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Dynamo generation of magnetic fields in fluid conductors

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1. Introduction

When electric currents flow in an electrically conducting fluid, the associated magnetic field $\underset{\sim}{B}(\underset{\sim}{x},t)$ evolves according to the induction equation

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \wedge (\underset{\sim}{u} \wedge \underset{\sim}{B}) + \lambda \nabla^2 \underset{\sim}{B}, \tag{1}$$

where λ is the magnetic diffusivity of the conductor, and $\underset{\sim}{u}(\underset{\sim}{x},t)$ is the fluid velocity. This equation holds in the fluid region; if this region is bounded by a non-conducting region, then in this non-conductor $\underset{\sim}{B}$ is a potential field satisfying $\nabla \cdot \underset{\sim}{B} = 0$, $\nabla \wedge \underset{\sim}{B} = 0$.

The kinematic dynamo problem consists in determining under what conditions the magnetic energy associated with the field $\underset{\sim}{B}$ can be prevented from decaying to zero through the action of ohmic diffusion. If $\underset{\sim}{u} \equiv 0$, the field decays in a time of order L^2/λ where L is the typical scale of the conducting region. It is now well known that if $\underset{\sim}{u}$ has suitable properties and λ is not too large, then this decay can be prevented and possibly reversed. Growth of the magnetic energy may be described as "dynamo action". This process is of crucial importance in explaining the origin and maintenance of the Earth's magnetic field (and also potentially of those other planets, Mercury, Mars and Jupiter, now known to have significant large-scale magnetic fields), and also in explaining the observed evolution of the large scale magnetic field of the sun (which appears to oscillate on roughly the same time-scale as that of the 22-year sunspot cycle - Parker 1970).

The literature of the dynamo problem is vast, and it is not possible to review it all at all adequately in a one hour lecture. I propose to concentrate on certain developments that have taken place over the last ten years or so stemming from ideas first put forward by Steenbeck, Krause & Radler (1966). These ideas in fact bear a close relation (which I shall endeavour to elucidate) with earlier approaches to the problem developed by Parker (1955) and Braginskii (1964) but it is only quite recently that the relationships between these theories have been fully appreciated.

Study of equation (1) for prescribed $\underset{\sim}{u}(\underset{\sim}{x},t)$ is of course a purely kinematic approach, and for a full treatment must naturally be supplemented by an appropriate equation of motion describing the evolution of $\underset{\sim}{u}$ itself. In the contexts described above the fluid can for the most part be treated as incompressible and as satisfying the Navier-Stokes equation in a rotating fluid - i.e. with Coriolis forces included. The Lorentz force $\underset{\sim}{j} \wedge \underset{\sim}{B}$ (where $\underset{\sim}{j}$ is the current) must also be included. It is however entirely legitimate to defer consideration of this equation at the outset; conclusions based on equation (1) alone can be of very wide generality, and provide useful guidelines as to necessary (or at least highly desirable) properties of any velocity field $\underset{\sim}{u}(\underset{\sim}{x},t)$ that may be expected to promote dynamo action.

2. Helicity

One of these 'highly desirable' properties turns out to be a 'lack of reflexional symmetry' in the velocity field, i.e. in some sense it must exhibit either a right-handedness or a left-handedness in its global properties. The simplest measure of the lack of reflexional symmetry of a localised motion of finite energy is its helicity

$$I = \int \underset{\sim}{u} \cdot \underset{\sim}{\omega} \, dV \qquad (2)$$

where $\underset{\sim}{\omega} = \nabla \wedge \underset{\sim}{u}$ is the vorticity and the integral is throughout the fluid volume V. This quantity is a pseudo-scolar whose sign changes under transformation from a right-handed to a left-handed frame of reference. Consequently a non-zero value for I certainly implies a lack of reflexional symmetry . I is a quantity that is invariant in an inviscid fluid moving under conservative body forces (Moffatt 1969); this invariance is associated with the interpretation of I as a topological invariant - the 'degree of knottedness' of the vortex lines of the motion - a quantity that is conserved essentially by virtue of the fact that under these conditions vortex lines are frozen in the fluid.

