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ROTATION OF A LIQUID METAL UNDER THE ACTION OF A ROTATING MAGNETIC FIELD

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Abstract

When a liquid metal in a closed container is placed in an applied magnetic field which is caused to rotate by suitably phased external current circuits, a rotational Lorentz force is established in the liquid, which drives a rotational flow. This problem is reviewed, and certain new results are obtained. In particular, it is shown that (i) in the weak field limit, and when the problem can be treated as a two-dimensional problem, the rotational part of the Lorentz force is steady; (ii) in the high frequency limit, flow with circular streamlines is possible only if the Hartmann number is not too large, and then there are *two* possible steady state solutions for the core angular velocity; (iii) in the case of an elliptic cylinder of large aspect ratio a/b, the torque distribution associated with the Lorentz force is concentrated near the points of maximum curvature on the boundary, and drives a core vorticity which increases in an unbounded manner as b/a decreases.

1. Introduction

One of the earliest papers on magnetohydrodynamics was that of Braunbeck (1932) who discussed the action of a rotating magnetic field on a small capsule of liquid metal suspended on a torsion fibre (Fig. 1). Rotation of the field (by the application of two alternating components out of phase and at right-angles) exerts a torque on the fluid which, under suitably restricted circumstances, is proportional to its electrical conductivity, and so measurement of the angular displacement of the capsule provides a means of determining the conductivity, without having to insert electrodes into the liquid. This technique has been greatly developed; see, for example, Ozelton and Wilson (1966).

There are now a number of different applications, both actual and potential, of this torque-producing action of a rotating magnetic field, which would appear to justify a review of certain fundamental aspects of the problem at this meeting. In a pipe flow, a field rotating in a plane perpendicular to the axis of the pipe will generate swirl; this swirl may be used (Hayes, Baum and Hobdell, 1971) to eliminate impurities in a liquid metal (Fig. 2a, b), bubbles being centrifuged towards the axis (where they may be siphoned off) and heavy particles towards the pipe wall (where they may be bled off). The swirl may also be used to cause dispersion in fine droplets (atomisation) of the liquid metal at the pipe exit (Alemany et al., 1977) (Fig. 2c). The use of rotating fields in continuous

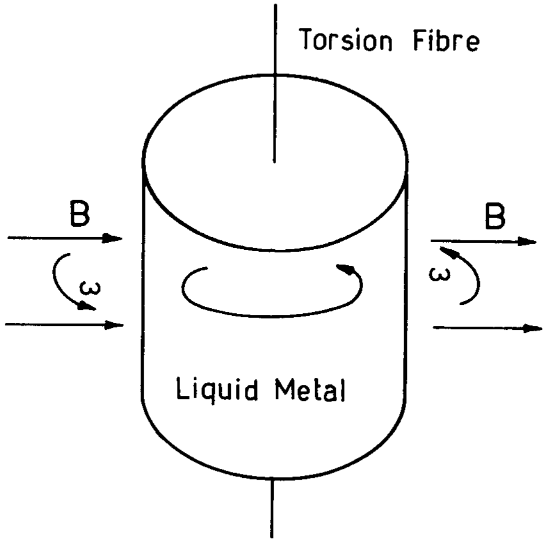

FIG. 1. The experiment of Braunbeck (1932). The horizontal applied field rotates about a vertical axis through the use of suitably phased external current circuits.

