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gemacht. Auf diesem Wege konnten z. B. Probleme der aktiven Dämpfung und der Optimierung von Kreiselsystemen erfolgreich behandelt werden. Insbesondere hat sich die Einführung des Begriffes der „durchdringenden Dämpfung" als überaus nützlich erwiesen. Man versteht darunter solche Dämpfungskräfte, die auch die Bewegungen in den nicht unmittelbar gedämpften Freiheitsgraden beeinflussen können.

Aktuelle Gebiete, auf denen zur Zeit gearbeitet wird und auf denen sicher auch weiterhin Forschungsarbeiten lohnend erscheinen, sind:

— Anpassung der analytischen Methoden an die numerischen Möglichkeiten modernen Rechenanlagen,
— Computereinsatz für ein zumindest teilweises Ableiten der Bewegungsgleichungen, besonders bei Vorliegen komplizierter kinematischer Bedingungen,
— Untersuchung aktiver Kreiselsysteme mit geregelten Teilsystemen,
— Erforschung von Starrkörper-Systemen mit kontinuierlichen, festen oder flüssigen Teilsystemen,
— Berücksichtigung nichtlinearer Effekte, insbesondere Ausweitung der für lineare Systeme erhaltenen Ergebnisse auf nichtlineare.

Ganz allgemein sollten globale Ergebnisse gegenüber den nur für Sonderfälle geltenden punktuellen Ergebnissen bevorzugt werden, damit der Einblick in das physikalische Verhalten komplizierter Systeme vertieft wird. Als Beispiel dieser Art seien die globalen Stabilitätssätze genannt, deren Brauchbarkeit auch zur Lösung konkreter Probleme in der letzten Zeit eindrucksvoll bestätigt werden konnte.

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Some Problems in the Magnetohydrodynamics of Liquid Metals

When electric currents are caused to flow in an electrically conducting fluid, either by the external application of time-periodic magnetic fields or by the application of large electric potential gradients at the boundary, the associated LORENTZ force is in general rotational and a fluid motion, which may be laminar or turbulent, is in general established. Three prototype problems, on which some progress has been made over the last decade, are reviewed: (i) the problem of the generation of rotation in a liquid metal by the application of a rotating magnetic field; (ii) the generation of cellular motion by the application of an alternating field of fixed direction; and (iii) the problem of the generation of fluid motion by the injection of steady current at a point electrode on the fluid boundary. All three problems are of importance in molten metal technology.

1. Introduction

I would like first to thank the Organising Committee for inviting me to give this general lecture. The title that was proposed to me was 'Magnetohydrodynamics' but I felt that it would be hard to do justice to such a wide subject in a single lecture, and I thought it more useful to select a few special problems of some practical importance within the field of liquid metal MHD and to give a brief review of progress on these problems.

It is usual to think of magnetohydrodynamics as a relatively young subject, and it has of course developed enormously over the last 20 years under the stimulus of thermonuclear fusion research, and of parallel research programmes in many centres on astrophysical and geophysical fluid dynamics. The problems that I propose to discuss are however much more mundane and were in fact known to metallurgists long before the word 'magnetohydrodynamics' was invented. Significant theoretical progress on these problems has been made only over the last ten years or so, and much remains to be done in bridging the gap between viable mathematical models and the practical realities of the situations considered.

Liquid metals such as molten aluminium or steel are good conductors of electricity and the electric current distribution $\boldsymbol{j}(\boldsymbol{x}, t)$ can interact with the associated magnetic field distribution $\boldsymbol{B}(\boldsymbol{x}, t)$ to give a LORENTZ force distribution $\boldsymbol{F} = \boldsymbol{j} \wedge \boldsymbol{B}$ which may have very strong dynamic effects. In general, this force is rotational and it therefore necessarily drives a rotational velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ whose distribution is of course controlled by inertia and viscous effects. Let u_0 be a velocity scale characterising $\boldsymbol{u}$, and L a length scale characterising the geometry of the container. Then the magnetic REYNOLDS number is defined by

$$R_m = \mu_0 \sigma L u_0 \,, \tag{1}$$

where σ is the electric conductivity of the fluid and μ_0 the permeability of free space. If $R_m \ll 1$, then the velocity field has only a weak perturbing effect on the magnetic field distribution, and this may be determined to good approximation by neglecting the fluid motion, i.e. by treating the conductor as if it were solid.

