean electromotive force $\varepsilon^2 \langle \mathbf{u}' \wedge \mathbf{b}' \rangle$ may compensate the erosive effects of the term $\lambda \nabla^2 \langle \tilde{\mathbf{B}} \rangle$ in the averaged induction equation. For this to be possible it is necessary that λ be not greater than $O(\varepsilon^2)$; we therefore put $\lambda = \lambda_0 \varepsilon^2$ and keep λ_0 fixed as $\varepsilon \to 0$. Expansion in powers of ε may then be expressed equivalently as expansion in powers of $R_m^{-1/2}$ where $R_m = U_0 L/\lambda$, where U_0 and L are scales characteristic of the mean velocity field $\langle \tilde{\mathbf{u}} \rangle$. This was the procedure adopted by Braginskii.

The mean velocity may be expressed as the sum of toroidal and poloidal parts:

$$\langle \tilde{\mathbf{u}} \rangle = U(s, z)\mathbf{i}_\phi + \mathbf{U}_P(s, z). \tag{6.12}$$

The toroidal part generates toroidal field from poloidal field by differential rotation (Section III,D) and this is certainly conducive to dynamo action (although, of course, not sufficient in itself). The poloidal part $\mathbf{U}_P(s, z)$ tends to redistribute the mean toroidal field; it also tends to wind up the mean poloidal field and to expel it (Section III,B) from regions of closed streamlines of $\mathbf{U}_P$ (i.e., from the whole fluid region for an enclosed flow) if a magnetic Reynolds number based on a typical value of $|\mathbf{U}_P|$ is large. This latter process is certainly *not* conducive to dynamo action, and the only way to control it is to suppose that $|\mathbf{U}_P|/U_0$ is at most $O(R_m^{-1})$, so that

$$U_0 L/\lambda = R_m, \qquad |\mathbf{U}_P| L/\lambda = O(1). \tag{6.13}$$

The dominant ingredient of the mean velocity field is then the toroidal ingredient $U\mathbf{i}_\phi$ whose typical order of magnitude is U_0. To emphasize this scaling, we rewrite (6.12) in the form

$$\langle \tilde{\mathbf{u}} \rangle = U(s, z)\mathbf{i}_\phi + \varepsilon^2 \mathbf{u}_P(s, z). \tag{6.14}$$

Note that, in proceeding in this way, attention is automatically restricted to velocity fields that have some a priori chance of success as potential dynamos.

The mean field $\langle\tilde{\mathbf{B}}\rangle$ may likewise be expressed as the sum of toroidal and poloidal parts:

$$\langle\tilde{\mathbf{B}}\rangle = B(s, z)\mathbf{i}_\phi + \mathbf{B}_P(s, z). \tag{6.15}$$

The dominant differential rotation $U\mathbf{i}_\phi$ generates $B\mathbf{i}_\phi$ from $\mathbf{B}_P$ and it may be anticipated (Section III,D) that $B = O(R_m)|\mathbf{B}_P|$, or equivalently $|\mathbf{B}_P| = O(\varepsilon^2)B$. Hence we may also rewrite (6.15) in the form

$$\langle\tilde{\mathbf{B}}\rangle = B(s, z)\mathbf{i}_\phi + \varepsilon^2\mathbf{b}_P(s, z). \tag{6.16}$$

It is, of course, implicit in the notation that B, $|\mathbf{b}'|$, and $|\mathbf{b}_P|$ are all of the same order of magnitude as $\varepsilon \to 0$; similarly for U, $|\mathbf{u}'|$, and $|\mathbf{u}_P|$.

C. Nearly Rectilinear Flows; Effective Fields

It is mathematically simpler to focus attention on the Cartesian analog of the situation described in Section VI,B, in which (s, ϕ, z) are replaced by (x, y, z), and $\langle\ \rangle$ in consequence now indicates an average with respect to the y variable. The additional difficulties of dealing with the cylindrical coordinate system are purely geometrical and need not concern us here.

The fact that the fluctuating fields of (6.10) and (6.11) are $O(\varepsilon)$ suggests that it may be possible to "accommodate" them through the use of a transformation function

$$\tilde{\mathbf{x}} = \mathbf{x} + \varepsilon\boldsymbol{\eta}(\mathbf{x}, t), \qquad \langle\boldsymbol{\eta}\rangle = 0, \quad \nabla\cdot\boldsymbol{\eta} = 0. \tag{6.17}$$

If this is possible, then the related fields $\mathbf{u}$ and $\mathbf{B}$ will have the simple form (the time dependence being for the moment understood)

$$\mathbf{u} = U(x, z)\mathbf{i}_y + \varepsilon^2\mathbf{u}_{eP}(x, z) + O(\varepsilon^3), \tag{6.18}$$

$$\mathbf{B} = B(x, z)\mathbf{i}_y + \varepsilon^2\mathbf{b}_{eP}(x, z) + O(\varepsilon^3), \tag{6.19}$$

where the effective fields $\mathbf{u}_{eP}$ and $\mathbf{b}_{eP}$ are related in some way (to be determined) to the fields $\mathbf{u}_P$ and $\mathbf{b}_P$.

Let us first obtain the relationship between the fields $\mathbf{b}_{eP}$ and $\mathbf{b}_P$. First, by expanding $\tilde{B}_i(\mathbf{x})$ in Taylor series about the point $\tilde{\mathbf{x}}$ and using $\tilde{B}_i(\tilde{\mathbf{x}}) = (\delta_{ik} + \varepsilon\,\partial\eta_i/\partial x_k)B_k(\mathbf{x})$, we obtain

$$\begin{aligned}\tilde{B}_i(\mathbf{x}) = {} & B_i(\mathbf{x}) + \varepsilon[\nabla\wedge(\boldsymbol{\eta}\wedge\mathbf{B}(\mathbf{x}))]_i \\ & + \varepsilon^2\left[\delta_{ik}\eta_j\frac{\partial\eta_m}{\partial x_j}\frac{\partial}{\partial x_m} - \eta_m\frac{\partial}{\partial x_m}\frac{\partial\eta_i}{\partial x_k} + \frac{1}{2}\delta_{ik}\eta_j\eta_i\frac{\partial^2}{\partial x_j\,\partial x_l}\right]B_k(\mathbf{x}) \\ & + O(\varepsilon^3).\end{aligned} \tag{6.20}$$

Comparison with (6.11) shows that

$$\mathbf{b}' = \nabla \wedge (\boldsymbol{\eta} \wedge \mathbf{B}) = B\, \partial\boldsymbol{\eta}/\partial y - (\boldsymbol{\eta} \cdot \nabla)B\mathbf{i}_y \tag{6.21}$$

(to leading order), and the mean poloidal ingredient of (6.20) gives

$$\mathbf{b}_P = \mathbf{b}_{eP} + \tfrac{1}{2}\nabla \wedge \langle \boldsymbol{\eta}_M \wedge \partial\boldsymbol{\eta}_M/\partial y \rangle B, \tag{6.22}$$

where $\boldsymbol{\eta}_M = \boldsymbol{\eta} - (\boldsymbol{\eta} \cdot \mathbf{i}_y)\mathbf{i}_y$ is the meridional projection of $\boldsymbol{\eta}$. In terms of the vector potentials a, a_e defined by

$$\mathbf{b}_P = \nabla \wedge (a\mathbf{i}_y), \qquad \mathbf{b}_{eP} = \nabla \wedge (a_e\, \mathbf{i}_y), \tag{6.23}$$

this result takes the simpler form

$$a = a_e - wB, \qquad w = -\tfrac{1}{2}\langle \boldsymbol{\eta}_M \wedge \partial\boldsymbol{\eta}_M/\partial y \rangle_y . \tag{6.24}$$

The relationship between $\mathbf{u}_{eP}$ and $\mathbf{u}_P$ is similarly obtained with the sole difference that it is $u_k - \partial x_k/\partial t$ (rather than u_k) that appears in the transformation relationship (6.1). This leads to the expression [analogous to (6.21)]

$$\mathbf{u}' = (\partial/\partial t + U\, \partial/\partial y)\boldsymbol{\eta} - (\boldsymbol{\eta} \cdot \nabla)U\mathbf{i}_y , \tag{6.25}$$

and the result [analogous to 6.22)]

$$\mathbf{u}_P = \mathbf{u}_{eP} + \tfrac{1}{2}\nabla \wedge \langle \boldsymbol{\eta}_M \wedge (\partial/\partial t + U\, \partial/\partial y)\boldsymbol{\eta}_M \rangle . \tag{6.26}$$

D. Dynamo Equations for Nearly Rectilinear Flows

Equations for the evolution of $B(x, z, t)$ and $\mathbf{b}_{eP}(x, z, t)$ may now be obtained by averaging (6.5) with respect to y. The y component (i.e., the "toroidal" ingredient) of the resulting equation is

$$\partial B/\partial t + \varepsilon^2\mathbf{u}_{eP} \cdot \nabla B = \varepsilon^2\mathbf{b}_{eP} \cdot \nabla U + (\nabla \wedge \mathscr{E}_0)_y + \lambda\nabla^2 B + O(\varepsilon^4), \tag{6.27}$$

where

$$\mathscr{E}_{0i} = \lambda\langle \alpha_{ij} \rangle B_j + \lambda\varepsilon_{ikp}\langle \beta_{pj} \rangle\, \partial B_k/\partial x_j . \tag{6.28}$$

Since $\lambda = O(\varepsilon^2)$ and α_{ij} and β_{pj} are both $O(\varepsilon^2)$, $\mathscr{E}_0$ is evidently $O(\varepsilon^4)$, so that to leading order (6.27) becomes

$$\partial B/\partial t + \varepsilon^2\mathbf{u}_{eP} \cdot \nabla B = \varepsilon^2\mathbf{b}_{eP} \cdot \nabla U + \lambda_0\varepsilon^2\nabla^2 B. \tag{6.29}$$

In this equation the effect of departures from exact rectilinearity are wholly absorbed through the introduction of the effective fields $\mathbf{u}_{eP}$ and $\mathbf{b}_{eP}$. Note that the time scale of evolution of the B field is $O(\varepsilon^{-2}L/U_0)$.