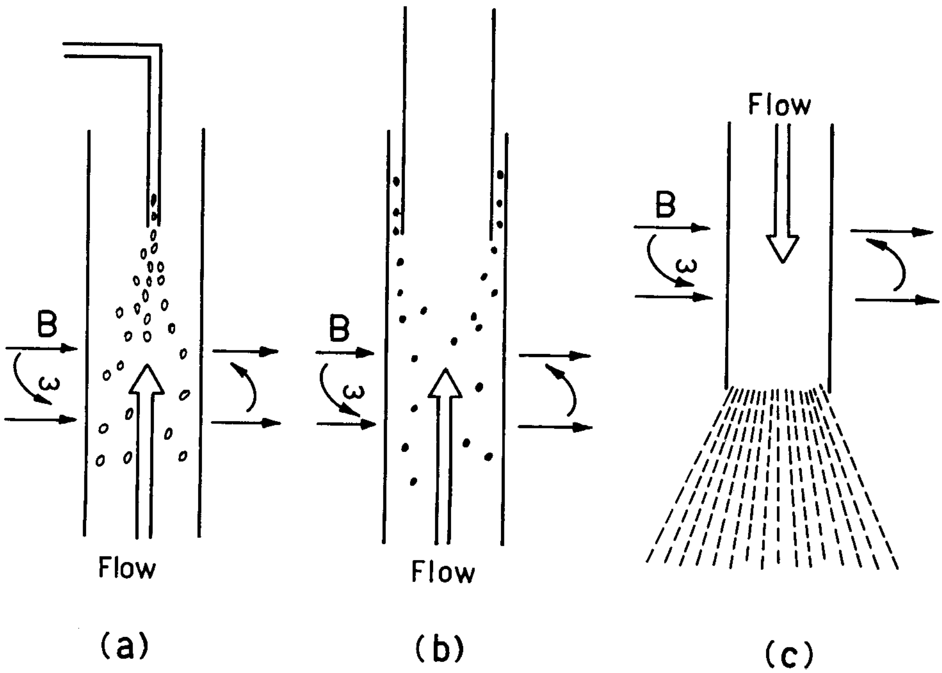

FIG. 2. Uses of swirl generated by a rotating field: (a) removal of bubbles, (b) removal of suspended particles, (c) production of spray.

casting processes has been studied by Kapusta and his colleagues at the Donets Institute (see Barinberg et al., 1975 for a summary of these contributions). Finally, since the flows that are generated turn out to be unstable in at least part of the flow regime (Richardson, 1974; Robinson, 1973), the rotating field has possibilities as a generator of turbulence when accelerated mixing in metallurgical reactions is desired.

Problems involving rotating fields have much in common with problems involving A.C fields of fixed direction as discussed by Moreau (see pp. 65–82). In both situations, electric currents are induced in the fluid; these interact with the field to produce a rotational Lorentz force which drives a rotational motion. When the field is sufficiently weak, or the field frequency sufficiently high, the electromagnetic problem is decoupled from the fluid dynamic problem, and the first step is the relatively easy one of determining the electromagnetic field in an effectively stationary conductor, due to a prescribed system of A.C. currents in external coils. The second step is the solution of the Navier-Stokes equation under the action of the (now known) Lorentz force; this is in general difficult, particularly when (as is almost invariably the case in practice) the Reynolds number of the resulting flow is large.

When the field is strong, the Lorentz force becomes dependent on the fluid velocity, i.e. the electromagnetic and dynamic problems become fully coupled. The problem is amenable to analysis only in the limit of large Hartmann number. In the case of the rotating field, in a frame of reference rotating with the field, the flow is generally characterised by Hartmann layers on the inside of the bounding container, and an effectively inviscid steady core motion.

2. The Lorentz force distribution when induction due to fluid motion is negligible

We may suppose that the source of the rotating field is a surface current

$$\mathbf{J}_s = J_0 \cos(m\theta - \omega t)\,\mathbf{k} \tag{1}$$

on the cylindrical surface $r = a$ where m is an integer (the polarity of the field). Here, (r, θ, z) are cylindrical polars, and $\mathbf{k}$ is a unit vector in the z-direction, i.e. parallel to the cylinder axis. In the absence of any conductor, the resulting field is given by

$$\mathbf{B}_0 = \nabla \wedge \left(A_0(r, \theta, t)\,\mathbf{k}\right) = -\mathbf{k} \wedge \nabla A_0, \tag{2}$$

where, for $r < a$,

$$A_0(r, \theta, t) = \frac{\mu_0 J a}{2m}\left(\frac{r}{a}\right)^m \cos(m\theta - \omega t), \tag{3}$$

i.e.

$$\left.\begin{aligned} B_r &= \frac{1}{r}\frac{\partial A_0}{\partial \theta} = -B_0\left(\frac{r}{a}\right)^{m-1} \sin(m\theta - \omega t) \\ B_\theta &= -\frac{\partial A_0}{\partial r} = -B_0\left(\frac{r}{a}\right)^{m-1} \cos(m\theta - \omega t) \end{aligned}\right\} \tag{4}$$