The current $\boldsymbol{j}$ and field $\boldsymbol{B}$ are related by AMPERE's law

$$\mu_0 \boldsymbol{j} = \nabla \wedge \boldsymbol{B} \,, \tag{2}$$

and if conditions are such that $\boldsymbol{B}$ (and so $\boldsymbol{j}$) are known functions of position, then $\boldsymbol{F}$ is also known. Suppose that the fluid is contained in a finite volume V with fixed rigid surface S, and that $\boldsymbol{F}(\boldsymbol{x})$ is steady. Then, if $\boldsymbol{u}(\boldsymbol{x})$ is the corresponding steady velocity field and $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$ the corresponding vorticity distribution, we have, from the NAVIER-STOKES equation for incompressible steady flow,

$$-\boldsymbol{u} \wedge \boldsymbol{\omega} = -\frac{1}{\varrho} \nabla \left(p + \frac{1}{2} \varrho u^2\right) + \boldsymbol{F} - \nu \nabla \wedge \boldsymbol{\omega} \,. \tag{3}$$

The streamlines within V may be closed, or they may cover surfaces (it is easy for example to imagine a situation in which each streamline covers a member of a family of nested toroids). Suppose first that C is a closed streamline. The line integral of (3) round C gives

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x} = \nu \int_C \mathrm{d}\boldsymbol{x} \cdot (\nabla \wedge \boldsymbol{\omega}) \,. \tag{4}$$

It is evident from this that, no matter how small the viscosity of the fluid may be, it is viscous effects alone that can limit the growth of circulation round C when $\boldsymbol{F}$ is rotational.

More generally, if J is a closed surface entirely within V, on which $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, we may easily deduce from (3) that

$$\int_J \boldsymbol{u} \cdot \boldsymbol{F} \, \mathrm{d}S = \nu \int_J \boldsymbol{u} \cdot (\nabla \wedge \boldsymbol{\omega}) \, \mathrm{d}S \,, \tag{5}$$

and if the left-hand side is non-zero it again follows that the kinetic energy of the motion generated is limited only by viscous effects.

2. The rotating field problem

Fluid contained within a closed surface S can be set in rotation by the application of a rotating magnetic field in the exterior region. This phenomenon was investigated by BRAUNBECK [1] (1932) with the object of devising a method for the measurement of liquid conductivity. The sample, enclosed in a small cylindrical container, is suspended with its axis vertical, and a rotating horizontal field is applied. When suitably calibrated, the rotation of the cylinder about the vertical axis provides a measure of the liquid conductivity (OZELTON and WILSON [2] 1966). The advantage of this method, over the more direct conventional method of simply passing a direct current through the sample, is that it avoids the need for contact between the liquid and inserted solid electrodes.

The rotating field can be regarded as the superposition of two uniform alternating fields out of phase and at right angles. When the field frequency ω is large (compared with $(\mu_0 \sigma L^2)^{-1}$), it penetrates only a small distance $\delta = O(\mu_0 \sigma \omega)^{-1/2}$ into the conductor (the skin effect). The associated LORENTZ force $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{t})$, which in general has a mean component $\boldsymbol{F}_0(\boldsymbol{x}) = \langle \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{t}) \rangle$ and a periodic component with frequency 2ω, is then confined to this thin magnetic boundary layer.