The x and z components of the averaged equation (6.5) may be "uncurled" to give

$$\varepsilon^2 \partial a_e/\partial t + \varepsilon^4 \mathbf{u}_{eP} \cdot \nabla a_e = \mathscr{E}_{0y} + \varepsilon^2 \lambda \nabla^2 a_e + O(\varepsilon^5), \tag{6.30}$$

and here it is clear that the term $\mathscr{E}_{0y}$ is of the same order as the other terms in the equation. Moreover, the dominant contribution to $\mathscr{E}_{0y}$ [from (6.28)] is evidently given by

$$\mathscr{E}_{0y} = \lambda \langle \alpha_{22} \rangle B + O(\varepsilon^6), \tag{6.31}$$

where, from (6.7) and (6.17),

$$\langle \alpha_{22} \rangle = \varepsilon^2 \varepsilon_{2kl} \langle \eta_{k,m} \eta_{l,m2} \rangle = 2\varepsilon^2 \langle \eta_{3,m} \eta_{1,m2} \rangle, \tag{6.32}$$

where a subscript after a comma indicates differentiation, and (6.30) now becomes

$$\partial a_e/\partial t + \varepsilon^2 \mathbf{u}_{eP} \cdot \nabla a_e = \lambda \Gamma B + \lambda \nabla^2 a_e, \tag{6.33}$$

where $\Gamma = \langle \alpha_{22} \rangle / \varepsilon^2$.

Equations (6.29) and (6.33) (with the correspondences $\varepsilon^2 \mathbf{u}_{eP} \to \mathbf{U}_P$, $\varepsilon^2 a_e \to A$, $\varepsilon^2 \mathbf{b}_{eP} \to \mathbf{B}_P$, $\lambda \langle \alpha_{22} \rangle \to \alpha$) are the Cartesian analogs of the $\alpha\omega$ dynamo Eqs. (5.55) and (5.56). The effective value of α, given by

$$\alpha = 2\lambda\varepsilon^2 \langle \eta_{3,m} \eta_{1,m2} \rangle, \tag{6.34}$$

where $\boldsymbol{\eta}$ is related to the velocity field by (6.25), is noteworthy. The frozen-field displacement merely gives a configuration of $\mathbf{u}'$ and $\mathbf{b}'$ that is *potentially* able to provide an α effect; it is the diffusive slip of $\mathbf{b}'$ relative to $\mathbf{u}'$ that realizes this potential and provides an α proportional to λ (cf. Section V,C).

The structure of (6.34) is elucidated by considering a particular displacement of the form

$$\boldsymbol{\eta}(\mathbf{x}, t) = \eta_0(\cos ky, 0, \sin ky), \tag{6.35}$$

for which $\nabla \wedge \boldsymbol{\eta} = k\boldsymbol{\eta}$ (so that the displacement has positive helicity) and for which (6.34) reduces to

$$\alpha = -\lambda\varepsilon^2 k^3 \eta_0^2. \tag{6.36}$$

So again, as in the random wave approach of Section V,C, positive helicity is associated with a negative value for α. More generally, as pointed out by Braginskii (1964a), a displacement of the form

$$\boldsymbol{\eta} = \boldsymbol{\eta}^{(c)}(x, z) \cos ky + \boldsymbol{\eta}^{(s)}(x, z) \sin ky \tag{6.37}$$

will provide a nonzero value for α [given by (6.34) only if the vectors $\boldsymbol{\eta}^{(c)}$ and

$\boldsymbol{\eta}^{(s)}$ are linearly independent]. Similarly the parameter w given by (6.24) has the form

$$w = -k(\boldsymbol{\eta}^{(c)} \wedge \boldsymbol{\eta}^{(s)})_y \tag{6.38}$$

when $\boldsymbol{\eta}$ is given by (6.37), and this also is nonzero only when $\boldsymbol{\eta}^{(c)}$ and $\boldsymbol{\eta}^{(s)}$ are linearly independent vectors.

If terms of order higher than ε^2 are retained in (6.29) and (6.33), then new effects are to be expected. These effects turn out, however, to be very much what might be anticipated from the background of mean field electrodynamics as expounded in Section V. To order ε^3, the form of Eqs. (6.29) and (6.33) is in fact unchanged (Tough, 1967; Tough and Gibson, 1969), but the expressions for w and Γ are modified; in particular, Γ takes the form $\Gamma_0 + \varepsilon\Gamma_1$, where now Γ_0 is the zeroth-order expression $(\lambda/\varepsilon^2)\alpha$ with α given by (6.34), and Γ_1 involves the mean of an expression cubic in $\boldsymbol{\eta}$—compare the form of expansion (5.24), which may also be regarded as an expansion in powers of the amplitude of the fluctuating velocity. At order ε^4 (Soward, 1972) effects analogous to the nonisotropic α and β effects of Section V,E also appear.

E. Comments on the General Approach of Soward†

Soward's approach undoubtedly provides the most natural framework for the treatment of nearly axisymmetric systems in the limit $\lambda \to 0$. However, Soward claims that his approach also provides the natural framework even when departures from axisymmetry are not small [i.e., when the transformation function $\tilde{\mathbf{x}}(\mathbf{x}, t)$ is not restricted by an assumption of the form (6.17)]. This claim, although plausible, requires some qualification. The reason is that the approach will work, as Soward acknowledges, only for velocity fields $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, t)$ for which a transformation $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(\mathbf{x}, t)$ exists such that the related field $\mathbf{u}(\mathbf{x}, t)$ given by (6.1a) is predominantly azimuthal, in the sense that

$$\mathbf{u}(\mathbf{x}, t) = U(s, z, t)\mathbf{i}_\phi + O(R_m^{-1}). \tag{6.39}$$

Now it can easily be shown that the streamlines of the field $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, t)$ map into streamlines of $\mathbf{u}(\mathbf{x}, t)$, which according to (6.39) are nearly the circles C_0: $s = s_0$, $z = z_0$; more precisely the topological measure analogous to (2.18) is

$$K = \int_V \mathbf{g} \cdot \mathbf{u}\, d^3\mathbf{x}, \tag{6.40}$$

† Soward (1972).

where $\mathbf{u} = \nabla \wedge \mathbf{g}$ and $\mathbf{u} \cdot \mathbf{n} = 0$ on the surface of V, and for a field of the form (6.39), $K = O(R_m^{-1})$. This means that unless the value of K for the field $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, t)$ is $O(R_m^{-1})$ (for every volume $\tilde{V}$ with surface $\tilde{S}$ on which $\mathbf{n} \cdot \tilde{\mathbf{u}} = 0$), the transformation function $\tilde{\mathbf{x}}(\mathbf{x}, t)$ simply does not exist. In simpler terms, if the streamlines of $\tilde{\mathbf{u}}(\tilde{\mathbf{x}})$ are linked, they cannot be unlinked by a continuous mapping $\tilde{\mathbf{x}} \to \mathbf{x}$. In general, a field having both toroidal and poloidal ingredients will have streamlines that are both knotted and linked (Moffatt, 1969) and so the restriction is in fact a severe one.

The criticism is not so serious when *weak* departures from axisymmetry are considered. In this situation Soward (1971) has proved that, given a velocity field of the form

$$\tilde{\mathbf{u}} = U(s, z)\mathbf{i}_\phi + \varepsilon\mathbf{u}'(s, \phi, z), \tag{6.41}$$

there exists an axisymmetric $\mathbf{V}(s, z)$ such that $\tilde{\mathbf{u}} + \varepsilon^2\mathbf{V}$ has streamlines that are closed at the $O(\varepsilon)$ level [i.e., if a streamline is followed as ϕ increases by 2π, the separation of the endpoints is $o(\varepsilon)$]. It seems likely that this same choice of $\mathbf{V}$ [which is identifiable with $\mathbf{u}_{eP} - \mathbf{u}_P$ in (6.26)] also guarantees that these same closed streamlines must be unlinked, although a convincing proof of this has yet to be given. A mapping $\tilde{\mathbf{x}} \to \mathbf{x}$ that maps the streamlines of $\tilde{\mathbf{u}} + \varepsilon^2\mathbf{V}$ onto the circles C_0 then exists, and the method can proceed with the necessary adjustments at order ε^2 in the meridional velocity represented by $\mathbf{u}_{eP} = \mathbf{u}_P + \mathbf{V}$.

Unfortunately there is no analog of Soward's closed streamline theorem when ε is not small; hence the limitation in the class of velocity fields to which the method may be applied when large departures from axisymmetry are considered.

F. Comparison between the Two-Scale and Nearly Axisymmetric Approaches

The many points of contact between Braginskii's theory and the theory of Section V are too numerous to be merely fortuitous and lead to the view that the two approaches are in fact different aspects of one and the same theory. What is essential in both is that it should be possible to define an average $\langle\ \rangle$. The existence of two scales, l and L, in Section V, lead to the conclusion that $\mathcal{E} = \langle \mathbf{u} \wedge \mathbf{b} \rangle$ depends only on the *local* values of $\mathbf{B}_0$ and its space derivatives and of the statistical properties of the velocity field. In Braginskii's theory, the two scales are also present in disguised form, the small scale being simply the distance that a fluid particle migrates from its mean position during its passage around the axis of symmetry. The same considerations as in mean field electrodynamics lead inexorably to the conclusion that

$\mathscr{E}$ must depend only on *local* values of $B(s, z)$, $U(s, z)$ (and possibly their derivatives), and the mean properties of the fluctuating motion. The crucial conclusion of both theories is that a contribution to $\mathscr{E}$ of the form $\alpha\langle\mathbf{B}\rangle$ can arise; and since α is a pseudoscalar in either theory, this effect necessarily requires a lack of reflectional symmetry in the motions that give rise to it; i.e., an observer would be able to distinguish (locally) either a right- or a left-handed character in the motions if the resulting value of α is nonzero.

The distinction between $\mathbf{u}_{eP}$ and $\mathbf{u}_P$ (and between $\mathbf{b}_{eP}$ and $\mathbf{b}_P$) must be regarded as merely of academic interest until a complete dynamic theory accounting for the structure both of the $\mathbf{u}_P$ field and of second-order mean quantities [like (6.34)] is available; in any case, if $\boldsymbol{\eta}$ vanishes at the boundary S of the region V of conducting fluid, then $\mathbf{b}_P = \mathbf{b}_{eP}$ on S, and so the matching conditions with the external field are unaffected by the distinction between $\mathbf{b}_P$ and $\mathbf{b}_{eP}$.