Analogously, when $\lambda = 0$, equation (1) implies conservation of the magnetic helicity

$$I_M = \int \underset{\sim}{A} \cdot \underset{\sim}{B} \, dV , \qquad \underset{\sim}{B} = \nabla \wedge \underset{\sim}{A} , \tag{3}$$

a result first recognized by Elsasser (1956). In this perfectly conducting limit, lines of force ('$\underset{\sim}{B}$-lines') are frozen in the fluid and the same interpretation of I_M as a topological invariant applies. It is a feature of most elementary dynamo systems (e.g. the homopolar disc dynamo described by Roberts 1967) that a lack of reflexional symmetry is apparent in the configuration considered; a non-zero value for I_M, though not inevitable, is a frequent consequence.

3. The mean electromotive force generated by turbulence

In order to see how lack of reflexional symmetry can be of relevance to the problem, it is necessary to develop the notation and approach of Steenbeck, Krause & Radler (1966), as done in Moffatt (1970, 1974). We conceive of motions on two length scales L and ℓ with $L \gg \ell$ i.e.

$$\underset{\sim}{u} = \underset{\sim}{U}(\underset{\sim}{x}, t) + \underset{\sim}{u}'(\underset{\sim}{x}, t), \tag{4}$$

where $\underset{\sim}{U}$ is the large-scale field and $\underset{\sim}{u}'$ the small-scale field (which may be thought of either as turbulence or as a random wave field with some definite dispersion relation). Similarly we write

$$\underset{\sim}{B} = \underset{\sim}{B}_0(\underset{\sim}{x},t) + \underset{\sim}{b}(\underset{\sim}{x},t). \tag{5}$$

We use angular brackets to denote averages over the scale ℓ, so that

$$\langle \underset{\sim}{u} \rangle = \underset{\sim}{U}\ , \quad \langle \underset{\sim}{u}' \rangle = 0, \quad \langle \underset{\sim}{B} \rangle = \underset{\sim}{B}_0\ , \quad \langle \underset{\sim}{b} \rangle = 0. \tag{6}$$

The average of equation (1) is then

$$\frac{\partial \underset{\sim}{B}_0}{\partial t} = \nabla \wedge (\underset{\sim}{U} \wedge \underset{\sim}{B}_0) + \nabla \wedge \underset{\sim}{\mathcal{E}} + \lambda \nabla^2 \underset{\sim}{B}_0\ , \tag{7}$$

where

$$\underset{\sim}{\mathcal{E}} = \langle \underset{\sim}{u}' \wedge \underset{\sim}{b} \rangle \tag{8}$$

is a mean electromotive force generated by the small-scale motion, and the equation for $\underset{\sim}{b}$ is evidently

$$\frac{\partial \underset{\sim}{b}}{\partial t} = \nabla \wedge (\underset{\sim}{U} \wedge \underset{\sim}{b}) + \nabla \wedge (\underset{\sim}{u}' \wedge \underset{\sim}{B}_0) + \nabla \wedge (\underset{\sim}{u}' \wedge \underset{\sim}{b} - \langle \underset{\sim}{u}' \wedge \underset{\sim}{b} \rangle) + \lambda \nabla^2 \underset{\sim}{b}\ . \tag{9}$$

If we assume that $\underset{\sim}{b} = 0$ at some reference instant $t = 0$, is is clear that this equation establishes a linear relation between $\underset{\sim}{b}$ and $\underset{\sim}{B}_0$, and so between $\underset{\sim}{\mathcal{E}} = \langle \underset{\sim}{u}' \wedge \underset{\sim}{b} \rangle$ and $\underset{\sim}{B}_0$. Since the scale L of is assumed large, it may be anticipated that this linear relationship has a series representation of the form

$$\mathcal{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + \gamma_{ijk\ell} \frac{\partial^2 B_{0j}}{\partial x_k \partial x_\ell} + \cdots . \tag{10}$$

Since $\underset{\sim}{B}$ is a pseudo-vector and $\underset{\sim}{\mathcal{E}}$ a vector, it is evident that $\alpha_{ij}, \beta_{ijk}, \cdots$ are pseudo-tensors. It is also evident that these pseudo-tensors are determined by the statistical properties of the velocity field (including its mean value $\underset{\sim}{U}(\underset{\sim}{x},t)$) and by the value of the parameter λ. In general $\alpha_{ij}, \beta_{ijk}, \cdots$ vary on the large length-scale L and also on any (large) time scale of variation of mean quantities.