The situation is very easily described for the idealised situation in which the cylinder containing the liquid is of circular cross-section and of infinite length (figure 1). The magnetic field is best represented in terms of its vector potential $A\boldsymbol{k}$, where $\boldsymbol{k}$ is the unit vector $(0, 0, 1)$ parallel to the cylinder axis. If a is the radius of the cylinder, then the equations and boundary conditions determining A are simply

$$\left.\begin{aligned} &\partial A/\partial t = \lambda \nabla^2 A && (r < a) \\ &\nabla^2 A = 0 && (r > a) \\ &A \sim B_0 r \sin(\theta - \omega t) \quad \text{as} \quad r \to \infty \\ &[A] = [\partial A/\partial r] = 0 \quad \text{across } r = a \,, \end{aligned}\right\} \tag{6}$$

where $\lambda = (\mu_0 \sigma)^{-1}$ is the magnetic diffusivity of the fluid, and the solution, when $\omega a^2/\lambda \gg 1$, (MOFFATT [3] 1965), is

$$A = \begin{cases} B_0(r - a^2/r) \sin(\theta - \omega t) & (r > a) \\ 2^{1/2} B_0 \delta \, \mathrm{e}^{-k(a-r)/\delta} \sin(\theta - \omega t + k(a - r) + \pi/4) & (r < a) \end{cases} \tag{7}$$

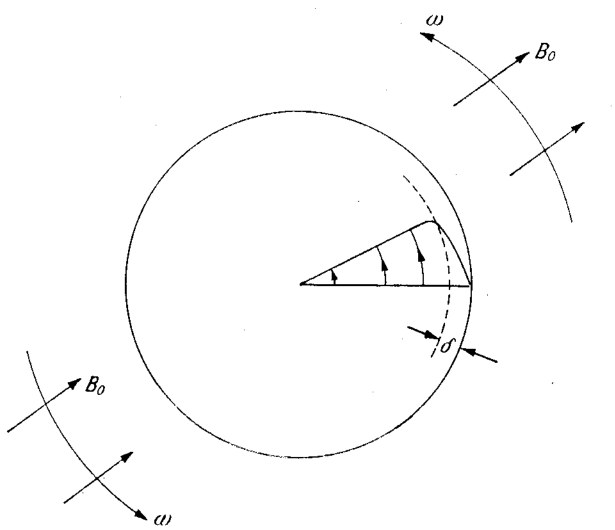

Fig. 1. The velocity distribution inside a cylinder generated by an externally applied magnetic field rotating at high frequency

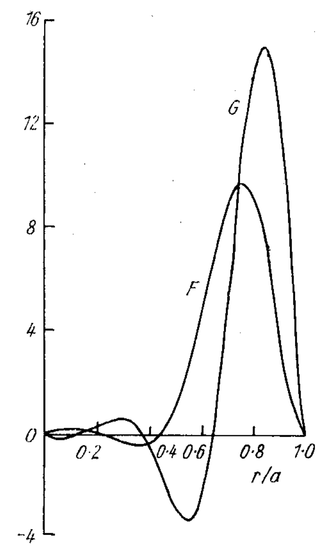

Fig. 2. Radial structure of radial and azimuthal components of the most unstable disturbance of the rotating flow (RICHARDSON [7] 1974)

where $\delta = (\omega/2\lambda)^{-1/2}$, exhibiting the expected boundary layer structure. From this solution, $\boldsymbol{B}$ and $\boldsymbol{j}$ and hence $\boldsymbol{F} = \boldsymbol{j} \wedge \boldsymbol{B}$ (in $r < a$) may be readily calculated. The rate of production of vorticity is given by

$$\nabla \wedge \boldsymbol{F} = -\frac{4B_0^2}{\mu_0 a\delta}\, \mathrm{e}^{-2(a-r)/\delta}\, \boldsymbol{k} . \tag{8}$$

The periodic ingredient vanishes for this idealised geometry. DAHLBERG (1972) has shown that this result remains true even if the frequency ω is low and a boundary layer approach is not applicable.