One is therefore bound to ask what advantage Braginskii's theory has over what appears to be the simpler and more illuminating approach of Section V. The answer is simply that it provides the means whereby the pseudotensors α_{ij} and β_{ijk} may be derived in the important situation when the mean flow is predominantly toroidal and axisymmetric, the fluctuating velocities are weak, and $U_0 L/\lambda$ is large. The theory may therefore be recognized as falling within the general framework of the mean field electrodynamics of Section V, but at the same time as occupying a particularly difficult and thorny corner of that subject. It is nevertheless necessary to grasp the thorns, since the most plausible models for both Earth and Sun do invoke strong toroidal velocity fields and one can hardly simply ignore the influence that this may have on α_{ij} and β_{ijk}, particularly when dynamical effects are subsequently considered.

Perhaps a more obvious attractive feature of Braginskii's approach is the very degree to which it places emphasis on the nonaxisymmetric ingredient $\varepsilon\mathbf{b}'$ of magnetic field. The Earth's field is of course nonaxisymmetric relative to the axis of rotation, the axis of the magnetic dipole moment being inclined at a slowly varying angle ψ (at present of order 10°) from the rotation axis. A perturbation field having a dependence exp $i(\phi - \omega t)$ on azimuth ϕ about the rotation axis (Braginskii, 1964c) gives a field with external potential of the form

$$Ar^{-2}P_1^1(\cos\theta)\cos(\phi - \omega t), \tag{6.42}$$

i.e., a dipole whose moment rotates in the equatorial plane with angular velocity ω. In conjunction with the axial dipole, there is here the basis of one (though certainly not the only) possible explanation for the tilt ψ, and the manner in which the direction of the dipole moment vector drifts relative to the rotation vector.

VII. Analytical and Numerical Solutions of the Dynamo Equations

A. The α^2 Dynamo with α Constant

We have already discussed in Section V,A the simplest exponentially growing solutions of the equation

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{U} \wedge \mathbf{B}) + \nabla \wedge (\alpha \mathbf{B}) + \lambda_e \nabla^2 \mathbf{B}, \tag{7.1}$$

in the case when $\mathbf{U} = 0$, $\alpha = \text{const}$, and the fluid is of infinite extent, namely the force-free mode for which $\nabla \wedge \mathbf{B} = K\mathbf{B}$, for which the growth rate is given by (5.9). Now that we have estimates of α and λ_e $(= \lambda + \beta)$, one further comment on this solution is called for. The growth rate ω is clearly maximal when $K = K_m = \alpha/2\lambda_e$. When $R_m = u_0 l/\lambda \ll 1$ (in the notation of Section V), $\alpha \sim lu_0^2/\lambda$ and $\lambda_e \sim \lambda$, so that

$$lK_m \sim R_m^2 \ll 1, \tag{7.2}$$

and so the initial assumption that the scale L of the mean field $(\sim K_m^{-1})$ be large compared with the scale l of the random $\mathbf{u}$ field is amply satisfied.

At the other extreme, when $R_m \gg 1$, we have seen that there is some uncertainty in estimates of α and β owing to uncertainty regarding convergence of the integral (5.39), and the unbounded behavior of β as given by (5.40). If we assume that α is independent of λ in the limit $\lambda \to 0$, then on dimensional grounds, $\alpha \sim u_0$ in this limit, and so $lK_m \sim lu_0/\lambda_e$. If λ_e is also independent of λ in the limit $\lambda \to 0$ (as maintained, e.g., by Parker, 1971b) then again on dimensional grounds $\lambda_e \sim u_0 l$. This would lead to the conclusion that $lK_m = O(1)$, i.e., that the medium is unstable to magnetic disturbances on scales right down to the scale l of the turbulence itself. Of course, if $Kl = O(1)$, then the two-scale approach ceases to be valid; nevertheless the increase in growth rate ω with decreasing scale K^{-1} calls for considerable delicacy in the interpretation of results in this limit.

This particular difficulty is removed if, as argued at the end of Section V,D, β is in fact large compared with lu_0 when $R_m \gg 1$. The inadequacy of the traditional estimate $\beta \sim lu_0$ in the solar context has been emphasized by Piddington (1972a); however, Piddington's main concern is with the process by which small-scale field fluctuations merge and annihilate each other, and the relevant eddy diffusivity for *this* process is not necessarily the same as the eddy diffusivity that operates on the global mean field. In this context, see also Parker (1972, 1973).

The α^2 dynamo in a spherical geometry was treated by Krause and Steenbeck (1967). This requires solution of (7.1) (again with $\mathbf{U} = 0$, $\alpha = \text{const}$) in a sphere of radius R, and with matching to a current-free field for $r > R$. The solutions in this case cannot be force-free, since as mentioned in Section II,D such fields do not exist in a finite geometry in the absence of external sources; the solutions nevertheless undoubtedly have a helical structure. Steady solutions are possible only if the parameter $\alpha R/\lambda_e$ takes one of a discrete set of eigenvalues; the lowest of these is 4.49 and the corresponding solution (in $r < R$) for A and B has the form

$$\left.\begin{aligned} A &= C\{k^{-1}r^{-1/2}J_{3/2}(kr) - \tfrac{1}{3}rJ_{1/2}(kr)\}\sin\theta,\\ B &= Cr^{-1/2}J_{3/2}(kr)\sin\theta, \end{aligned}\right\} \qquad (7.3)$$

which matches to a dipole field for $r > R$. If $\alpha R/\lambda_e \gg 1$ (as would, for example, be the case if $\alpha \sim u_0$, $\lambda_e \sim u_0 l$, and $l \ll R$) then a wide range of modes become unstable, and the *most* unstable is not that having the simplest structure (7.3) but rather a high harmonic having a much smaller length scale (as suggested by the results for an infinite geometry).

B. α^2 Dynamos with Antisymmetric α

The assumption of constant α is unrealistic in a rotating body such as the Earth or the Sun in which Coriolis forces are responsible for the generation of helicity. We referred in the introductory section to Parker's (1955a) concept of rising, twisting blobs of fluid in the convection zone of the Sun surrounded by connected regions of falling fluid. The rising blob tends to entrain fluid from the sides and conservation of angular momentum then leads to spin-up or positive helicity in the northern hemisphere, and similarly to negative helicity in the southern hemisphere. Correspondingly, α would in this picture be negative in the northern hemisphere and positive in the southern. [It is relevant to note that Steenbeck *et al.* (1966) came to precisely the opposite conclusion in arguing that a blob that rises through several scale heights will expand and therefore "spin down" in the northern hemisphere, generating negative helicity there. Viscous entrainment and compressibility effects are in competition here and the true sign of α—which is of crucial importance for $\alpha\omega$ dynamos discussed below—is presumably determined by which of these effect dominates.] The simplest reasonable assumption for $\alpha(\mathbf{x})$ on this type of physical picture is

$$\alpha(\mathbf{x}) = \alpha_0 f(r/R)\cos\theta. \qquad (7.4)$$

Unfortunately, even with this simplest choice, the eigenvalue problem

$$\nabla \wedge (\alpha\mathbf{B}) + \lambda_e \nabla^2 \mathbf{B} = 0 \qquad (7.5)$$

(with the matching conditions to a current-free field on $r = R$) is not amenable to simple analysis, and the problem must be solved numerically (Steenbeck and Krause, 1966, 1969b; P. H. Roberts, 1972), using series expansions for $\mathbf{B}$ and truncating after a few terms. Roberts found that six terms were in fact sufficient to give 0.1 % accuracy in the determination of eigenvalues. The most striking feature of the numerical results is that the eigenvalues for fields of dipole and quadrupole symmetry are almost indistinguishable, i.e., these fields can be excited with almost equal ease! The reason for this was given by Steenbeck and Krause (1969b): the α effect given by (7.4) operates most effectively near the poles, where $\cos\theta = \pm 1$, and the toroidal current distribution (which gives rise to the poloidal field) is correspondingly concentrated in rings near the poles. The mutual inductance between these current rings is small, and so one of the current rings can be reversed without greatly affecting conditions near the other. This operation transforms a field of dipole symmetry into one of quadrupole symmetry or vice versa.

C. Local Behavior in $\alpha\omega$ Dynamos

The Cartesian analog of (5.55) and (5.56) [cf. (6.29) and (6.33)] is

$$\partial A/\partial t + \mathbf{u}_P \cdot \nabla A = \alpha B + \lambda \nabla^2 A, \tag{7.6}$$

$$\partial B/\partial t + \mathbf{u}_P \cdot \nabla B = \mathbf{B}_P \cdot \nabla U + \lambda \nabla^2 B, \tag{7.7}$$

where the variables are functions only of x and z. It is illuminating (Parker, 1955a) to consider the local modes of behavior of solutions of these equations in regions where $\mathbf{u}_P$, α, and ∇U may all be regarded as constant. In interpreting these solutions in the solar context, the axis $0x$ is south, $0y$ is east, and $0z$ is vertically upward, relative to some origin 0 in the convective zone in, say, the northern hemisphere. The equations admit solutions proportional to $\exp(\omega t + i\mathbf{K}\cdot\mathbf{x})$ (where $\mathbf{K}$ is a real two-dimensional wave vector in the x-z plane), the dispersion relation being

$$(\omega + \lambda K^2 + i\mathbf{u}_P \cdot \mathbf{K})^2 = -i\alpha(\mathbf{K} \wedge \nabla U)_y = 2i\gamma, \tag{7.8}$$

say. If $\gamma < 0$, the roots of (7.8) are given by

$$\omega = -\lambda K^2 - i\mathbf{u}_P \cdot \mathbf{K} \pm |\gamma|^{1/2}(1 - i), \tag{7.9}$$

and there is an exponentially growing solution of $\mathbf{K}$ satisfies

$$\alpha |\mathbf{K} \wedge \nabla U|_y > \lambda^2 K^4. \tag{7.10}$$

This growing mode has the phase factor $\exp i(\mathbf{K}\cdot\mathbf{x} - \mathbf{u}_P \cdot \mathbf{K}t - |\gamma|^{1/2}t)$. If $\mathbf{u}_P = 0$ and $U = U(z)$ only, then this result implies that if $\alpha\, \partial U/\partial z < 0$ then a growing mode with space dependence $\exp(iKx)$ can propagate in the positive x direction, i.e., toward the equatorial plane. (Alternatively, we could regard ω as real and K as complex, in which case the growth of the mode is spatial rather than temporal.) Conversely, if $\alpha\ \partial U/\partial z > 0$, then the growing mode propagates away from the equatorial plane. The sign of the product of α and the vertical shear is therefore of crucial importance in determining the character of these local solutions, and this carries over also to properties of global solutions of (7.6) and (7.7) in a spherical geometry (Roberts, 1972).