An important and illuminating (though idealised) special case is however that in which $\underset{\sim}{U} = 0$ and the statistical properties of $\underset{\sim}{u}'$ (e.g. $\langle \underset{\sim}{u}'^2 \rangle$) are stationary and homogeneous - i.e. do not vary with t or with $\underset{\sim}{x}$. In this case, α_{ij}, β_{ijk}, --- must be constants, and it may be noted that since α_{ij}, β_{ijk}, --- do not depend on $\underset{\sim}{B}_0$, α_{ij} may be derived on the assumption that $\underset{\sim}{B}_0$ is uniform, β_{ijk} on the assumption that $\partial B_{0j}/\partial x_k$ is uniform, etc., if this should prove convenient.

A further idealisation consists in the assumption of 'no preferred direction' in the statistical properties of $\underset{\sim}{u}'$. If these statistical properties are invariant under rotations of the frame of reference, then α_{ij}, β_{ijk}, --- share this property, so that necessarily,

$$\alpha_{ij} = \alpha\,\delta_{ij}\,, \quad \beta_{ijk} = -\beta\epsilon_{ijk}\,, \quad \gamma_{ijkl} = \gamma_1\,\delta_{ij}\delta_{kl} + \gamma_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),\;--- \tag{11}$$

where α is pseudo-scalar, β is a scalar γ_1, γ_2 , are pseudo-scalars, and so on. At this stage, the crucial importance of the concept of reflexional symmetry (or lack of it) becomes evident. If the turbulence is reflexionally symmetric, then all statistical properties of the turbulence must be invariant under change from a right-handed to a left-handed frame of reference. In particular, α , which is (implicitly) a statistical property of the turbulence (dependent also on λ) must in this case vanish. (Similarly γ_1 and γ_2 must vanish, but there is no need for β to vanish (and it will in fact invariably be non-zero)).

Turbulence that exhibits no preferred direction but that lacks reflexional symmetry may be described as <u>pseudo-isotropic</u>. Deferring for the moment the important question of how such a condition may be realised let us first look at some of the consequencies. Equations (10) and (11) together give

$$\underset{\sim}{\mathcal{E}} = \alpha \underset{\sim}{B_0} - \beta \nabla \wedge \underset{\sim}{B_0} + \cdots \tag{12}$$

The effect of subsequent terms in this series have not been investigated and are probably unimportant. Substitution of (12) in (7) (with for the moment $\underset{\sim}{U} = 0$) gives

$$\frac{\partial \underset{\sim}{B_0}}{\partial t} = \alpha \nabla \wedge B_0 + (\lambda + \beta) \nabla^2 \underset{\sim}{B_0} . \tag{13}$$

The β -term evidently represent a turbulent diffusivity effect, and it may confidently be asserted that β is always positive (although no general proof of this appears to be yet available). The α -term has a profoundly different character; the associated electromotive force in (12) is parallel to the local field $\underset{\sim}{B_0}$ - this is the α -effect of Steenbeck, Krause & Radler (1966) - and the resulting term in (13) dominates over the diffusion term when the scale L of $\underset{\sim}{B_0}$ is sufficiently large. Many of the consequencies of this phenomenon are brougth together in the article by Krause & Radler 1971.

The simplest physical interpretation of this effect is that originally due to Parker (1955), and illustrated in figure 1. Consider a localised motion having non-zero helicity, say positive (a 'cyclonic event' in Parker's terminology). This motion generates a loop in a mean field line, and twists the loop so that its normal has a component in the direction of the original undistorted field. Such a loop of field can be thought of as associated with a current in the direction of the normal. A random superposition of such events with positive mean helicity may then be expected to generate a mean current anti-parallel to the mean field. Diffusion tends to eliminate the field distortion so that the effect will be small when λ is large; in this limit, α will certainly be negative when the mean helicity is positive. The situation is not so clear when λ is small; in this limit loops may be twisted any number of times, and a twist of one loop through $2n\pi + \frac{\pi}{2}$ will tend to be cancelled by a twist of another loop through $2m\pi + 3\pi/2$ where m and n are integers If however the lifetime of the events is very short (as assumed by Parker) then only the limited twist picture of figure 1 will be relevant and again one would expect α to have the opposite sign from that of the mean helicity.