The velocity field satisfying (3) in this situation is very simple; its streamlines are circular, and the radial distribution is given by

$$v(r) = \Omega(r - a\,\mathrm{e}^{-2(a-r)/\delta}) , \tag{9}$$

where

$$\Omega = B_0^2\lambda/\mu_0\varrho\nu a^2\omega . \tag{10}$$

Inside the boundary layer, the fluid rotates rigidly with angular velocity Ω. Note that when ν is small, Ω is large, since viscosity is the only mechanism limiting the angular acceleration of the fluid.

The analysis as described here neglects the effect of the fluid motion on the field. If the applied field is very strong (so that Ω as given by (10) becomes of the same order as ω) this neglect is no longer justified. In the limit of an infinitely strong field, it is clear that (again except in boundary layers which are now of the HARTMANN layer type) the fluid effectively acquires rigidity because of the infinite tension in the field lines, and field and fluid then rotate with the same angular velocity ω. This situation has been investigated by ALEMANY [4] (1976), who also considers effects associated with applied rotating fields of more complicated structure.

The low-frequency weak-field situation was studied by SMITH [5] (1964) and by DAHLBERG [6] (1972); the resulting velocity field, analogous to (9), but now valid for $\omega a^2/\lambda \ll 1$, is given by

$$v(r) = \Omega_1(r - r^3/a^2) , \tag{11}$$

where

$$\Omega_1 = a^2 B_0^2\omega/16\mu_0\lambda\varrho\nu . \tag{12}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}r}(rv(r)) = \Omega_1(2r - 4r^3/a^2) , \tag{13}$$

so that the circulation is a decreasing function of r in the outer region $0.707 < r/a < 1$. The flow is therefore potentially unstable in this outer region. The stability of the profile (11) has been investigated by RICHARDSON [1] (1973). Defining a TAYLOR number

$$T_1 = a\Omega_1^2\delta_1^3/\nu^2 , \tag{14}$$

where $\delta_1 = 0.293a$ represents the radial extent of the unstable region, RICHARDSON's criterion for instability (obtained numerically) is

$$T_1 > T_{1c} \approx 3344 , \tag{15}$$

and the length of the unstable cell in the z-direction under critical conditions is $0.476a$. The radial and azimuthal perturbations have radial structures given by the functions $F(r)$, $G(r)$ reproduced in figure 2. Note the expected concentration of the disturbance in the unstable region, the maximum for $G(r)$ occurring almost precisely at the centre of this region. The disturbance does of course penetrate weakly into the stable region $r \lessapprox 0.707a$, but vortices here must clearly be regarded as 'driven' rather than spontaneous.

The profile (9) may likewise be expected to be unstable if the relevant TAYLOR number

$$T = a\Omega^2\delta^3/\nu^2$$

exceeds a value of order 10^3. This speculation (MOFFATT [3] 1965) has not yet been subjected to analytical or numerical verification.

There are three principal industrial applications of the centrifuging action of a rotating magnetic field: (i) as a straightforward centrifuge for liquid sodium (or other liquid metal) to remove gas bubbles or other contaminants (HAYES et al. [8] 1971); (ii) as a large scale stirrer in the metal casting process (KAPUSTA [9] 1969); and (iii) as a generator of turbulence to accelerate mixing in metallurgical reactions. The scale of these operations is such that the flows generated are almost inevitably turbulent, and laminar theories can at best provide only a qualitative indication of the results to be expected. An experiment under fully turbulent conditions has been carried out by ROBINSON [10] (1973), using a constant-temperature hot-film anemometer to measure both mean and fluctuating components $V(r)$ and $v(r)$ of the azimuthal velocity. A semi-empirical description of the turbulence was devised by LARSSON [10] (1973), and shows the right qualitative trends, although the difference between predicted and measured values of V and v range up to about 40%.

3. Induction furnace problems

Similar semi-empirical methods have been adopted by TARAPORE and EVANS [11] (1976) in a study of the velocities generated in the melt of an induction furnace. Similar calculations have been carried out by HODGKINS [12] (1972). The principle is illustrated in figure 3. An alternating current in the external coils induces a vertical component of magnetic field which diffuses into the melt. The primary purpose is to heat the melt by joule dissipation, and this purpose is of course helped by convection currents driven by buoyancy forces and by the LORENTZ force. The latter is predominantly radial and is maximal near the centre of the system, as indicated in the figure, and a two-cell axisymmetric flow is generated. The upper free surface is generally perturbed, an effect that can be a limiting factor in the operation of such furnaces.