Note that when $\mathbf{u}_\mathrm{P} = 0$, the above solutions are necessarily oscillatory; the phase velocity can, however, be reduced to zero if $\mathbf{u}_\mathrm{P} \cdot \mathbf{K} = -|\gamma|^{1/2}$, and we then have an exponentially growing field whose spatial structure remains constant. This property again carries over to global $\alpha\omega$ dynamos (Braginskii, 1964b; P. H. Roberts, 1972). Suitable choice of meridional circulation can convert a situation in which an oscillatory dynamo (as described in Section II,A) is preferred into one in which a steady dynamo is preferred, i.e., excited at a lower value of $|D|$ [see (7.12)]. If this meridional circulation is too strong, however, the dynamo fails altogether, presumably because of the consequent expulsion of poloidal flux from the region where the differential rotation would otherwise be generating toroidal field from it. Since differential rotation and meridional circulation are dynamically coupled, the question of whether a given system will exhibit steady or oscillatory dynamo behavior inevitably demands consideration of the governing dynamical equations.

D. Global Behavior of $\alpha\omega$ Dynamos

When α is given by (7.4) and ω is assumed to vary only radially, say

$$\omega'(r) = \omega'_0 g(r/R), \tag{7.11}$$

the crucial dimensionless parameter is what Parker called the dynamo number

$$D = \alpha_0 \omega'_0 R^4/\lambda^2. \tag{7.12}$$

Generally, when f and g are reasonably smooth overlapping functions and $\mathbf{u}_\mathrm{P} = 0$, (5.55) and (5.56) yield (numerically) only oscillatory solutions (P. H. Roberts, 1972), a solution of dipole symmetry being more easily excited than one of quadrupole symmetry when $D < 0$. In such oscillatory solutions, there is a tendency during a period of the oscillation for toroidal field to propagate from polar regions toward the equatorial plane (consistent with the discussion of Section VII,C); it is this type of behavior that lends credence to the picture relating sunspot formation to the behavior of the underlying toroidal field, as outlined in the introduction. Indeed, many authors (e.g., Steenbeck and Krause, 1969a; Leighton, 1969; Roberts and Stix, 1972) have succeeded, on the basis of such solutions, in constructing butterfly diagrams, which bear at least a plausible resemblance to those based on observation (Maunder, 1922; see, e.g., Kiepenheuer, 1953). Such diagrams depict the appearance of sunspots as a function of latitude and time and are extracted from solutions of the dynamo equations by plotting in the θ-t plane the "isotors" $B(r, \theta, t) = \text{const}$, for some fixed value of r, on the view (Parker, 1955b) that sunspots form by eruption when the toroidal field below the surface exceeds some critical value.

When the functions f and g are nonoverlapping, the field most easily excited can be steady rather than oscillating. The case

$$f(\rho) = \delta(\rho - \rho_1), \qquad g(\rho) = \delta(\rho - \rho_2), \qquad \rho_1 \neq \rho_2, \tag{7.13}$$

has been studied by Dienzer *et al.* (1974), who conclude that the steady field is favored when $|\rho_1 - \rho_2|$ is sufficiently large.

As an alternative to the eigenvalue approach, Jepps (1975) has carried out a step-by-step numerical integration of (5.55) and (5.56) again for particular choices of the function f and g and for varying values of D. Numerical experimentation leads to determination of the critical value of D at which there exist solutions varying periodically in time without long-term growth or decay. An advantage of this approach is that dynamical effects can be readily incorporated. In particular, Jepps investigated the effect of making α a decreasing function of the local value of B (see Section VIII,A); he found that the sinusoidal time behavior of the poloidal field was distorted by this sort of nonlinear effect toward a "spiked" rather than a "flattened" wave form.

Finally, we note that Parker (1971b) has proposed that Eqs. (7.6) and (7.7) with $\mathbf{u}_P = 0$ may also provide a correct description of the process of generation of the galactic magnetic field (a view that is strongly contested by Piddington, 1972b). With origin 0 at an arbitrary point of the plane of symmetry of the galactic disk (of thickness $2z_0$), $0z$ normal to the plane, $0x$ radial, and with

$$U = Gx, \qquad \alpha = \begin{cases} \alpha_0, & 0 < z < z_0, \\ -\alpha_0, & -z_0 < z < 0, \end{cases} \tag{7.14}$$

Parker obtains the dispersion relation for nonoscillatory modes proportional to $\exp(\omega t + ikx)$: for large positive values of the dynamo number, which in this case is

$$D = \alpha G z_0^3/\lambda^2, \tag{7.15}$$

and for modes in which $\mathbf{B}$ is symmetric about $z = 0$, this takes the form

$$\omega \sim -(\lambda D/k z_0^3) \exp(-\tfrac{1}{2}D^{1/3}) \cos \tfrac{1}{2}\sqrt{3}\, D^{1/3}, \tag{7.16}$$

indicating exponential growth if

$$(4n - 1)\pi < \sqrt{3}\, D^{1/3} < (4n + 1)\pi, \tag{7.17}$$

where n is any large positive integer. This is a surprising result since it indicates that for large values of D, a relatively small change (positive or negative) in D can transform a situation that permits exponential growth of a magnetic field mode into one in which the same mode decays. Of course, it may be that oscillatory modes are more easily excited (Parker, 1971e), as is the case for the spherical geometry; but if not, and if Parker's model is applicable, then the nature of the relation (7.16) between ω and D would suggest that the galactic field depends for its existence on a precarious interaction between effects of horizontal shear and cyclonic turbulence of rather critically defined intensity.

VIII. Dynamic Effects and Self-Equilibration

A. Waves Influenced by Coriolis Forces, and Associated Dynamo Action

A suitable starting point for the consideration of dynamic effects is an analysis of the waves that can propagate in a rotating incompressible fluid permeated by a locally uniform magnetic field $\mathbf{B}_0$. In this context, it is convenient to use the local Alfvén velocity $\mathbf{V} = (\mu_0 \rho)^{-1/2}\mathbf{B}_0$ as the measure of the field strength. The linearized equations governing velocity perturbations $\mathbf{u}$ about a state of rest and magnetic perturbations $\mathbf{b} = (\mu_0 \rho)^{1/2}\mathbf{v}$ are then

$$\partial\mathbf{u}/\partial t + 2\mathbf{\Omega} \wedge \mathbf{u} = -\nabla P + \mathbf{V} \cdot \nabla\mathbf{v} + \nu\nabla^2\mathbf{u}, \tag{8.1}$$

$$\partial\mathbf{v}/\partial t = \mathbf{V} \cdot \nabla\mathbf{u} + \lambda\nabla^2\mathbf{v}, \tag{8.2}$$

$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = 0, \tag{8.3}$$

where

$$P = p/\rho + (\mathbf{V} + \mathbf{v})^2/2\rho - (\mathbf{\Omega} \wedge \mathbf{x})^2, \tag{8.4}$$

and ν is the kinematic viscosity of the fluid. These equations (first considered by Lehnert, 1954) admit solutions of the form

$$(\mathbf{u}, \mathbf{v}, P) = (\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{P}) \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t), \tag{8.5}$$

and (8.2) gives the important relation between $\hat{\mathbf{v}}$ and $\hat{\mathbf{u}}$ already implicit in Section V,C:

$$\hat{\mathbf{v}} = -(\omega + i\lambda k^2)^{-1}(\mathbf{k} \cdot \mathbf{V})\hat{\mathbf{u}}. \tag{8.6}$$

Equation (8.1) then gives

$$-i\sigma\hat{\mathbf{u}} + 2\boldsymbol{\Omega} \wedge \hat{\mathbf{u}} = -i\mathbf{k}\hat{P}, \tag{8.7}$$

where

$$\sigma = \omega + i\nu k^2 - (\omega + i\lambda k^2)^{-1}(\mathbf{V} \cdot \mathbf{k})^2. \tag{8.8}$$

By crossing (8.7) with the vector $\mathbf{k}$ and then repeating this operation, we obtain two equations from which $\hat{\mathbf{u}}$ and $\mathbf{k} \wedge \hat{\mathbf{u}}$ may be eliminated, giving the dispersion relation

$$\sigma = \mp 2(\mathbf{k} \cdot \boldsymbol{\Omega})/k, \tag{8.9}$$

and the corresponding relation

$$i\mathbf{k} \wedge \hat{\mathbf{u}} = \pm k\hat{\mathbf{u}} \tag{8.10}$$

between the (complex) components of $\hat{\mathbf{u}}$. This latter relation implies that such a wave is circularly polarized [cf. the spatial structure (6.35)] and that its helicity is

$$\langle \mathbf{u} \cdot \boldsymbol{\omega} \rangle = \pm \tfrac{1}{2}k|\hat{\mathbf{u}}|^2. \tag{8.11}$$

If, therefore, for any reason waves corresponding to, say, the upper sign are present in greater proportion than those with the lower sign, then the net mean helicity will be positive, and the medium will be unstable to the growth of large-scale field fluctuations, by the α^2 effect discussed in Section V,A. If now we imagine that the field $\mathbf{V}$ itself varies on a large length scale, then large-scale Fourier components of force-free structure [like (2.25) but with $|\mathbf{K}|$ much smaller than typical values of $|\mathbf{k}|$ in the spectrum of $\mathbf{u}$] will grow exponentially.