Let us look more closely at these two limits. If λ is in some sense large, then the diffusion term $\lambda \nabla^2 \underset{\sim}{b}$ in (9) will dominate over all other terms in the equation linear in $\underset{\sim}{b}$. As mentioned above, in determining α, we may assume that $\underset{\sim}{B}_o$ is uniform, and the appropriate limiting form of (9) becomes simply

$$\lambda \nabla^2 \underset{\sim}{b} = - \underset{\sim}{B}_o . \nabla \underset{\sim}{u}'. \tag{14}$$

In terms of Fourier transforms defined by

$$\underset{\sim}{u}' = \int \underset{\sim}{p}(\underset{\sim}{k}, t) e^{i\underset{\sim}{k}.\underset{\sim}{x}} d^3\underset{\sim}{k}, \quad \underset{\sim}{b} = \int \underset{\sim}{q}(\underset{\sim}{k}, t) e^{i\underset{\sim}{k}.\underset{\sim}{x}} d^3\underset{\sim}{k}, \tag{15}$$

this becomes

$$\lambda k^2 \underset{\sim}{q} = i(\underset{\sim}{B}_o \cdot \underset{\sim}{k}) \underset{\sim}{p} \tag{16}$$

and the mean electromotive force is

$$\mathcal{E}_i = \langle \underset{\sim}{u}' \wedge \underset{\sim}{b} \rangle_i = \epsilon_{ijk} \iint \langle p_j(\underset{\sim}{k}, t)\, q_k(\underset{\sim}{k}', t)\, e^{i(\underset{\sim}{k}+\underset{\sim}{k}').\underset{\sim}{x}} d^3\underset{\sim}{k}\, d^3\underset{\sim}{k}'. \tag{17}$$

Using (16), and translating into spectral terminology, this becomes

$$\mathcal{E}_i = \alpha_{ij} B_{oj} \tag{18}$$

where

$$\alpha_{ij} = i\epsilon_{ikl} \lambda^{-1} \int k^{-2} k_j\, \Phi_{kl}(\underset{\sim}{k})\, d^3\underset{\sim}{k}, \tag{19}$$

where $\Phi_{kl}(\underset{\sim}{k})$ is the usual Eulerian spectrum tensor of the turbulence

(Batchelor 1953). For pseudo-isotropic turbulence,

$$\Phi_{ij}(\underset{\sim}{k}) = \frac{E(k)}{4\pi k^4}\left(k^2\delta_{ij} - k_i k_j\right) + \frac{iF(k)}{8\pi k^4}\epsilon_{ijk}k_k , \tag{20}$$

where $E(k)$ is the energy spectrum function and $F(k)$ is the <u>helicity spectrum function</u> (a pseudo-scalar) satisfying

$$\langle \underset{\sim}{u}' \cdot \underset{\sim}{\omega} \rangle = \int_0^\infty F(k)\, dk . \tag{21}$$

Substitution of (20) in (19) leads to $\alpha_{ij} = \alpha\delta_{ij}$ where

$$\alpha = -\frac{1}{3\lambda}\int_0^\infty k^{-2} F(k)\, dk . \tag{22}$$

Note the dependence of α on the helicity spectrum and the (expected) appearance of the minus sign in (22). Note also that, as expected, $\alpha \to 0$ as $\lambda \to \infty$.

Consider now the opposite limit $\lambda \to 0$. If we ignore entirely the effects of diffusion (and it must be admitted that this is a dangerous procedure) the classical Cauchy solution if (1) is relevant. Let $\underset{\sim}{x}(\underset{\sim}{a},t) = \underset{\sim}{a} + \underset{\sim}{\xi}(\underset{\sim}{a},t)$ be the position at time t of the fluid particle that started at position $\underset{\sim}{a}$ at time $t=0$. Then, in Lagrangian notation,

$$B_i(\underset{\sim}{x},t) = B_j(\underset{\sim}{a},0)\, \partial x_i/\partial a_j , \tag{23}$$

and so

$$\begin{aligned} \mathcal{E}_i(\underset{\sim}{x},t) &= \langle \underset{\sim}{u}' \wedge \underset{\sim}{b} \rangle_i = \langle \underset{\sim}{u}' \wedge \underset{\sim}{B} \rangle_i \\ &= \langle \epsilon_{ijk}\, v_j(\underset{\sim}{a},t)\, B_\ell(\underset{\sim}{a},0)\, \partial x_k/\partial a_\ell \rangle , \end{aligned} \tag{24}$$