A simpler prototype problem has been studied by SNEYD [13] (1971). Again the fluid domain is idealised by the infinite cylindrical geometry, to which an alternating transverse field is applied. At high frequencies, the skin effect again allows a simple analysis, The LORENTZ force is radially inwards and is greatest at the points of the cylinder where its tangents are parallel to the applied field. This generates a flow with a four-cell structure as illustrated in figure 4. Again the result (4) holds on each closed streamline C.

It is difficult to carry out any analysis (other than computational) for any geometry other than the simple cylinder as described above. In some circumstances however, a local analysis is possible and illuminating. In particular, if the rigid boundaries of the fluid domain have any sharp corners (as they do have in a typical induction furnace) then a local analysis near the corner is indicated. Suppose for example that the fluid is bounded by plane walls $\theta = \pm\alpha$ (figure 5). Then in the notation of § 2, the general solution of $\nabla^2 A = 0$ in the external region for which $A = 0$ on $\theta = \pm\alpha$ is

$$A = Cr^p \cos p(\theta - \pi)\, e^{i\omega t}, \qquad p = \frac{\pi}{2(\pi - \alpha)} > 0\,, \tag{16}$$

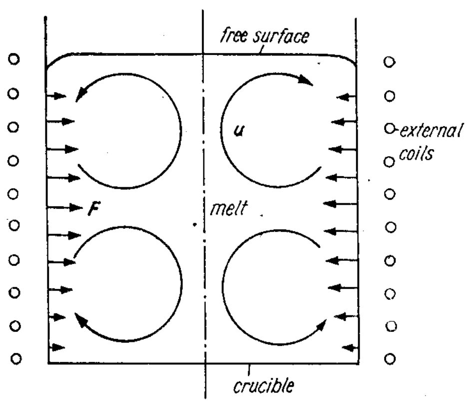

Fig. 3. Sketch of the induction furnace configuration, as studied by TARAPORE & EVANS [11] (1976)

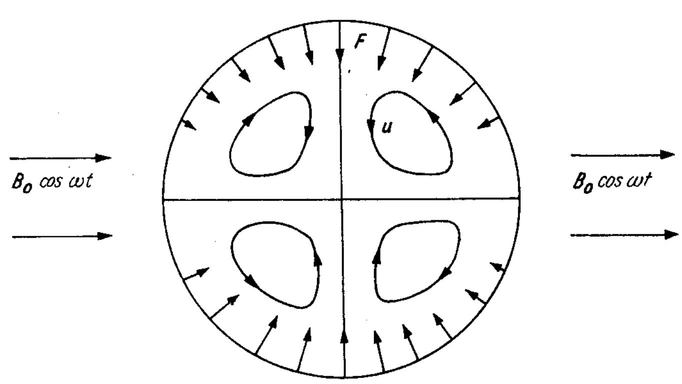

Fig. 4. The idealised model of SNEYD [13] (1971); the LORENTZ force distribution near the circumference of the cylinder drives a flow with a four-cell structure

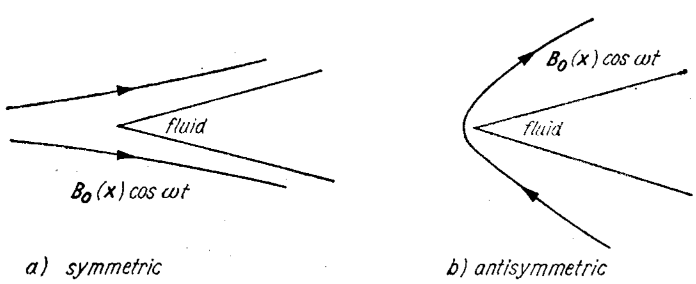

Fig. 5. Symmetric and asymmetric configurations for high frequency field near a sharp corner

or

$$A = Cr^p \sin p(\theta - \pi)\, e^{i\omega t}\,, \qquad p = \frac{\pi}{\alpha - \pi} < 0\,. \tag{17}$$