What then ultimately limits the growth of this type of field structure? To see this, it is only necessary to consider the detailed form of the integral (5.27) for α_{ij}, which is, of course, still applicable. As $\mathbf{V}$ grows locally in strength, the dispersion relation $\omega = \omega(\mathbf{k})$ as given by (8.8) and (8.9) is progressively modified in such a way that all components of α_{ij} (which is severely anisotropic in this situation) tend to decrease. Field intensification ceases when these components are reduced to such a level that α effect generation is just compensated by ohmic dissipation. If there is no source of energy for the small-scale motion, then both velocity and magnetic fields must ultimately die away through both ohmic and viscous effects; on the other hand, if random body forces are invoked as a means of maintaining a random wave field, then a steady magnetohydrodynamic equilibrium may be anticipated. This type of process has been analyzed in detail (Moffatt, 1970b, 1972; Soward, 1975). Of particular interest is the level to which the magnetic energy density $M = \frac{1}{2}\bar{\mathbf{V}}^2$ can grow in this situation (the overbar indicating an average over the large wavelength $2\pi/K$ of the **V** field). The result obtained (Moffatt, 1972) in the case when waves of typical frequency are maintained by a random force distribution is that in the ultimate steady state the ratio of M to the kinetic energy density E in the random wave field is

$$M/E \sim (\Omega/\omega_0)^{1/2}(\nu/\lambda)^{1/2}(L/l), \tag{8.12}$$

where l is as usual the scale of the **u** field and L the scale of the **V** field. [Viscosity enters into this calculation because, under random forcing, there are some modes (those with $\mathbf{k} \cdot \mathbf{V} = 0$) that are unaffected by the magnetic field, and it is only viscosity that can limit the growth of such modes when excited at their resonant frequencies.] Clearly M can be large compared with E if L/l is sufficiently large: there is no question of equipartition of energy in a dissipative system of this kind.

If we neglect dissipative effects in the dispersion relation (8.8) and (8.9) (i.e., put $\lambda = \nu = 0$), then when $|\mathbf{V} \cdot \mathbf{k}| \ll 4(\mathbf{\Omega} \cdot \mathbf{k})^2/k^2$, the two roots of the resulting quadratic for ω are approximately

$$\omega_f \approx 2\Omega \cos\theta + (2\Omega \cos\theta)^{-1}(\mathbf{V} \cdot \mathbf{k})^2, \qquad \omega_s \approx -(2\Omega \cos\theta)(\mathbf{V} \cdot \mathbf{k})^2 \tag{8.13}$$

where θ is the angle between $\mathbf{\Omega}$ and $\mathbf{k}$. The root ω_f corresponds to an inertial wave slightly modified by the Lorentz force. The root ω_s ($|\omega_s| \ll |\omega_f|$) corresponds to a very slow wave (relative to the time scale Ω^{-1}) in which inertia effects are almost negligible. Indeed, the root ω_s may be most simply obtained from (8.1)–(8.3) by first dropping the inertia term $\partial\mathbf{u}/\partial t$. The resulting force balance between Coriolis, Lorentz, and pressure forces is known as *magnetostrophic* balance [a term introduced by Malkus (1959), although not quite with this same meaning], and we shall therefore describe the waves corresponding to the root ω_s as magnetostrophic waves. [Hide and Acheson (1973) in their review of the hydromagnetics of rotating fluids use the term "hydromagnetic–inertial" waves, but insofar as inertia plays no part (except via the Coriolis force), the term is perhaps misleading.]

B. Magnetostrophic Flow and the Taylor Constraint

There is good reason to believe (Hide and Roberts, 1961) that the global force balance in the core of the Earth is approximately magnetostrophic, i.e., that to a good approximation,

$$2\rho\mathbf{\Omega} \wedge \mathbf{U} = -\nabla p + \mathbf{J} \wedge \mathbf{B}, \tag{8.14}$$

where for the moment we use capital letters **U**, **J**, **B** to emphasize that we have large-scale fields in mind. Suppose that the fluid is contained within a sphere S, center 0, with $\mathbf{U}\cdot\mathbf{n}=0$ on S, and let $C(s_0)$ be the cylinder $s=s_0$, where (s, ϕ, z) are cylindrical polar coordinates with $0z$ parallel to $\mathbf{\Omega}$. Then integration of the ϕ component of (8.14) over $C(s_0)$ (Taylor, 1963) gives

$$\iint_{C(s_0)} (\mathbf{J}\wedge\mathbf{B})_\phi \, d\phi \, dz = 0, \qquad (8.15)$$

the integral being over that part of $C(s_0)$ contained within S. Under conditions of magnetostrophic balance, therefore, the magnetic field **B** and associated current $\mathbf{J}=\mu_0^{-1}\nabla\wedge\mathbf{B}$ must satisfy the Taylor constraint (8.15). The result is unaffected by the inclusion of buoyancy forces in (8.14), which of course have no ϕ component. We shall refer to important implications of this constraint in Section VIII,D.

C. Excitation of Magnetostrophic (MAC) Waves by Unstable Stratification

One of the great difficulties in making firm progress on the dynamics of the Earth's interior is the absence of direct knowledge concerning the dominant source of energy for core motions. One possibility, however, is that at least some part of the liquid core is unstably stratified (either thermally or through coexistence of mixed ingredients of different densities—Braginskii, 1964d); the question of thermal stratification has been discussed by Higgins and Kennedy (1971) and Kennedy and Higgins (1973) on the basis of existing knowledge concerning the melting point temperature of iron under high pressure, and the conclusion of the latter study is that the inner one-third of the liquid core may have a temperature distribution that allows "adiabatic" convection, whereas the outer two-thirds is (in this sense) stably stratified.

Whatever the actual situation, the following idealized problem (Braginskii, 1964d, 1967, 1970; Eltayeb, 1972, 1975; Roberts and Stewartson, 1974, 1975) provides a basis for careful analysis of the effect of unstable stratification. Suppose that fluid is contained between planes $z=\pm z_0$ on which the temperature θ is $\theta_0 \mp \beta z_0$, respectively. Suppose further that $\mathbf{\Omega}=(0, 0, \Omega)$ and $\mathbf{V}=(0, V, 0)$ (other directions also for $\mathbf{\Omega}$ and $\mathbf{V}$ are considered in the papers of Eltayeb, 1972, 1975); then, on the Boussinesq approximation, Eq. (8.1) is modified only by the inclusion of a buoyancy force $\alpha\mathbf{g}\theta$ on the right-hand side, where α is the coefficient of thermal expansion, and θ satisfies the linearized heat conduction equation

$$\partial\theta/\partial t + \beta u_z = \kappa\nabla^2\theta, \qquad (8.16)$$

with κ the thermal diffusivity. Equations (8.2) and (8.3) continue to hold unaltered.

These equations admit solutions proportional to $\exp[i(lx + my) + \omega t]$, with further u_x, u_y, v_x, and v_y proportional to $\cos nz$, and u_z, v_z, and θ proportional to $\sin nz$. If viscous effects are neglected and if the planes are assumed perfect conductors of heat and of electric current (conditions that are again varied in the papers by Eltayeb, 1972, 1975), then all the boundary conditions are satisfied provided nz_0/π is an integer. If $\partial\mathbf{u}/\partial t$ is also dropped from the equations (so that attention is focused on the slow magnetostrophic modes) the following cubic equation determining possible values of ω is obtained:

$$Y^2(\omega + \lambda k^2)^2(\omega + \kappa k^2) + (\omega + \kappa k^2) - X(\omega + \lambda k^2) = 0, \qquad (8.17)$$

where

$$k^2 = l^2 + m^2 + n^2, \qquad X = \alpha\beta g(m^2 + l^2)/m^2k^2V^2, \qquad Y = 2n\Omega/V^2m^2k. \qquad (8.18)$$

In the nondissipative limit $\lambda = \kappa = 0$ to which Braginskii (1964d, 1967) restricted attention, the roots of (8.17) become $\omega = 0$ and

$$\omega = \pm(X - 1)^{1/2}/Y, \qquad (8.19)$$

indicating instability ($\omega^2 > 0$) whenever $\mathbf{k} = (l, m, n)$ is such that $X > 1$. In the geophysical context, as in previous sections, we interpret the y direction as east (or azimuth), so that m (like n) can take only discrete values. The most unstable mode is that for which m takes its smallest nonzero value m_1 (corresponding to $\exp i\phi$ dependence on azimuth), and as pointed out by Roberts and Stewartson (1974), if $g\alpha\beta/V^2$ exceeds m_1^2 by any amount, no matter how small, an infinite number of modes (corresponding to large values of l) become unstable, since as $l \to \infty$, $X \sim \alpha\beta g/V^2m_1^2$. Of course, as l increases, these modes are increasingly affected by diffusion and also by inertia [neglected in obtaining (8.17)].

Weak diffusion plays a more important role in shifting the root $\omega = 0$ away from the origin of the complex plane. Naive linearization of (8.17) in the small quantities λ, κ, and ω gives for this root the expression

$$\omega = -\lambda k^2(X - q)/(X - 1), \qquad q = \kappa/\lambda. \qquad (8.20)$$

If $0 < q < X < 1$, then the modes given by (8.19) are stable, but that given by (8.20) is unstable with a slow growth rate determined by the weak diffusion effects. This is a "resistive" instability in the terminology of Furth *et al.* (1963); its existence underlines the possible importance of diffusion effects, no matter how weak these may be. It is not clear that the arguments of Higgins and Kennedy (1971) concerning the impossibility of radial convection with no diffusive heat exchange apply with equal force when resistive instabilities of this kind are considered.

In general, the three roots of the cubic (8.17) are either all real or one is real and the other two are complex conjugates. In this latter case, if the complex conjugate roots are given by $\omega = \omega_r \pm i\omega_i$ with $\omega_r > 0$, the corresponding disturbances propagate as magnetostrophic waves of increasing amplitude. Roberts and Stewartson (1974) show the region of the plane of the variables q and $Q = 2\Omega\lambda/V^2$ in which such unstable modes are excited in preference to "steady" modes for which $\omega_i = 0$.

