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NOTES ON THE 1978 SUMMER STUDY PROGRAM ON DYNAMO MODELS OF GEOMAGNETISM IN GEOPHYSICAL FLUID DYNAMICS AT THE WOODS HOLE OCEANOGRAPHIC INSTITUTION

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The oxymoronic role of molecular diffusivity in the dynamo process

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Abstract

The delicate question concerning the behaviour of the regeneration coefficient α and the turbulent diffusivity β in the limit of vanishing molecular diffusivity ($\eta \to 0$) in helical turbulence is discussed, in the light of an exact result of Bondi & Gold (1950) viz. when $\eta = 0$ the external dipole moment of a current distribution in a sphere is permanently bounded.

§1. The oxymoron is a figure of speech which embodies an apparent contradiction; e.g. creative destruction, relaxed tension, devastating triviality, etc. The oxymoronic role of molecular diffusivity η $(= (\mu_0\sigma)^{-1})$ is this: that while non-zero diffusivity ($\eta > 0$) is directly responsible for the natural ohmic processes of dissipation and decay, it is also indirectly responsible for the means of regeneration of the magnetic field; the dynamo process may be described as a process of 'regenerative decay', or perhaps better 'reinvigorating dissipation'.

§2. Consider the dipole moment $\underset{\sim}{\mu}(t)$ associated with a current distribution $\underset{\sim}{j}(\underset{\sim}{x},t) = \mu_0^{-1}\nabla \wedge \underset{\sim}{B}$ in a conducting sphere $V: r < a$. This is given by various alternative expressions:

$$8\pi\underset{\sim}{\mu}(t) = \mu_0 \int_V \underset{\sim}{x} \wedge \underset{\sim}{j}\, dV = 3\int_V \underset{\sim}{B}\, dV = 3\int_S \underset{\sim}{x}(\underset{\sim}{B}.\underset{\sim}{n})\, dS\ , \tag{1}$$

where S is the surface $r = a$; and its rate of change is given by

$$\frac{8\pi}{3}\frac{d\underset{\sim}{\mu}}{dt} = \int_V \frac{\partial \underset{\sim}{B}}{\partial t}\, dV = \int_S (\underset{\sim}{n} \wedge \underset{\sim}{E})\, dS. \tag{2}$$

With $\underset{\sim}{E} = -\underset{\sim}{u} \wedge \underset{\sim}{B} + \eta\nabla \wedge \underset{\sim}{B}$, and $\underset{\sim}{n}.\underset{\sim}{u} = 0$ on S, this gives

$$\frac{8\pi}{3}\frac{d\underset{\sim}{\mu}}{dt} = \int_S \underset{\sim}{u}(\underset{\sim}{n}.\underset{\sim}{B})\, dS - \eta\int \underset{\sim}{n} \wedge (\nabla \wedge \underset{\sim}{B})\, dS. \tag{3}$$

The first term on the right describes the mechanism identified by Bondi & Gold (1950) for increase of the dipole moment: field sweeping towards the magnetic poles (defined by the instantaneous direction of the vector $\underset{\sim}{\mu}$) can increase $|\underset{\sim}{\mu}|$, but, as emphasised by Bondi & Gold, this mechanism is strictly limited when $\eta = 0$, since $|\underset{\sim}{\mu}|$ then attains a finite maximum when all the flux of $\underset{\sim}{B}$ is concentrated at opposite ends of a diameter of the sphere (as in an elementary bar magnet). To see this explicitly from the above equations, let $S_\pm$ denote those parts of S on which $\underset{\sim}{n}.\underset{\sim}{B} >$ or < 0 respectively, and let

$$\underset{\sim}{\mu}_\pm = \frac{3}{8\pi}\int_{S_\pm} \underset{\sim}{x}(\underset{\sim}{n}.\underset{\sim}{B})\, dS \tag{4}$$

so that $\underset{\sim}{\mu} = \underset{\sim}{\mu}_+ + \underset{\sim}{\mu}_-$. We then have

$$|\underset{\sim}{\mu}_+| \leq \frac{3}{8\pi} a\Phi , \qquad |\underset{\sim}{\mu}_-| \leq \frac{3}{8\pi} a\Phi, \tag{5}$$

where $$\Phi = \int_{S_+} (\underset{\sim}{n}.\underset{\sim}{B})dS = -\int_{S_-} (\underset{\sim}{n}.\underset{\sim}{B})dS. \tag{6}$$