Equation (16) gives field lines (A = const.) that are symmetrically disposed with respect to the wedge bisector, while (17) gives the antisymmetric configuration. In either case, the equation $\partial A/\partial t = \lambda \nabla^2 A$ may be readily solved in the fluid domain coupled with the condition that the tangential component of $\boldsymbol{B}$ be continuous across $\theta = \pm\alpha$. We again have a skin effect except in the immediate neighbourhood ($r \lessapprox (\lambda/\omega)^{1/2}$) of the vertex. Outside this region the rate of production of vorticity $\nabla \wedge \boldsymbol{F}$ may be calculated; we find

$$\nabla \wedge \boldsymbol{F} = -\frac{1}{\delta}|C|^2 p^2 (p-1)\, x^{2p-3}\, e^{-2y/\delta}\,, \tag{18}$$

where $\delta = (2\lambda/\omega)^{1/2}$ as before, and y is a coordinate normal to the boundary into the fluid. In the symmetric case given by (16),

$$2p - 3 = \frac{3\alpha - \pi}{\pi - \alpha} \gtrless 0 \quad \text{acc. as} \quad \alpha \gtrless \frac{\pi}{3}. \tag{19}$$

Vorticity production apparently increases as the corner is approached if $\alpha < \pi/3$, a singularity being thwarted only in the small excluded region $r \lessapprox (\lambda/\omega)^{1/2}$.

In the antisymmetric case,

$$2p - 3 = \frac{3\alpha - 4\pi}{\pi - \alpha} < 0 \quad \text{for all } \alpha \tag{20}$$

and in this case vorticity production inevitably increases as the corner is approached for all values of α.

4. The weld-pool problem

A closely related class of problem is that in which a steady current is injected into a volume of conducting fluid by prescription of the electrostatic potential distribution φ over its boundary. Again neglecting the weak perturbing effect of the fluid motion, the current field is then simply given by

$$\boldsymbol{j} = -\sigma \nabla \varphi \tag{21}$$

and if this current is the only source of magnetic field, $\boldsymbol{B}$ is determined by

$$\nabla \cdot \boldsymbol{B} = 0\,, \qquad \nabla \wedge \boldsymbol{B} = \mu_0 \boldsymbol{j} = -\mu_0 \sigma \nabla \varphi\,. \tag{22}$$

The prototype situation, which has been studied by Zhigulev (1960) and Shercliff (1970), is that in which current is injected from a point electrode into a half-space of conducting fluid (figure 6). This can be regarded as a primitive model of what happens in the neighbourhood of the contact both in the arc welding process, and in the arc furnace in which a container of liquid metal is heated by just this method, viz. the injection of a large steady current at a point on its surface. In this latter context, uniform heating of the melt depends critically on the convection currents induced, and for this reason again the dynamics of the system have to be considered.

In the problem as formulated by Shercliff, in spherical polar coordinates (r, θ, φ) the current is given by

$$\boldsymbol{j} = (J/2\pi r^2, 0, 0) \tag{23}$$

and, by Ampere's law, the field $\boldsymbol{B}$ is then purely azimuthal and has the form

$$\boldsymbol{B} = \big(0, 0, \mu_0 J \sin\theta / 2\pi r\,(1 + \cos\theta)\big)\,. \tag{24}$$

Hence

$$\boldsymbol{F} = \boldsymbol{j} \wedge \boldsymbol{B} = \big(0, -\mu_0 J^2 (\sin\theta)/4\pi^2 r^3 (1 + \cos\theta)\,, 0\big)\,, \tag{25}$$

and

$$\nabla \wedge \boldsymbol{F} = \left(0, 0, \frac{\mu_0 J^2}{2\pi^2 r^4} \frac{\sin\theta}{(1 + \cos\theta)}\right). \tag{26}$$