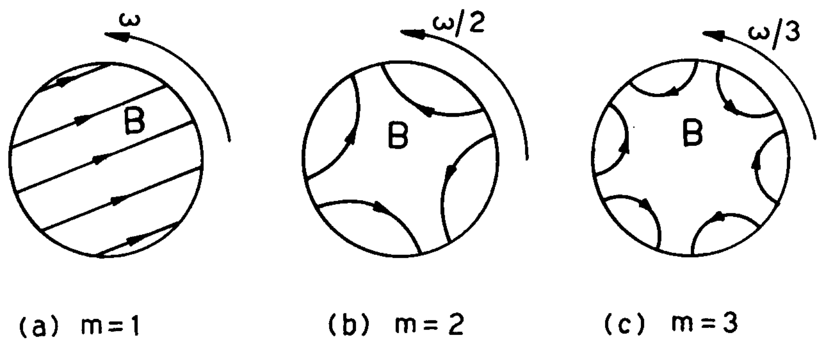

FIG. 3. Magnetic field produced by a rotating current sheet in the cases $m = 1, 2, 3$.

where $B_0 = \mu_0 J_0/2$. The lines of force are given by $A_0(r,\theta,t) = \text{const.}$, and the field pattern evidently rotates with angular velocity ω/m. When $m = 1$, the field is uniform for $r < a$, when $m = 2$ the field lines are rectangular hyperbolae, and for $m > 2$ they have m-fold symmetry about the lines

$$\theta - \frac{\omega}{m} t = (2n + 1)\pi/2m, \qquad (n = 0, 1, 2, \ldots, m - 1) \tag{5}$$

(Fig. 3).

Suppose now that a liquid metal contained inside an insulating closed rigid boundary S is placed in such a field in the region $r < a$ (Fig. 4). If $\mathbf{j}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ are the resulting current and field distributions within the conductor, and $\mathbf{F} = \mathbf{j} \wedge \mathbf{B}$, then in general $\nabla \wedge \mathbf{F} \neq 0$ and a motion $\mathbf{u}(\mathbf{x}, t)$ must ensue. Let V be the interior of S, and let L be a typical scale of V (e.g. the maximum diameter). Suppose first that

$$|\mathbf{u}|_{\max} \ll \omega L, \tag{6}$$

an inequality that will clearly be satisfied if the field strength is sufficiently weak (or, as will emerge, if the frequency ω is sufficiently large). Then Ohm's law

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \wedge \mathbf{B}) \tag{7}$$

becomes approximately

$$\mathbf{j} \approx \sigma \mathbf{E}, \tag{8}$$

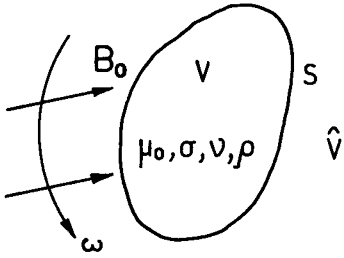

FIG. 4. The general configuration considered.

i.e. the current distribution in V is the same as if the conductor were motionless. Introducing a vector potential $\mathbf{A}$ such that

$$\mathbf{B} = \nabla \wedge \mathbf{A}, \qquad \mathbf{E} = -\partial \mathbf{A}/\partial t \tag{9}$$

and

$$\frac{\partial \mathbf{A}}{\partial t} = \lambda \nabla^2 \mathbf{A}, \tag{11}$$

where $\lambda = (\mu_0 \sigma)^{-1}$. Also $\nabla^2 \mathbf{A} = 0$ in the region $\hat{V}$ outside S, and both $\mathbf{A}$ and its normal gradient $\partial \mathbf{A}/\partial n$ are continuous across S.