Now, when $\eta = 0$, Φ is constant, since flux through <u>every</u> closed material circuit is conserved, and so

$$|\underset{\sim}{\mu}| \leq |\underset{\sim}{\mu}_+| + |\underset{\sim}{\mu}_-| \leq \frac{3}{4\pi} a\Phi \tag{7}$$

the maximum being attained only when the flux is entirely concentrated at the poles, as mentioned above.

§3. There can therefore be no doubt that, when $\eta = 0$, exponential increase of the dipole moment is impossible, no matter what the complexity (laminar or turbulent) of the velocity field in V may be. The situation is transformed if $\eta > 0$, because then <u>diffusive increase</u> in the dipole moment (represented by the second term of (3) is possible, provided the velocity field is such as to maintain a field with a suitably negative gradient near the boundary $r = a$.

§4. The impossibility of <u>sustained</u> dynamo action (in the sense of an exponentially increasing external dipole moment) applied equally to such basic systems as the homopolar disc dynamo. If the disc conductivity is infinite, then the magnetic flux across it <u>cannot</u> change with time, and exponential growth of the magnetic field associated with the device is impossible no matter how fast we rotate the disc or how ingeniously we twist the wire, and whatever conventional wisdom may tell us to the contrary. In terms of growth rate, if, in general, $B \propto e^{pt}$, then p must depend on the disc Reynolds number R_m in the manner indicated in figure 1. It is reasonable to conjecture that fluid dynamos also must behave in this manner.

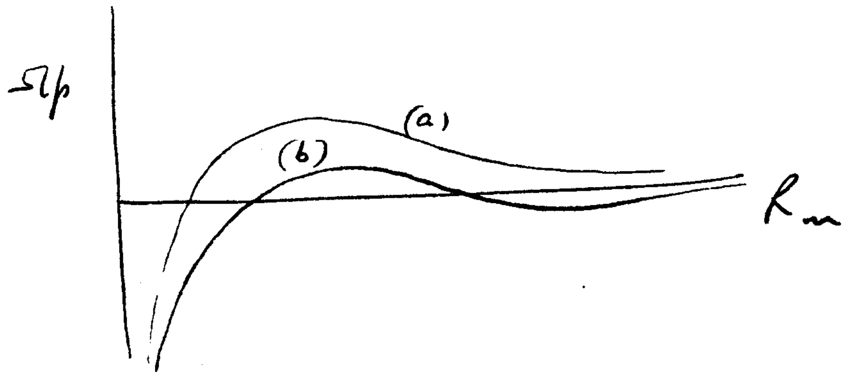

Figure 1. Possible dependence of p on R_m for homopolar disc dynamo. (a): wire resistance zero; (b): wire resistance non-zero. In either case, $p \to 0$ as $R_m \to \infty$.

§5. Consider now the situation in mean-field electrodynamics, in which, in conventional notation,

$$ {}_i = \langle \underset{\sim}{u} \wedge \underset{\sim}{b} \rangle_i = \alpha_{ij} B_{oj} + \beta_{ijk}\, \partial B_{oj} / \partial x_u + \cdots \tag{8}$$

where $\underset{\sim}{B}_0(\underset{\sim}{x},t) = \langle \underset{\sim}{B}(\underset{\sim}{x},t)\rangle$ is the large-scale (mean) field, and $\underset{\sim}{b} = \underset{\sim}{B} - \underset{\sim}{B}_0$. Under first-order smoothing theory (Moffatt 1978 - hereafter referred to as M - chap.7) we have the results

$$\alpha \equiv \frac{1}{3}\alpha_{ii} = -\frac{1}{3}\eta \iint \frac{k^2 F(k,\omega)}{\omega^2 + \eta^2 k^4}\, dk\, d\omega \,, \tag{9}$$

$$\beta \equiv \frac{1}{6}\epsilon_{ijk}\beta_{ijk} = \frac{2}{3}\eta \iint \frac{k^2 E(k,\omega)}{\omega^2 + \eta^2 k^4}\, dk\, d\omega, \tag{10}$$