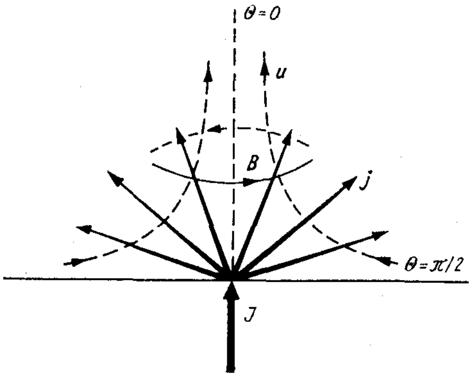

Fig. 6. Current injection into a half-space, as analysed by Shercliff [15] (1970)

Note that $F_\theta < 0$ for all $\theta < \pi/2$, and the magnitude of F_θ decreases as θ decreases from $\pi/2$ to 0. The force field therefore drives a jet-type motion along the axis of symmetry, the fluid being drawn in laterally by a 'pinch effect' which is naturally strongest near to the current source.

The fluid particles that pass very near the source singularity acquire a very large vorticity (due to the r^{-4} dependence in (26)). SHERCLIFF showed, on the basis of an inviscid analysis that if the flow 'upstream' is irrotational, then the flow 'downstream' is necessarily singular on the axis of symmetry. As pointed out by SOZOU (1971), viscous effects must be included if a well-behaved solution is to be found; this may be seen again from equation (1): although the streamlines are not closed (the flow domain being infinite) the result (1) still holds on any streamline C provided the pressure is uniform and the velocity zero at infinity; a steady state satisfying these conditions is therefore not possible unless viscous effects are taken into account.

Dimensional analysis led SOZOU to seek a similarity solution to the problem in which the STOKES stream function is given by

$$\psi = \nu r g(\mu, K)\,, \tag{27}$$

where $\mu = \cos\theta$ and

$$K = \mu_0 J_0^2/\varrho\nu^2\,. \tag{28}$$

This form of solution is suggested by the fact that there is no natural length-scale in the problem, and K is the sole dimensionless parameter that can be constructed. The situation is closely analogous to the classical round jet problem, for which a flow is generated by the application of a point force (or equivalently a point source of momentum) on a plane boundary. In the present context, the force is distributed rather than concentrated at a point, but its radial dependence ($\sim r^{-3}$) is just such as to be in possible equilibrium with both the inertia force $\varrho\boldsymbol{u}.\nabla\boldsymbol{u}$ and the viscous force $\varrho\nu\nabla^2\boldsymbol{u}$ when $\boldsymbol{u} \propto r^{-1}$, i.e. when ψ is given by (27).

There is however a difficulty in pursuing this analogy that does not appear to have been fully appreciated. For, on the one hand, a streamfunction of the form (27) is associated with a definite flux of momentum F_0 in the direction of the axis of symmetry (see, for example, BATCHELOR 1967, § 4.6). If $g(\mu, k)$ is known, then integration of the associated momentum flux over any hemisphere $r = R$, $\theta < \pi/2$, gives a definite relationship of the form

$$F_0 = 2\pi\varrho\nu^2 G(K) \tag{29}$$

for some function G, and this value of F_0 must be interpreted as the equivalent point force at $r = 0$ which will generate a jet flow having the same momentum flux as the flow given by (27).

On the other hand, we can easily calculate the total force imparted to the fluid in the region $r_0 < r < R$ on the basis of (25), viz.

$$\hat{F}_0 = -\int_{r_0}^{R}\int_{0}^{\pi/2} F_\theta \sin\theta \cdot 2\pi r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta = \frac{\mu_0 J^2}{4\pi}\log\left(\frac{R}{r_0}\right), \tag{30}$$

which diverges logarithmically as $R/r_0 \to \infty$. Unless this divergence is compensated by similar divergence in the suction exerted by the plane $\theta = \pi/2$ as $R \to \infty$, this implies an unbounded flux of momentum in the fluid as $r \to \infty$. SOZOU did in fact find, by integration of the non-linear ordinary differential equation for $g(\mu, K)$, that singularities appeared on the axis $\mu = 1$ for large values of K (> 300). The above argument raises questions concerning the physical realisability of solutions of the form (27) for a n y value of K.