The distinction between overstable modes and steady modes is of potential importance in the context of the main problem of regeneration of the large-scale field. We have seen in earlier sections that lack of reflectional symmetry, as measured by the mean helicity $\langle \mathbf{u} \cdot \nabla \wedge \mathbf{u} \rangle$ is vital in this context; the mean helicity of the unstable motions of the present problem can be calculated on the basis of the linearized equations, and it is found that $\langle \mathbf{u} \cdot \nabla \wedge \mathbf{u} \rangle$ (the average being over horizontal planes) is nonzero only if $\omega_i \neq 0$, i.e., only if the modes are of the overstable variety. It is possible that the nonlinear effects studied by Roberts and Stewartson (1974, 1975), and in particular the inclusion of mean shear in the y direction, may modify this conclusion, but this is something that requires further investigation; in this context, see also the discussion of Roberts and Soward (1972, p. 148).

At any rate, the picture conceived by Braginskii (1967) is the following: unstable stratification in the liquid core of the Earth in the presence of the predominantly toroidal field excites magnetostrophic waves that are essentially nonaxisymmetric and that propagate in the azimuthal direction. Braginskii called these waves "MAC waves" (M for magnetic, A for Archimedean, C for Coriolis). These nonaxisymmetric motions provide the α effect that is necessary to generate poloidal field from toroidal field, which in turn is generated from the poloidal field by dominant differential rotation. We have here the possibility of a dynamically consistent dynamo driven by buoyancy forces; to be completely consistent, however, the stability analysis described in this section should include the effects of a mean flow in the y direction with strong vertical shear (i.e., the Cartesian analog of strong differential rotation). Roberts and Stewartson (1975) take an important step in this direction. At the same time, a fully consistent model should include the dynamics of the mean flow itself, an important topic on which we focus attention in the following section.

Before leaving the topic of convection-driven dynamo action, however, we should refer to the papers by Childress and Soward (1972), Soward (1974), and Busse (1973), in which different approaches to the determination of a dynamically consistent dynamo are considered. Busse considers a combination of two-dimensional convection rolls, together with superimposed shear flow parallel to the rolls. Childress and Soward consider convection in a strongly rotating system in which the horizontal scale of the convection cells is small $[O(\Omega^{-1/3})]$ relative to the vertical scale. This permits the use of the two-scale methods described in Section V, and leads (Soward, 1974) to the determination of a dynamo in which the magnetic energy density fluctuates about a weak average value.

In these studies involving both magnetic field and rotation, it is well known (Chandrasekhar, 1961) that while rotation and magnetic field are separately stabilizing, the two effects can work against each other in such a way that a flow that is stable under the action of rotation alone becomes unstable when a magnetic field is also introduced, while a flow that is unstable under the action of rotation alone becomes even more unstable with a magnetic field. This consideration led Malkus (1959), on the basis of his appealing conjecture that a convective system with infinitely many degrees of freedom will take advantage of this freedom to maximize the transport of heat in the direction of the imposed temperature gradient, to argue that a magnetic field *must* grow by dynamo action, if only to release the constraints of rotation, to permit more vigorous convection, and so to increase the transport of heat. Soward's work should in principle provide a test of whether Malkus's conjecture is true or false; but this still remains a choice target for future investigation.

D. Mean Flow Equilibration

A magnetic field that is growing in intensity as a result of the α effect will in general not be force-free and will therefore tend to drive a mean velocity, or to modify any preexisting velocity distribution. This effect has been studied by Childress (1969), Malkus and Proctor (1975), and Proctor (1975). It is supposed in these investigations that background turbulence, present for whatever reason, gives rise to an isotropic α effect, with α a prescribed function of position in a sphere, and that there is initially zero mean velocity. It is further supposed that the level of the α effect is just a little greater than that at which field excitation takes place. The growing field $\mathbf{B}$ will then approximate to the axisymmetric eigenfunction corresponding to the lowest eigenvalue for a steady α^2 dynamo (Sections VII,A and B). A mean axisymmetric velocity field $\mathbf{U}(\mathbf{x}, t)$ will then develop according to the equation

$$\partial\mathbf{U}/\partial t + \mathbf{U}\cdot\nabla\mathbf{U} + 2\boldsymbol{\Omega}\wedge\mathbf{U} = -\nabla P + (\mu_0\rho)^{-1}(\nabla\wedge\mathbf{B})\wedge\mathbf{B} + \nu\nabla^2\mathbf{U}, \quad (8.21)$$

and subject to appropriate boundary conditions on the surface of the sphere. The existence of a mean velocity $\mathbf{U}$ then modifies the structure of the field $\mathbf{B}$, which evolves according to the equation

$$\partial\mathbf{B}/\partial t = \nabla\wedge(\mathbf{U}\wedge\mathbf{B}) + \nabla\wedge(\alpha\mathbf{B}) + \lambda\nabla^2\mathbf{B} \quad (8.22)$$

in the fluid region. It is to be expected that this modification in structure will lead to increased ohmic dissipation, and hence to curtailment of growth of magnetic energy. This expectation is borne out by numerical integration of (8.21) and (8.22) (Proctor, 1975) in the geophysically relevant case when $\alpha = \alpha_0 \cos\theta$. This problem is characterized by three dimensionless numbers,

$$E = \nu/\Omega R^2, \qquad E_M = \lambda/\Omega R^2, \qquad \hat{\alpha} = \alpha_0 R/\lambda, \tag{8.23}$$

where R is the sphere radius, and the lowest eigenvalue of the kinematic α^2 dynamo problem (with $\mathbf{U} = 0$) is in this case (P. H. Roberts, 1972) $\hat{\alpha}_c = 7.65$. E represents the ratio of viscous forces to Coriolis forces, and, since λ/L is the relevant velocity scale, E_M represents the ratio of inertia forces to Coriolis forces. Both E and E_M are small in the geophysical context; in the limit $E = 0$, $E_M = 0$, (8.21) degenerates to (8.14) and the balance of forces is then magnetostrophic.

In each of the cases studied, Proctor found that the growth of magnetic energy $M(t)$ was arrested by the mean flow effect, the level of equilibrium magnetic energy increasing with $\hat{\alpha} - \hat{\alpha}_c$. Provided E and E_M were not too small, $M(t)$ settled down to a constant level $M(\infty)$, after some damped oscillations about this level. Figure 4 shows the structure of the magnetic

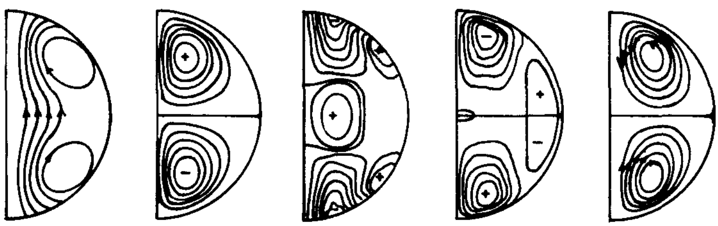

FIG. 4. Field structures in the Proctor (1975) steady state dynamically equilibrated dynamo; $E = 0.01$, $E_M = 0.04$, $\hat{\alpha} = 8.0$. (a) Poloidal field lines, (b) isotors $B =$ const, (c) lines of constant angular momentum about axis of symmetry, (d) lines of constant vorticity of meridional circulation, (e) streamlines of meridional circulation.

field and the velocity field in the ultimate steady state when $E = 0.01$, $E_M = 0.04$, and $\hat{\alpha} = 8.0$. The poloidal field is shown only for $r < R$, although it of course extends outside the sphere. There is an indication of a "cylindrical structure" in the patterns of angular momentum and vorticity, which is presumably an indication of the influence of the Taylor constraint (8.15), which is nearly satisfied for these low values of E and E_M.

At even lower values of E and E_M ($E = 0.005$, $E_M = 0.0025$), Proctor found that the oscillations of $M(t)$ about its ultimate mean level did not show any tendency to die out with time. These oscillations of $M(t)$ are associated with torsional oscillations about an equilibrium in which the Taylor constraint is satisfied. As pointed out by Roberts and Soward (1972) such torsional oscillations will be damped by Ekman suction effects in boundary layers on the sphere $r = R$, but when E and E_M are very small, this damping is evidently not sufficient to suppress the oscillations. It appears that the steady equilibrium state satisfying the Taylor constraint is unstable or in some other sense unattainable; a related phenomenon was found by Roberts and Stewartson (1975) in the treatment of nonlinear aspects of the stability problem discussed in Section VIII,C.

Neglect of the suppression of the α effect by the growing magnetic field is justified in Proctor's investigation for small values of $\hat{\alpha} - \hat{\alpha}_c$ since then the mean flow equilibrium mechanism causes a leveling off of $M(t)$ at a low level at which the turbulence suppression effect is still negligible. The two effects would have to be considered in conjunction if $\hat{\alpha} - \hat{\alpha}_c$ were large; but then one would also have the complication that more than one mode of the linearized kinematic problem might be unstable.

The model is also open to criticism on the grounds that the reflectionally asymmetric turbulence giving rise to the α effect is simply assumed present, without any consideration of the means whereby it may be maintained. A marriage of the models of Sections VIII,C and D may in the longer term provide something approaching the complete picture as far as maintenance of the Earth's field is concerned.

ACKNOWLEDGMENTS

The writing of this review was completed during my tenure of a Visiting Professorship at the Institut de Mécanique Théorique et Appliquée, Université Pierre et Marie Curie, Paris, and I wish to express my gratitude to Professor Cabannes and his colleagues for generous support and hospitality during my visit.

I am indebted to Dr. R. H. Kraichnan who drew my attention to an erroneous conclusion in the original draft of Section V,C; this led to an important modification of the penultimate paragraph of that section.

REFERENCES

Babcock, H. W., and Babcock, H. D. (1955). The Sun's magnetic field. *Astrophys. J.* **121**, 349–366.

Backus, G. E. (1958). A class of self-sustaining dissipative spherical dynamos. *Ann. Phys. New York* **4**, 372–447.

Batchelor, G. K. (1950). On the spontaneous magnetic field in a conducting liquid in turbulent motion. *Proc. Roy. Soc., Ser. A* **201**, 405–416.

Batchelor, G. K. (1953). "Homogeneous Turbulence." Cambridge Univ. Press, London and New York.

BATCHELOR, G. K. (1956). Steady laminar flow with closed streamlines at large Reynolds numbers. *J. Fluid Mech.* **1**, 177–190.