where $v_j(\underset{\sim}{a},t)\ (= u_j(\underset{\sim}{x},t))$ is the Lagrangian velocity. In order to derive an expression for α_{ij} in this limit, we may again assume that $\underset{\sim}{B_0}$ is uniform and that $\underset{\sim}{b} = 0$ at $t=0$. Under this latter assumption it naturally takes time for $\langle \underset{\sim}{u} \wedge \underset{\sim}{b} \rangle$ to build up from zero, and so α_{ij} will be time-dependent (in the previous limit this build-up was effectively instantaneous). From (24) we then have $\mathcal{E}_i = \alpha_{il} B_l$, where

$$\alpha_{il} = \epsilon_{ijk} \langle v_j(\underset{\sim}{a},t)\, \partial \xi_k / \partial a_l \rangle = \epsilon_{ijk} \int_0^t \langle v_j(\underset{\sim}{a},t)\, \partial v_k(\underset{\sim}{a},\tau)/\partial a_l \rangle \, d\tau \qquad (25)$$

In this form the close relation between this expression and the corresponding expression for the turbulent diffusion tensor for a scalar field (Taylor 1921)

$$D_{ij} = \int_0^t \langle v_i(\underset{\sim}{a},t)\, v_j(\underset{\sim}{a},\tau) \rangle \, d\tau \qquad (26)$$

is noteworthy; (this expression can be obtained by a method closely related to that described above for α_{il}) . In the pseudo-isotropic situation, again $\alpha_{il} = \alpha \delta_{il}$ and from (25),

$$\alpha = -\tfrac{1}{3} \int_0^t \langle \underset{\sim}{v}(\underset{\sim}{a},t) \cdot \nabla_{\underset{\sim}{a}} \wedge \underset{\sim}{v}(\underset{\sim}{a},\tau) \rangle \, d\tau. \qquad (27)$$

Note again the appearance of a type of helicity correlation, but this time in terms of the Lagrangian variables; and it may be remarked that one of the most intractable problems of turbulence is that of expressing Lagrangian correlations in terms of the more traditional Eulerian statistical quantities.

The expression (27) has a recognizably similar structure to the expression obtained by Parker (1971) in a reexamination and reformulation of his earlier theory. Parker's expression is meaningful however only within the framework of his representation of the turbulence by random short-lived cyclonic events and there is no guarantee that turbulence in general admits such a representation.

An expression for β comparable with the expression (27) for α has been obtained by Moffatt (1974). However there are difficulties which have not yet been fully resolved associated with the convergence of integrals such as (27) as $t \to \infty$; (the difficulties are more acute for the integrals appearing in the expression for β). If the integral in (27) does converge as $t \to \infty$, then the limiting value is undoubtedly the appropriate value for α in the limit $\lambda \to 0$. If the integral does not converge, then molecular diffusion effects must at some stage intervene, and the appropriate value of α the depends on λ in a non-trivial way in the limit $\lambda \to 0$. It is difficult to see how this question can be resolved by theoretical means, but numerical experimentation could perhaps be used to settle the matter.

We can now comment on the relation of the above approach to that of Braginskii (1964). Braginskii used a decomposition of the form (4), (5) for velocity and magnetic field in a spherical conductor, but with the angular brackets representing an average over the azimuthal angle φ , so that $\underset{\sim}{U}$ and $\underset{\sim}{B}_0$ are axisymmetric fields and $\underset{\sim}{u}'$ and $\underset{\sim}{b}$ are perturbation (non-axisymmetric) fields. Bragniskii assumed that these departures from axisymmetry were weak, and he concentrated furthermore on the weak diffusion limit $\lambda \to 0$. In solving the induction equation he developed a perturbation scheme in powers of $\lambda^{\frac{1}{2}}$ in which the magnetic field at leading order was purely toroidal (i.e. in the φ-direction). On the basis of the general arguments presented above it is to be expected that a mean electromotive force linearly related to the mean magnetic field will be generated. In fact Braginskii found a mean toroidal electromotive force $\mathcal{E}_\varphi$ given by $\mathcal{E}_\varphi = \alpha B_\varphi$ where $\alpha = \lambda \Gamma$ and Γ is a (pseudo-scalar) quadratic functional of the velocity perturbation field $\underset{\sim}{u}'$. In other words, Braginskii also found an α-effect, but with the very significant difference that his expression for α vanishes in the limit $\lambda \to 0$ (unlike the expression (27) above which does not show any dependence on λ , and unlike the similar expression given by Parker (1971)).