The most reasonable way to resolve this sort of difficulty is of course to return to the problem of a finite fluid domain, and at the same time to replace the point electrode by an electrode of finite size. A step in this direction has been taken by SOZOU & PICKERING (1976) who consider the effect of the force distribution (25) within a hemispherical container $r < R$, $\theta < \pi/2$, and obtain steady streamline patterns by integrating the nonlinear equation for the stream-function $\psi(r, \theta)$ numerically. There is still however in this situation a difficulty in the overall momentum balance. The force $\hat{F}_0$ given by (30) becomes infinite as $r_0 \to 0$ for fixed R, and this infinite volume force imparted to the fluid must be imparted (by momentum conservation) to the boundaries containing the fluid. The only point of the boundary at which this infinity can reasonably be accounted for is the point electrode itself; at this point, the boundary must exert an infinite suction, a situation that would inevitably lead to local cavitation, and intermittency in the resulting current passed to the liquid. SOZOU & PICKERING actually suppose that the surface $\theta = \pi/2$ is a free surface (as is appropriate in the technological applications mentioned above); but the assumption of a point source of current on the boundary must then inevitably lead to a singularity in the surface deformation at this point also. (This may equally be appreciated in terms of the infinite magnetic pressure at the origin.)

The alternative way to try to resolve the difficulty is to accept that where the velocity is large, neglect of its effect on the magnetic field distribution may no longer be tenable. Allowance for field convection and diffusion introduces one new physical parameter into the problem, viz. the magnetic diffusivity λ. We now have the very curious situation of a problem defined in terms of t h r e e dimensional parameters $(\mu_0/\varrho)^{1/2} J$, ν and λ all having the same dimensions (length)^2 (time)$^{-1}$, from which we still cannot construct a natural length-scale. The current lines must therefore still be radial, so that instead of (23) we can have only

$$\boldsymbol{j} = ((J/2\pi r^2)\, f(\theta), 0, 0) \tag{31}$$

for some function $f(\theta)$ satisfying

$$\int_0^{\pi/2} f(\theta)\sin\theta\,\mathrm{d}\theta = 1\,. \tag{32}$$

An easy calculation now leads to the appropriate modification of (30) viz

$$\hat{F}_0 = \frac{\mu_0 J^2}{2\pi} \log\left(\frac{R}{r_0}\right) \int_0^{\pi/2} f(\theta) \cos\theta \sin\theta \, d\theta \,. \tag{33}$$

The only way that $\hat{F}_0$ can remain finite as $R/r_0 \to \infty$ is if

$$\int_0^{\pi/2} f(\theta) \cos\theta \sin\theta \, d\theta = 0 \,. \tag{34}$$

This can be satisfied (in conjunction with (32)) only if $f(\theta)$ is negative for some values of θ in the range $0 < \theta < \pi/2$. For example, the function

$$f(\theta) = 6 \cos\theta - 4 \tag{35}$$

satisfies both (32) and (34). Indications of reversed current flow have in fact been found in numerical computation incorporating induction (or 'field sweeping') effects by SOZOU & ENGLISH (1972), but the extent and intensity of the reversed current does not appear sufficient for satisfaction of the condition (34). The possibility of current reversal was also noted by SHERCLIFF [15].

It must be concluded that, in spite of the conceptual simplicity of the idealised problem as posed by SHERCLIFF, the solutions that have so far been proposed have internal inconsistencies that have yet to be fully resolved. It should perhaps be noted, moreover, that even if the laminar problem were fully understood, the flow is very likely to be unstable when K exceeds some critical value, and a turbulent state is then the most likely outcome.

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