where $F(k,\omega)$, $E(k,\omega)$ are the helicity and energy spectrum functions of the random $\underset{\sim}{u}$-field. If

$$F(k,\omega) = O(\omega^2), \quad E(k,\omega) = O(\omega^2) \quad \text{as} \quad \omega \to 0, \tag{11}$$

then clearly

$$\alpha \sim \alpha_0{}'\eta \,, \quad \beta \sim \beta_0{}'\eta \quad \text{as} \quad \eta \to 0, \tag{12}$$

where $\alpha_0{}'$ and $\beta_0{}'$ are in general non-zero constants ($\beta_0{}' > 0$). This is clearly the situation when the $\underset{\sim}{u}$-field is a field of random waves with no zero-frequency ingredients. In this case, the regenerative process normally associated with the pseudo-scalar α vanishes as $\eta \to 0$, consistent with the remarks of §1. It may be noted that the theory of Bragniskii (M, chap.8) gives an expression for the regenerative coefficient very similar to (9), and again with the property $\alpha = O(\eta)$ as $\eta \to 0$.

§6. Difficulties arise however if the $\underset{\sim}{u}$-field has non-zero spectral density at $\omega = 0$, as is the case for conventional turbulence. The zero-frequency ingredients of the turbulence are precisely those that are responsible for the dispersion of particles in a turbulent flow, and they are of vital importance also in the field-line-stretching context. It must be noted however that results such as $\langle \underset{\sim}{\xi}^2 \rangle \sim 2Dt$ for the relative dispersion of two particles separated by vector distance $\underset{\sim}{\xi}(t)$ is ultimately limited by the physical dimensions of the fluid domain; and care may then be needed in carrying over asymptotic results from strictly homogeneous turbulence to turbulence in a finite domain, particularly when these results are sensitive to the limiting $(t \to \infty)$ behaviour.

§7. When $\eta = 0$, there is an alternative approach to the determination of the coefficients α and β using Lagrangian averages. If at some instant $t = 0$, the $\underset{\sim}{u}$ and $\underset{\sim}{b}$ fields are uncorrelated, then α and β are functions of t (which clearly vanish at $t = 0$). The Lagrangian procedure (M, §7.10) leads to the expressions

$$\alpha(t) = -\frac{1}{3}\int_0^t \langle \underset{\sim}{v}(t).\nabla \wedge \underset{\sim}{v}(\tau)\rangle d\tau \tag{13}$$

$$\beta(t) = \frac{1}{3}\int_0^t \langle \underset{\sim}{v}(t).\underset{\sim}{v}(\tau)\rangle d\tau + \int_0^t \alpha(t)\alpha(\tau)d\tau$$

$$+ \frac{1}{6}\int_0^t\int_0^t \langle \underset{\sim}{v}(t).\underset{\sim}{v}(\tau_2)\nabla_{\underset{\sim}{a}}.\dot{\underset{\sim}{v}}(\tau_1) - (\underset{\sim}{v}(t).\nabla_{\underset{\sim}{a}}\underset{\sim}{v}(\tau_1))\underset{\sim}{v}(\tau_2)\rangle d\tau_1 d\tau_2 , \tag{14}$$

where $\underset{\sim}{v}(t)$ is the velocity of the fluid particle initially at position $\underset{\sim}{a}$. The difficulty here is to determine how these expressions behave for a typical field of homogeneous turbulence as $t \to \infty$. Kraichan (1976 a,b) has argued that, in the case of turbulence with non-zero helicity,

$$\alpha(t) \sim \alpha_0 , \quad \beta(t) \sim \beta_0 \quad \text{as} \quad t \to \infty , \tag{15}$$

the apparent positive divergence in the second term of (14) being cancelled by an equal negative divergence in the third term, which involves the awkward triple Lagrangian correlations. Kraichnan's arguments rest in part on comparison with the results of first-order smoothing theory in situations where both approaches (first order smoothing and Lagrangian) may be expected to be valid, and in part on numerical evaluation of $\alpha(t)$ and $\beta(t)$ for velocity fields with prescribed Eulerian statistics. Further numerical experimentation is needed however, before the results (15) can be regarded as absolutely and definitively established. Let us nevertheless accept (15), and pursue the consequences in the context of α^2- and $\alpha\omega$-dynamo models.