Braginskii's theory has been greatly elucidated by Soward (1972) who has shown how Braginskii's results may be obtained in terms of a Lagrangian forumlation. In Soward's formulation, the reason for the above dependence of α on λ is traced directly to the transformation of the diffusion term in the induction equation under the change to Lagrangian variables. The variables that Bragniskii introduced and described as "effective variables" also appear in a much more natural way from Soward's approach.

The crucial role that diffusion may play may be illustrated in very simple terms as follows. Consider the effect of the velocity field

$$\underset{\sim}{u} = u_0 \left(0 , \sin(kx - \sigma t), \cos(kx - \sigma t) \right) \tag{28}$$

on the magnetic field $(B_0, 0, 0)$. The vorticity associated with (28) is

$$\underset{\sim}{\omega} = k \underset{\sim}{u} . \tag{29}$$

This is in fact a motion of 'maximal helicity' (Kraichnan 1973). If λ is equal to zero, the corresponding magnetic perturbation $\underset{\sim}{b}$ is given by $\partial \underset{\sim}{b}/\partial t = B_0 \partial \underset{\sim}{u}/\partial x$ and is

$$\underset{\sim}{b} = -(k/\sigma) \underset{\sim}{u} , \tag{30}$$

so that $\langle \underset{\sim}{u} \wedge \underset{\sim}{b} \rangle = 0$, and there is no α -effect. If however $\lambda \neq 0$ then the phase of $\underset{\sim}{b}$ is shifted slightly relative to that of $\underset{\sim}{u}$ and $\underset{\sim}{u} \wedge \underset{\sim}{b}$ no longer vanishes. It may be verified that in this case

$$\underset{\sim}{u} \wedge \underset{\sim}{b} = \alpha \underset{\sim}{B}_0 \quad , \quad \alpha = - \frac{u_0^2 k^3 \lambda}{\lambda^2 k^4 + \sigma^2} . \tag{31}$$

The value of α does indeed vanish as $\lambda \to 0$. Some dissipation appears essential if a wave motion of the form (25) (or equally a random superposition of such waves) is to provide an α -effect. Braginskii's velocity fields are more akin to this type of motion than to the sort of turbulence in which any pair of particles drift further and further apart with increasing time.

4. Solution of the dynamo equations

If we now focus attention on the equations for the mean field $\underset{\sim}{B_0}$, we may forget about the background turbulence except in so far as it provides an α-effect and an enhanced diffusivity through the β-term. In the simplest case described by equation (13) it is easy to see how exponentially growing modes can appear. If α is say positive (corresponding most probably to negative mean helicity) then magnetic modes having a 'force-free' structure satisfying

$$\nabla\wedge \underset{\sim}{B_0} = K\underset{\sim}{B_0} \quad , \quad \nabla^2\underset{\sim}{B_0} = -K^2\underset{\sim}{B_0} \tag{32}$$

evidently evolve like $e^{\omega t}$ where

$$\omega = \alpha K - (\lambda+\beta)K^2 \tag{33}$$

and we have dynamo action provided

$$K < (\lambda+\beta)^{-1}\alpha, \tag{34}$$

i.e. provided the scale of the field $\underset{\sim}{B_0}$ is sufficiently large. For example the field given at time $t=0$ by

$$\underset{\sim}{B_0} = B_0\,(0, \cos Kx, \sin Kx) \tag{35}$$

evolves in this way.

Such a field applies only to a conducting fluid of infinite extent, and serves merely to indicate how easily the α-effect can overcome the effects of ohmic and turbulent diffusion in promoting dynamo action. The situation of greater relevance in the planetary and solar contexts is that in which the fluid is confined to a sphere, α is antisymmetric about the equatorial plane (for reasons that stem from simple dynamical considerations) and the mean velocity $\underset{\sim}{U}$ may well be non-zero. The