Suppose now that

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{p}(\mathbf{x}) \cos \omega t + \mathbf{q}(\mathbf{x}) \sin \omega t, \tag{12}$$

as will certainly be the case when the applied field has the form described by equations (2) and (3). (Actually, (12) will apply for more general A.C. fields also.) Then from (11)

$$\nabla^2 \mathbf{p} = \frac{\omega}{\lambda} \mathbf{q} \quad \text{and} \quad \nabla^2 \mathbf{q} = -\frac{\omega}{\lambda} \mathbf{p}, \tag{13}$$

and so

$$\mu_0 \mathbf{j} = -\nabla^2 \mathbf{A} = \frac{\omega}{\lambda} (\mathbf{p} \sin \omega t - \mathbf{q} \cos \omega t), \tag{14}$$

and

$$\begin{aligned} \mathbf{F} &= \mathbf{j} \wedge \mathbf{B} \\ &= \frac{\omega}{\mu_0 \lambda} [\mathbf{p} \wedge (\nabla \wedge \mathbf{q}) - \mathbf{q} \wedge (\nabla \wedge \mathbf{p}) \\ &\quad + \{\nabla(\mathbf{p} \cdot \mathbf{q}) - (\mathbf{p} \cdot \nabla) q - (\mathbf{q} \cdot \nabla) \mathbf{p}\} \cos 2\omega t \\ &\quad + \{\nabla(\mathbf{p}^2 - \mathbf{q}^2) - (\mathbf{p} \cdot \nabla) \mathbf{p} - (\mathbf{q} \cdot \nabla) \mathbf{q}\} \sin 2\omega t]. \end{aligned} \tag{15}$$

Note, in general, the presence of a steady ingredient and a periodic ingredient, with period π/ω. These will drive steady and periodic flow ingredients, although of course the inertial response to the periodic part of the force will be small when the frequency is large.

There is, however, an important simplification when the surface S is a long cylinder (of arbitrary cross section) such that end-effects may be neglected. In this case,

$$\mathbf{p} = p(x, y)\, \mathbf{k}, \qquad \mathbf{q} = q(x, y)\, \mathbf{k}, \tag{16}$$

where (x, y) are Cartesian coordinates in the plane perpendicular to the cylinder axis, and the expression (15) then reduces to

$$\mathbf{F} = \frac{\omega}{2\mu_0 \lambda} [p \nabla q - q \nabla p + \nabla \{pq \cos 2\omega t + (p^2 - q^2) \sin 2\omega t\}]. \tag{17}$$

Note now that the periodic part is irrotational, so that this will not drive a flow ingredient, but will merely generate a periodic pressure fluctuation. The steady part of (17) is, however, rotational; in fact

$$\nabla \wedge \mathbf{F} = \frac{\omega}{\mu_0 \lambda} \nabla p \wedge \nabla q. \tag{18}$$

This result, that the curl of the Lorentz force is steady, was noted by Moffatt (1965) for the case of a cylinder of circular cross section in the limit of high field frequency. The restriction to high frequency was removed by Dahlberg (1972) and the restriction to circular cross section was removed by Sneyd (1979). The above treatment shows that the result is in fact quite general, provided only that the field exhibits the two-dimensionality described by (16).

3. The case of a circular cylinder neglecting end effects

If S is a cylinder of circular cross section, radius a, which just fits inside the field-producing current sheet (1) (Fig. 5), then the field satisfies jump conditions across $r = a$:

$$[B_r] = 0, \qquad [B_\theta] = -B_0 \cos(m\theta - \omega t) \text{ on } r = a, \tag{19}$$

where $B_0 = \mu_0 J_0$, and the vector potential is given by

$$\mathbf{A} = \mathrm{Re} f(r)\, e^{i(m\theta - \omega t)} \mathbf{k}, \tag{20}$$

where $f(r)$ satisfies

$$-i\omega f = \lambda \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m}{r^2} \right) f. \tag{21}$$

The solution satisfying (19), and finite at $r = 0$, is

$$f(r) = D J_m\big((1 + i)\, r/\delta\big), \qquad \delta = (2\lambda/\omega)^{1/2}, \tag{22}$$

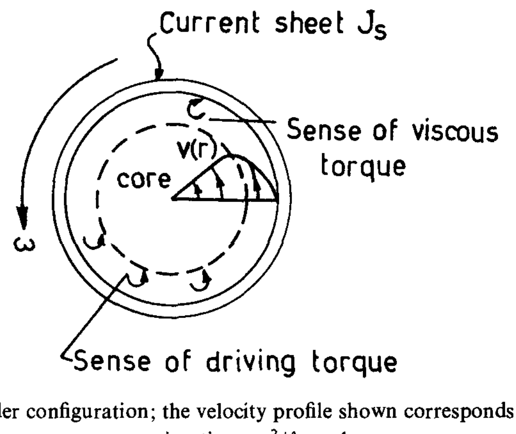

FIG. 5. Circular cylinder configuration; the velocity profile shown corresponds to the high frequency situation $\omega a^2/\lambda \gg 1$.