BRAGINSKII, S. I. (1964a). Self excitation of a magnetic field during the motion of a highly conducting fluid. *Sov. Phys.—JETP* **20**, 726–735.

BRAGINSKII, S. I. (1964b). Theory of the hydromagnetic dynamo. *Sov. Phys.—JETP* **20**, 1462–1471.

BRAGINSKII, S. I. (1964c). Kinematic models of the Earth's hydromagnetic dynamo. *Geomag. Aeron.* **4**, 572–583.

BRAGINSKII, S. I. (1964d). Magnetohydrodynamics of the Earth's core. *Geomag. Aeron.* **4**, 698–712.

BRAGINSKII, S. I. (1967). Magnetic waves in the Earth's core. *Geomag. Aeron.* **7**, 851–859.

BRAGINSKII, S. I. (1970). Oscillation spectrum of the hydromagnetic dynamo of the Earth. *Geomag. Aeron.* **10**, 172–181.

BULLARD, E. C. (1968). Reversals of the Earth's magnetic field. *Phil. Trans. Roy. Soc. London, Ser. A* **263**, 481–524.

BULLARD, E. C., and GELLMAN, H. (1954). Homogeneous dynamos and terrestrial magnetism. *Phil. Trans. Roy. Soc. London, Ser. A* **247**, 213–278.

BUSSE, F. H. (1973). Generation of magnetic fields by convection. *J. Fluid Mech.* **57**, 529–544.

BUSSE, F. H. (1975). A necessary condition for the geodynamo. *J. Geophys. Res.* **80**, 278–280.

CHANDRASEKHAR, S. (1956). On force-free magnetic fields. *Proc. Nat. Acad. Sci. U.S.* **42**, 1–5.

CHANDRASEKHAR, S. (1961). "Hydrodynamic and Hydromagnetic Stability." Oxford Univ. Press (Clarendon), London and New York.

CHILDRESS, S. (1969). A class of solutions of the magnetohydrodynamic dynamo problem. *In* "The Application of Modern Physics to the Earth and Planetary Interiors" (S. K. Runcorn, ed.), pp. 629–648, Wiley, New York.

CHILDRESS, S., and SOWARD, A. M. (1972). Convection driven hydromagnetic dynamo. *Phys. Rev. Lett.* **29**, 837–839.

COWLING, T. G. (1934). The magnetic field of sunspots, *Mon. Notic. Roy. Astron. Soc.* **94**, 39–48.

DIENZER, W., VON KUSSEROW, H. U., and STIX, M. (1974). Steady and oscillatory $\alpha\omega$-dynamos. *Astron. Astrophys.* **36**, 69–78.

DILLON, R. F. (1975). Gravity and magnetic field correlation and related geomagnetic topics. Ph.D. Thesis, Cambridge Univ., Cambridge, England.

DOLGINOV, S. (1975). "On Magnetic Dynamo Mechanism of the Planets," Prep. N9a (124). Inst. Terr. Magn., Ionosphere, Radio Wave Propagation, Moscow.

DOLGINOV, S., YEROSHENKO, Y., and ZHUZGOV, L. (1973). Magnetic field in the very close neighbourhood of Mars according to data from the Mars 2 and Mars 3 Spacecraft. *J. Geophys. Res.* **78**, 4779–4786.

DROBYSHEVSKI, E. M., and YUFEREV, V. S. (1974). Topological pumping of magnetic flux by three-dimensional convection. *J. Fluid Mech.* **65**, 33–44.

ELTAYEB, I. A. (1972). Hydromagnetic convection in a rapidly rotating fluid layer. *Proc. Roy. Soc., Ser. A* **326**, 229–254.

ELTAYEB, I. A. (1975). Overstable hydromagnetic convection in a rotating fluid layer. *J. Fluid Mech.* **71**, 161–179.

FURTH, H. P., KILLEEN, J., and ROSENBLUTH, M. N. (1963). Finite-resitivity instabilities of a sheet pinch. *Phys. Fluids* **6**, 459–484.

GAILITIS, A. (1970). Self-excitation of a magnetic field by a pair of annular vortices. *Mag. Gidrod.* **6**, 19. [*Magnetohydrodynamics* **6**, 14 (1972).]

GIBSON, R. D. (1968). The Herzenberg dynamo II. *Quart. J. Mech. Appl. Math.* **21**, 257–267.

GOLITSYN, G. S. (1960). Fluctuations of the magnetic field and current density in a turbulent flow of a weakly conducting fluid. *Sov. Phys. Dokl.* **5**, 536–539.

GUBBINS, D. (1974). Theories of the geomagnetic and solar dynamos. *Rev. Geophys. Space Phys.* **12**, 137–154.

HERZENBERG, A. (1958). Geomagnetic dynamos. *Phil. Trans. Roy. Soc. London, Ser. A* **250**, 543–585.

HIDE, R. (1974). Jupiter and Saturn. *Proc. Roy. Soc., Ser. A* **336**, 63–84.

HIDE, R., and ACHESON, D. J. (1973). Hydromagnetics of rotating fluids. *Rep. Progr. Phys.* **36**, 159–221.

HIDE, R., and ROBERTS, P. H. (1961). The origin of the main geomagnetic field. *Phys. Chem. Earth* **4**, 25–98.

HIGGINS, G. H., and KENNEDY, G. C. (1971). The adiabatic gradient and the melting point gradient in the core of the Earth. *J. Geophys. Res.* **76**, 1870–1878.

IBBETSON, A., and TRITTON, D. J. (1975). Experiments on turbulence in a rotating fluid. *J. Fluid Mech.* **68**, 639–672.

JEPPS, S. A. (1975). Numerical models of hydromagnetic dynamos. *J. Fluid Mech.* **67**, 625–646.

KENNEDY, G. C., and HIGGINS, G. H. (1973). The core paradox, *J. Geophys. Res.* **78**, 900–904.

KIEPENHEUER, K. O. (1953). Solar activity. *In* "The Sun" (G. P. Kuiper, ed.), pp. 322–465. Univ. of Chicago Press, Chicago, Illinois.

KOVASZNAY, L. S. G. (1960). Plasma turbulence. *Rev. Mod. Phys.* **32**, 815–822.

KRAICHNAN, R. H. (1973). Helical turbulence and absolute equilibrium. *J. Fluid Mech.* **59**, 745–752.

KRAICHNAN, R. H., and NAGARAJAN, S. (1967). Growth of turbulent magnetic fields. *Phys. Fluids* **10**, 859–870.

KRAUSE, F. (1968). Eine Lösung des Dynamo problems auf der Grundlage einer linearen Theorie der magnetohydrodynamischen Turbulenz. Habilitationschrift, Univ. of Jena, Jena. (See Roberts and Stix, 1971.)

KRAUSE, F., and ROBERTS, P. H. (1973). Some problems of mean field electrodynamics. *Astrophys. J.* **181**, 977–992.

KRAUSE, F., and STEENBECK, M. (1967). Some simple models of magnetic field regeneration by non-mirror-symmetric turbulence. *Z. Naturforsch. A* **22**, 671–675. (See Roberts and Stix, 1971.)

LEHNERT, B. (1954). Magnetohydrodynamic waves under the action of the Coriolis force. *Astrophys. J.* **119**, 647–654.

LEIGHTON, R. B. (1969). A magneto-kinematic model of the solar cycle. *Astrophys. J.* **156**, 1–26.

LERCHE, I. (1971a). Kinematic-dynamo theory. *Astrophys. J.* **166**, 627–638.

LERCHE, I. (1971b). Kinematic-dynamo theory, II. Dynamo action in infinite media with isotropic turbulence. *Astrophys. J.* **166**, 639–649.

LIEPMANN, H. W. (1952). Aspects of the turbulence problem. *Z. Angew. Math. Phys.* **3**, 321–342.

LIGHTHILL, M. J. (1959). "Introduction to Fourier Transforms and Generalized Functions." Cambridge Univ. Press, London and New York.

LORTZ, D. (1968). Impossibility of steady dynamos with certain symmetries. *Phys. Fluids* **11**, 913–915.

LOWES, F. J., and WILKINSON, I. (1963). Geomagnetic dynamo: A laboratory model. *Nature (London)* **198**, 1158–1160.

LOWES, F. J., and WILKINSON, I. (1968). Geomagnetic dynamo: An improved laboratory model. *Nature (London)* **219**, 717–718.

LUMLEY, J. L. (1962). The mathematical nature of the problem of relating Lagrangian and Eulerian statistical functions in turbulence. *In* "Mécanique de la Turbulence," No. 108, pp. 17–26. CNRS, Paris.

MALKUS, W. V. R. (1959). Magnetoconvection in a viscous fluid of infinite electrical conductivity. *Astrophys. J.* **130**, 259–275.

MALKUS, W. V. R., and PROCTOR, M. R. E. (1975). The macrodynamics of α-effect dynamos in rotating fluids. *J. Fluid Mech.* **67**, 417–444.

MAUNDER, E. W. (1922). The Sun and sunspots 1820–1920. *Mon. Not. Roy. Astron. Soc.* **82**, 534–544.

MOFFATT, H. K. (1961). The amplification of a weak applied magnetic field by turbulence in fluids of moderate conductivity. *J. Fluid Mech.* **11**, 625–635.

MOFFATT, H. K. (1963). Magnetic eddies in an incompressible viscous fluid of high electrical conductivity. *J. Fluid Mech.* **17**, 225–239.

MOFFATT, H. K. (1969). The degree of knottedness of tangled vortex lines. *J. Fluid Mech.* **35**, 117–129.

MOFFATT, H. K. (1970a). Turbulent dynamo action at low magnetic Reynolds number. *J. Fluid Mech.* **41**, 435–452.

MOFFATT, H. K. (1970b). Dynamo action associated with random inertial waves in a rotating conducting fluid. *J. Fluid Mech.* **44**, 705–719.

MOFFATT, H. K. (1972). An approach to a dynamic theory of dynamo action in a rotating conducting fluid. *J. Fluid Mech.* **53**, 385–399.

MOFFATT, H. K., ed. (1973). Report on the NATO Advanced Study Institute on magnetohydrodynamic phenomena in rotating fluids. *J. Fluid Mech.* **57**, 625–649.