The effective induction equation is then

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \wedge (\alpha \underset{\sim}{B}) + \nabla \wedge (\underset{\sim}{U} \wedge \underset{\sim}{B}) + \lambda_e \nabla^2 \underset{\sim}{B}, \tag{36}$$

where we have dropped the suffix on $\underset{\sim}{B}$ and where $\lambda_e = \lambda + \beta$ is assumed uniform. This equation, or minor variants of it, emerges as described in §3 above from the theories of Parker, of Braginskii, and of Ste nbeck, Krause and Radler, and solutions have been extensively explored by computational means for prescribed choices of the functions $\alpha(\underset{\sim}{x})$ and $\underset{\sim}{U}(\underset{\sim}{x})$ - (see particularly Roberts 1972). For the most part in such investigations $\underset{\sim}{U}$ and $\underset{\sim}{B}$ are assumed axisymmetric. Among Roberts's most interesting conclusions are the following:

(i) If the velocity field is purely toridal and if the dynamo operates according to the scheme first proposed by Parker (1955) in which toroidal field is generated from poloidal field by differential rotation, while poloidal field is regenerated from toroidal field by the α - effect, then the most easily excited dynamo mode is not steady but is oscillatory in time; when the sign of α and the radial gradient of differential rotation are suitably related, this mode has a dipole structure and the periodic evolution consists of amplifying waves propagating from the poles towards the equator. Such oscillatory modes can be used to explain features of the Sun's 22-year sunspot cycle if it is assumed that sunspots appear by eruption of a toroidal field below the surface when it reaches a certain critical level (see for example Steenbeck and Krause (1969)).

(ii) If in addition to the differential rotation, there is a meridional circulation of sufficient strength and structure, then the most easily excited mode is steady; the relative steadiness of the Earth's dipole field suggests that meridional circulation may be important in this context. If the meridional circulation becomes too strong, then the dynamo fails, presumably because the poloidal flux is then excluded from the region of regeneration by the flux expulsion mechanism studied by Weiss (1966).

It should be mentioned however that these conclusions are quite sensitive to the particular assumptions made regarding the assumed forms of α and $\underset{\sim}{U}$. As Deinzer (1974) has shown, the question of whether the preferred mode of excitation is oscillatory or steady can depend critically on the degree of spatial separation of the regions of α-activity and differential rotation activity, as well as on the influence of meridional circulation. It is of course dynamical considerations that determine $\alpha(\underset{\sim}{x})$ and $\underset{\sim}{U}(\underset{\sim}{x})$, and the ultimate question of whether a dynamo will be oscillatory or steady in character cannot therefore be wholly separated from these dynamical considerations.

5. Dynamical equilibration

When a magnetic field grows exponentially as in the simple situation described in the first paragraph of §4, this situation cannot of course persist indefinitely, but only for so long as the velocity field remains unaffected by Lorentz forces. These forces, being quadratic in the magnetic field, must eventually intervene and modify the motion in such a way as to provide some kind of equilibrium, either steady or oscillatory. The Lorentz force $\underset{\sim}{j} \wedge \underset{\sim}{B}$ may have two effects:

(i) It may have a damping effect on the small-scale motions thus tending to decrease the α-effect; this effect has been studied in detail, (Moffatt 1970b, 1972, Soward 1975) in the situation where the small-scale velocity field consists of a random superposition of inertial waves in a rotating conducting fluid. It may be noted that each constituent wave has a circularly polarised structure like that of the velocity field (28), and that such waves are therefore particularly conducive to dynamo action.

(ii) It may also influence the mean velocity field $\underset{\sim}{U}(\underset{\sim}{x})$. A field of simple structure such as that given by (35) may be force-free $((\nabla \wedge \underset{\sim}{B}_0) \wedge \underset{\sim}{B}_0 = 0)$ in which case this effect does not arise. But a field in a conductor of finite extent and with no external sources cannot be everywhere force-free, and in this case the effect must be present. Even if we start with a situation in which $\underset{\sim}{U}=0$, the mean Lorentz force associated with a growing magnetic field will drive a growing mean velocity $\underset{\sim}{U}(\underset{\sim}{x},t)$ which ultimately influences the growth of the field through its appearance in the mean induction equation (7). The formalism necessary to handle such an effect has been developed by Malkus & Proctor (1975), and convincing computational evidence for the approach to equilibration has been givenbby Proctor (1975).