where

$$D = \frac{B_0 a}{mJ_m(\zeta) + \zeta J'_m(\zeta)}, \qquad \zeta = (1 + i)\, a/\delta \tag{23}$$

From (18), we then have

$$\nabla \wedge \mathbf{F} = \frac{\omega m}{2\mu_0 \lambda r} \frac{d}{dr} |f|^2\, \mathbf{k}, \tag{24}$$

and hence (Dahlberg, 1972)

$$\mathbf{F} = \frac{\omega m}{2\mu_0 \lambda r} |f|^2\, \hat{\mathbf{e}}_\theta + \mathbf{F}_1, \tag{25}$$

where $\mathbf{F}_1$ is an irrotational contribution which affects only the pressure field.

Clearly this force drives a velocity field of the form

$$\mathbf{u} = v(r)\, \hat{\mathbf{e}}_\theta. \tag{26}$$

The associated inertial acceleration $\mathbf{u} \cdot \nabla \mathbf{u}$ is irrotational and therefore compensated by a radial pressure adjustment. The θ-component of the Navier-Stokes equation is simply

$$\rho v \frac{d}{dr} \frac{1}{r} \frac{d}{dr}(rv) = -\frac{\omega m}{2\mu_0 \lambda r} |f|^2. \tag{27}$$

High frequency limit (Moffatt, 1965)

If $\omega a^2/\lambda \gg 1$, then $\delta/a \ll 1$ and $|\zeta| \gg 1$, and we may use the asymptotic form

$$J_m(\zeta) \sim \left(\frac{2}{\pi\zeta}\right)^{1/2} \cos\left(\zeta - \frac{m\pi}{2} - \frac{\pi}{4}\right). \tag{28}$$

Hence (provided $|\zeta| \gg m$),

$$|f|^2 \sim \tfrac{1}{2} B_0^2 \delta^2 e^{-2(a-r)/\delta}. \tag{29}$$

The driving force is here concentrated in a skin, thickness $O(\delta)$. To the same degree of approximation, the solution of (27) is

$$v(r) = \Omega r (1 - e^{-2(a-r)/\delta}), \qquad \Omega = \frac{2B_0^2 m \lambda}{\mu_0 \rho v a^2 \omega}. \tag{30}$$

The velocity field therefore consists of a rigid body rotation in the 'core' ($a - r \gg \delta$), together with a very simple boundary-layer adjustment to satisfy the no-slip condition on $r = a$ (as indicated in Fig. 5).

Neglect of the $\mathbf{u} \wedge \mathbf{B}$ term in Ohm's law is justified only if $\Omega \ll \omega$, i.e. only if

$$2mM^2 \ll (\omega a^2/\lambda)^2 = R_m^2, \tag{31}$$

where $M = B_0 a(\lambda/\mu_0 \rho v)^{1/2}$ is the Hartmann number, and R_m the magnetic Reynolds number based on the field frequency. However, the very simple form of the velocity field (30) suggests that 'field-sweeping' effects embodied in the $\mathbf{u} \wedge \mathbf{B}$ term may be easily incorporated in the above analysis. In fact, if a velocity field of the form (26) is included in (7), then (11) is replaced by

$$\frac{\partial \mathbf{A}}{\partial t} + \frac{v(r)}{r}\frac{\partial \mathbf{A}}{\partial \theta} = \lambda \nabla^2 \mathbf{A}, \tag{32}$$

and (21) is replaced by

$$-i\left(\omega - \frac{v(r)}{r}\right) f = \lambda\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\omega^2}{r^2}\right) f. \tag{33}$$

In the core region, $v(r) = \Omega r$, and so ω is effectively replaced by $\omega - \Omega$ in the analysis leading to the determination of the Lorentz force. Physically, it is the *relative* motion between field and fluid that is important, rather than the absolute motion of the field relative to the laboratory frame of reference.