§8. For an α^2-dynamo in a sphere $r < a$ (M. chap.9), the growth rates have the form

$$p = \frac{\eta_e}{a^2} F(R_\alpha) \tag{16}$$

where $\eta_e = \eta + \beta$, and

$$R_\alpha = |\alpha| a^2/\eta_e , \tag{17}$$

and dynamo action occurs when $F(R_\alpha) > 0$. This generally occurs for the simplest mode of dipole symmetry when

$$R_\alpha > R_{\alpha c} \tag{18}$$

where $R_{\alpha c}$ is a positive number of order unity which depends on the precise assumption made about any large-scale variation of α throughout the sphere. Let us suppose that, as $\eta \to 0$, the relevant behaviour of α and β (cf 15) is

$$\alpha \sim \alpha_0 \,, \quad \beta \sim \beta_0 \quad \text{as} \quad \eta \to 0. \tag{19}$$

Then (16) becomes

$$p \sim \frac{\beta_0}{a^2} F(\bar{R}_\alpha), \quad \bar{R}_\alpha = \frac{|\alpha_0| a^2}{\beta_0} \; . \tag{20}$$

The condition $R_\alpha > R_{\alpha c}$ is certainly satisfied if a is large enough, and then p tends to a strictly positive value as $\eta \to 0$, implying exponential increase of the mean field, and in particular of the external dipole moment. This appears to be in fundamental conflict with the Bondi & Gold result (7), which applies when $\eta = 0$ whatever the complications of the velocity field, and whether laminar or turbulent.

The conflict does not arise under the alternative limiting behaviour (12). In this case,

$$p \sim \frac{\eta(1 + \beta_0')}{a^2} F\left(\frac{|\alpha_0'| a^2}{1 + \beta_0}\right) \to 0 \quad \text{as} \quad \eta \to 0 \tag{21}$$

and the dipole moment does not grow exponentially in the limit $\eta = 0$.

§9. For dynamos of $\alpha\omega$-type, growth rates are generally given by

$$p = \frac{\eta_e}{a^2} F(X) \;, \quad X = \frac{|\alpha| \; a^3}{\eta_e^{\;2}} \;, \tag{22}$$

where is a measure of the shear associated with differential rotation. The condition for dynamo action is now of the form

$$X > X_c \;, \tag{23}$$

where X_c is model-dependent, but generally of order unity. Again under the behaviour (19), as $\eta \to 0$,

$$X \to \bar{X}_c = |\alpha_0| \;\; a^3/\beta_0^{\;2} \;, \tag{24}$$

and

$$p \sim \frac{\beta_0}{a^2} f(\bar{X}) > 0 \quad \text{if} \quad \bar{X} > X_c \quad , \tag{25}$$

and we encounter the same fundamental conflict with the Bondi & Gold result.

Under the alternative behaviour (12),

$$p \sim \eta \frac{(1+\beta_0{}')}{a^2} F\left(\frac{|\alpha_0{}'| \quad a^3}{\eta(1+\beta_0{}')^2}\right). \tag{26}$$

To determine the behaviour of p as $\eta \to 0$, we need to know the behaviour of $F(X)$ as $X \to \infty$. If $F(X) = o(X)$ as $X \to \infty$, then $p \to 0$ as $\eta \to 0$, and conflict with Bondi & Gold is avoided. The asymptotic behaviour of $F(X)$ as $X \to \infty$ does not appear to have been investigated for $\alpha\omega$-dynamos in a spherical geometry. A clue is however provided by the results for an $\alpha\omega$-dynamo in a Cartesian geometry (modelling the galactic disc). For this case, which can be solved completely (M, §9.9),

$$f(X) \sim \log X \quad \text{as} \quad X \to \infty \tag{27}$$

and so $p \to 0$ as $\eta \to 0$ as required.

§10. It is hard to escape the conclusion that the result (19) cannot be correct, or that, if it is correct in homogeneous turbulence, it is, for some deep reason, not applicable when the turbulence is confined to a finite region (see the remarks of §6).

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Acknowledgement

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