Oneextremely interesting aspect of this study is the way in which a constraint due to J.B. Taylor (1963) is incorporated in the analysis. This constraint arises under circumstances when Coriolis and Lorentz forces dominate over inertia and viscous forces so that the controlling dynamic equation for the mean flow is

$$2\rho\,\underset{\sim}{\Omega}\wedge\underset{\sim}{u} = -\nabla p + \underset{\sim}{j}\wedge\underset{\sim}{B}\,. \tag{37}$$

Here ρ is the fluid density, $\underset{\sim}{\Omega}$ the global rotation rate, p is pressure, and $\underset{\sim}{j}$ and $\underset{\sim}{B}$ the mean current and mean field respectively. When this equation holds in fluid contained in any cavity axisymmetric about the direction of $\underset{\sim}{\Omega}$, as shown by Taylor, the field must satisfy for each s the constraint

$$\int_{C(s)} \left[(\nabla\wedge\underset{\sim}{B})\wedge\underset{\sim}{B}\right]_{\varphi}\, d\varphi\, dz = 0\,, \tag{38}$$

where (s,φ,z) are cylindrical polar co-ordinates, and $C(s)$ is the surface of a cylinder of radius s intersecting the cavity boundary, and with axis parallel to $\underset{\sim}{\Omega}$.

Now in general a field of the form $\underset{\sim}{B}(\underset{\sim}{x})e^{\omega t}$ that results from solution of an equation of the form (36) with appropriate boundary conditions will <u>not</u> satisfy the constraint (38). In this case the mean Lorentz force must so modify the mean velocity $\underset{\sim}{U}$ (and hence the mean field $\underset{\sim}{B}$) that the constraint is more nearly satisfied. This will lead either to a force balance of the form (37) (magnetogeostrophic balance), or possibly to torsional oscillations about such a state. Some aspects of this problem were antidpated by Childress (1969) and also by Braginskii (1967,1970). Proctor's (1975) work is limited to the situation where $\underset{\sim}{U}$ is driven entirely by the mean Lorentz force and where the initial growth of the field is due entirely to an α -effect with α antisymmetric about the equatorial plane. Within this limitation he has found convincing evidence an approach to equilibration under the effects of the growing mean Lorentz force as described above. Further work along these lines is still required to determine whether such phenomena as sudden reversals of the Earth's magnetic field may also be embraced by the model.

Finally, I should refer to parallel attacks on dynamic aspects of the dynamo problem from the point of view of convective instability theory. When fluid between two horizontal plates is subjected to a temperature gradient by heating the lower plate, thermal convection sets in when the Rayleigh number is sufficiently large. The nature of the instability and the structure of the convective cells are of course sensitive both to Coriolis forces if the fluid is rotating about an axis normal to the planes and to Lorentz forces if, say, a horizontal magnetic field is present. Different aspects of this stability problem have been given recent close study by Braginskii (1970), Bugse (1970,1976), Eltayeb (1972,1975), and Roberts and Stewartson (1974,1975). A feature of the convection pattern when the rotation $\underset{\sim}{\Omega}$ is strong (and yet the Rayleigh number is still sufficiently large for convection to occur) is that the horizontal scale is small compared with the vertical scale (the ratio being $O(\Omega^{-\frac{1}{3}})$). Soward (1974) has exploited this fact in order to develop, by the methods of the mean field electrodynamics of §4, a detailed treatment of the dynamo action associated with such motions, and he provides evidence for the existence of such dynamo action in which the magnetic energy density fluctuates about an average value. Busse (1973) similarly demonstrated the existence of a convectively driven dynamo; in this case there was no overall rotation, and the motion consisted of convection in roll-type cells with horizontal axes together with an imposed shear flow parallel to these axes.

These studies (and others too numerous to mention) provide convincing evidence that the magnetic field of the Earth (and by inference of those other planets that have magnetic fields) may indeed be explained in terms of internal dynamo action due to motion in a liquid core having both large-scale and small-scale ingredients. The ultimate source of energy for these motions is still a matter for intense debate - whether convective as in the models mentioned above, or gravitational due to bulk transfer of matter in the radial direction (Braginskii 1964b), or

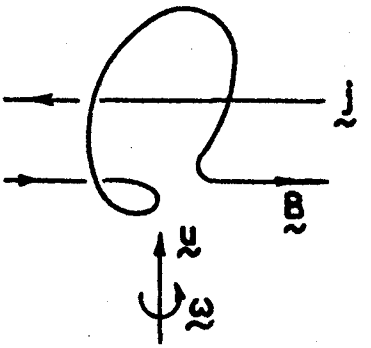

Figure 1. Effect of a localised motion with helicity on a magnetic line of force.

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