MOFFATT, H. K. (1974). The mean electromotive force generated by turbulence in the limit of perfect conductivity. *J. Fluid Mech.* **65**, 1–10.

NESS, N., BEHANNON, K. W., LEPPING, R. P., WHANG, Y. C., and SCHATTEN, K. H. (1974). Magnetic field observations near Mercury: Preliminary results from Mariner 10. *Science* **185**, 151–160.

PARKER, E. N. (1955a). Hydromagnetic dynamo models. *Astrophys. J.* **122**, 293–314.

PARKER, E. N. (1955b). The formation of sunspots from the solar toroidal field. *Astrophys. J.* **121**, 491–507.

PARKER, E. N. (1963). Kinematical hydromagnetic theory and its applications to the low solar photosphere. *Astrophys. J.* **138**, 552–575.

PARKER, E. N. (1970a). The origin of Solar magnetic fields. *Ann. Rev. Astron. Astrophys.* **8**, 1–30.

PARKER, E. N. (1970b). The generation of magnetic fields in astrophysical bodies. I. The dynamo equations. *Astrophys. J.* **162**, 665–673.

PARKER, E. N. (1971a). The generation of magnetic fields in astrophysical bodies. II. The galactic field. *Astrophys. J.* **163**, 255–278.

PARKER, E. N. (1971b). The generation of magnetic fields in astrophysical bodies. III. Turbulent diffusion of fields and efficient dynamos. *Astrophys. J.* **163**, 279–285.

PARKER, E. N. (1971c). The generation of magnetic fields in astrophysical bodies. IV. The Solar and terrestrial dynamos. *Astrophys. J.* **164**, 491–509.

PARKER, E. N. (1971d). The generation of magnetic fields in astrophysical bodies. V. Behaviour at large dynamo numbers. *Astrophys. J.* **165**, 139–146.

PARKER, E. N. (1971e). The generation of magnetic fields in astrophysical bodies. VI. Periodic modes of the galactic field. *Astrophys. J.* **166**, 295–300.

PARKER, E. N. (1971f). The generation of magnetic fields in astrophysical bodies. VIII. Dynamical considerations *Astrophys. J.* **168**, 239–249.

PARKER, E. N. (1972). Topological dissipation and the small-scale fields in turbulent gases. *Astrophys. J.* **174**, 499–510.

PARKER, E. N. (1973). The reconnection rate of magnetic fields. *Astrophys. J.* **180**, 247.

PARKER, E. N. (1975). Generation of magnetic fields in astrophysical bodies, X. Magnetic buoyancy and the solar dynamo. *Astrophys. J.* **198**, 205–209.

PARKER, R. L. (1966). Reconnexion of lines of force in rotating spheres and cylinders. *Proc. Roy. Soc., Ser. A* **291**, 60–72.

Pichakhchi, L. D. (1966). Theory of the hydromagnetic dynamo. *Sov. Phys.—JETP* **23**, 542–543.

Piddington, J. H. (1972a). Solar dynamo theory and the models of Babcock and Leighton. *Sol. Phys.* **22**, 3–19.

Piddington, J. H. (1972b). The origin and form of the galactic magnetic field, I. Parker's dynamo model. *Cosmic Electrodyn.* **3**, 60–70.

Proctor, M. R. E. (1975). Non-linear mean field dynamo models and related topics. Ph.D. Thesis, Cambridge Univ., Cambridge, England.

Rädler, K.-H. (1969a). On the electrodynamics of turbulent fluids under the influence of Coriolis forces. *Monats. Dt. Akad. Wiss. Berlin* **11**, 194–201 (see Roberts and Stix, 1971).

Rädler, K.-H. (1969b). A new turbulent dynamo, I. *Monatsber. Deut. Akad. Wiss. Berlin* **11**, 272–279. (See Roberts and Stix, 1971.)

Roberts, G. O. (1970). Spatially periodic dynamos. *Phil. Trans. Roy. Soc. London, Ser. A* **266**, 535–558.

Roberts, G. O. (1972). Dynamo action of fluid motions with two-dimensional periodicity. *Phil. Trans. Roy. Soc. London, Ser. A* **271**, 411–454.

Roberts, P. H. (1967). "An Introduction to Magnetohydrodynamics." Longmans, Green, New York.

Roberts, P. H. (1971). Dynamo theory. *In* "Mathematical Problems in the Geophysical Sciences" (W. H. Reid, ed.), pp. 129–206. Amer. Math. Soc., Providence Rhode Island.

Roberts, P. H. (1972). Kinematic dynamo models. *Phil. Trans. Roy. Soc. London, Ser. A* **272**, 663–698.

Roberts, P. H., and Soward, A. M. (1972). Magnetohydrodynamics of the Earth's core. *Annu. Rev. Fluid Mech.* **4**, 117–154.

Roberts, P. H., and Soward, A. M. (1975). A unified approach to mean field electrodynamics. *Astron. Nachr.* **296**, 49–64.

Roberts, P. H., and Stewartson, K. (1974). On finite amplitude convection in a rotating magnetic system. *Phil. Trans. Roy. Soc. London, Ser. A* **277**, 287–315.

Roberts, P. H., and Stewartson, K. (1975). On double roll convection in a rotating magnetic system. *J. Fluid Mech.* **68**, 447–466.

Roberts, P. H., and Stix, M. (1971). "The Turbulent Dynamo: A Translation of a Series of Papers by F. Krause, K.-H. Rädler and M. Steenbeck," Tech. Note NCAR/TN/1A-60. Nat. Cent. Atmos. Phys., Boulder, Colorado.

Roberts, P. H., and Stix, M. (1972). α-Effect dynamos by the Bullard-Gellman formalism. *Astron. Astrophys.* **18**, 453–466.

Saffman, P. G. (1960). On the effect of the molecular diffusivity in turbulent diffusion. *J. Fluid Mech.* **8**, 273–283.

Saffman, P. G. (1963). On the fine-scale structure of vector fields convected by a turbulent fluid. *J. Fluid Mech.* **16**, 545–572.

Schlüter, A., and Biermann, L. (1950). Interstellare Magnetfelder. *Z. Naturforsch. A* **5**, 237–251.

Smith, E., Davis, L., Jones, D. E., Colburn, D. S., Coleman, P. J., Dyal, P., and Sonett, C. P. (1974). Magnetic field of Jupiter and its interaction with the Solar wind. *Science* **183**, 305–306.

Soward, A. M. (1971). Nearly symmetric advection. *J. Math. Phys. (N.Y.)* **12**, 2052–2062.

Soward, A. M. (1972). A kinematic theory of large magnetic Reynolds number dynamos. *Phil. Trans. Roy. Soc. London, Ser. A* **272**, 431–462.

Soward, A. M. (1974). A convection-driven dynamo I. The weak field case. *Phil. Trans. Roy. Soc. London, Ser. A* **275**, 611–651.

SOWARD, A. M. (1975). Random waves and dynamo action. *J. Fluid Mech.* **69**, 145–177.

STEENBECK, M., and KRAUSE, F. (1966). The generation of stellar and planetary magnetic fields by turbulent dynamo action. *Z. Naturforsch. A.* **21**, 1285–1296. (See Roberts and Stix, 1971.)

STEENBECK, M., and KRAUSE, F. (1969a). On the dynamo theory of stellar and planetary magnetic fields, I. A.C. dynamos of solar type. *Astron. Nachr.* **291**, 49–84. (See Roberts and Stix, 1971.)

STEENBECK, M., and KRAUSE, F. (1969b). On the dynamo theory of stellar and planetary magnetic fields, II. D.C. dynamos of planetary type. *Astron. Nachr.* **291**, 271–286. (See Roberts and Stix, 1971.)

STEENBECK, M., KRAUSE, F., and RÄDLER, K.-H. (1966). A calculation of the mean electromotive force in an electrically conducting fluid in turbulent motion, under the influence of Coriolis forces. *Z. Naturforsch. A* **21**, 369–376. (See Roberts and Stix, 1971.)

TAYLOR, G. I. (1922). Diffusion by continuous movements. *Proc. London Math. Soc., A* **20**, 196–211. (Collect. pap., Vol. II, pp. 172–184. Cambridge Univ. Press, London and New York, 1960.)

TAYLOR, J. B. (1963). The magnetohydrodynamics of a rotating fluid and the Earth's dynamo problem. *Proc. Roy. Soc. London, Ser. A* **274**, 274–283.

TOUGH, J. G. (1967). Nearly symmetric dynamos. *Geophys. J. Roy. Astron. Soc.* **13**, 393–396.

TOUGH, J. G., and GIBSON, R. D. (1969). The Braginskii dynamo. *In* "The Application of Modern Physics to the Earth and Planetary Interiors" (S. K. Runcorn, ed.), pp. 555–569. Wiley, New York.

VAINSHTEIN, S. I., and ZEL'DOVICH, Y. B. (1972). Origin of magnetic fields in astrophysics. *Sov. Phys.—Usp.* **15**, 159–172.

VENEZIAN, G. (1967). "Magnetohydrodynamics of the Earth's Magnetic Field," Rep. No. 85-37, Div. Eng. Appl. Sci., California Inst. Tech., Pasadena, California.

WARWICK, J. (1963). Dynamic spectra of Jupiter's decametric emission. *Astrophys. J.* **137**, 41–60.

WEISS, N. O. (1966). The expulsion of magnetic flux by eddies. *Proc. Roy. Soc., Ser. A* **293**, 310–328.

WEISS, N. O. (1971). The dynamo problem. *Quart. J. Roy. Astron. Soc.* **12**, 432–446.

WEISS, N. O. (1974). Dynamo maintenance of magnetic fields in stars. *In* "Magnetohydrodynamics" (L. Mestel, F. Meyer, and N. O. Weiss, eds.). Swiss Soc. Astron. Astrophys., Geneva.

WOLTJER, L. (1958). A theorem on force-free magnetic fields. *Proc. Nat. Acad. Sci. U.S.* **44**, 489–491.

ZEL'DOVICH, Y. B. (1957). The magnetic field in the two-dimensional motion of a conducting turbulent liquid. *Sov. Phys.—JETP* **4**, 460–462.