This leads to a rather surprising conclusion concerning possible steady states. If ω is replaced by $\omega - \Omega$ in (30b) then we have

$$\Omega = \frac{\omega_c^2}{4(\omega - \Omega)} \quad \text{where} \quad \omega_c^2 = \frac{8mM^2\lambda^2}{a^4}. \tag{34}$$

We now have a *quadratic* equation for Ω, with roots

$$\Omega_1 = \tfrac{1}{2}(\omega - \sqrt{\omega^2 - \omega_c^2}), \qquad \Omega_2 = \tfrac{1}{2}(\omega + \sqrt{\omega^2 - \omega_c^2}). \tag{35}$$

There would appear to be *two* solutions if $\omega > \omega_c$, and *no* (real) solutions if $\omega < \omega_c$! Although the argument is not conclusive, it is at least strongly indicative that when $\omega < \omega_c$, or equivalently when

$$2\sqrt{2m}\,M_c > R_m, \tag{36}$$

motion with circular streamlines is no longer possible. We shall return to this 'strong field' situation in §5 below.*

Physical interpretation of the two solutions when $\omega > \omega_c$

To be specific, suppose that $\omega = 2\omega_c$; then the roots (35) are

$$\Omega_1 = \tfrac{1}{4}(2 - \sqrt{3})\,\omega \quad \text{and} \quad \Omega_2 = \tfrac{1}{4}(2 + \sqrt{3})\,\omega \tag{37}$$
$$\doteqdot 0.067\omega \qquad\qquad \doteqdot 0.933\omega$$

* A related problem, with plane geometry, has been studied by M. R. E. Proctor (private communication) with similar conclusions concerning possible multiplicity of solutions. See also Gimblett and Peckover (1979).

The skin-effect analysis for both motions is justified if

$$\frac{\omega a^2}{\lambda} = 4\sqrt{2mM} \gg 1. \tag{38}$$

Suppose that we start at time $t = 0$ from a state of rest (zero viscous dissipation) and allow the fluid to accelerate under the action of the Lorentz force. Then the *smaller* equilibrium core velocity will be attained, and in this equilibrium, Joule dissipation in the boundary layer may be expected to dominate over viscous dissipation.

Suppose now that we start at time $t = 0$ from a state of corotation, in which the cylinder, fluid and field *all* rotate with angular velocity ω, and that we impulsively bring the cylindrical container to rest at this instant. The initial dissipation is then entirely viscous. The core velocity will then decrease from ω to the *larger* equilibrium value Ω_2 and it is to be expected that viscous dissipation will dominate over Joule dissipation in this equilibrium.

If $\omega \gg \omega_c$, then the thin skin analysis is valid only for the first solution Ω_1, and the above argument concerning the existence of two steady state solutions must be treated with caution.

Low frequency situation $\omega a^2/\lambda \ll 1$ (Dahlberg, 1972)

If $\omega a^2/\lambda \ll 1$, then $|\zeta| \ll 1$, and

$$|f(r)|^2 \approx \left(\frac{B_0 a}{2m}\right)^2 \left(\frac{r}{a}\right)^{2m}. \tag{39}$$

Hence, from (27),

$$v(r) \approx \Omega_1 r\left(1 - (r/a)^{2m}\right), \tag{40}$$

where

$$\Omega_1 = \frac{M^2}{16m^2(m+1)}\omega. \tag{41}$$

Clearly, neglect of $\mathbf{u} \wedge \mathbf{B}$ in this limit is valid only if $\Omega_1 \ll \omega$, i.e.

$$M \ll 4m(m+1)^{1/2}. \tag{42}$$

If this condition is not satisfied, then again significant departures from circular streamlines are to be expected.

Flow stability

It is evident from (40) that the circulation $2\pi r v(r)$ is a decreasing function of r when r is near to a, and that the flow may therefore be expected to be prone to instability, by Rayleigh's criterion. In fact

$$\frac{d}{dr}\left(rv(r)\right) = 2\Omega_1 r\left(1 - (m+1)(r/a)^{2m}\right), \tag{43}$$

so that instability may be anticipated in the region

$$r/a > (m + 1)^{-m/2}. \tag{45}$$

The stability has been investigated, for the case $m = 2$, when (45) becomes $r/a > 0.707$, by Richardson (1974) using small perturbation analysis. Defining $\delta_1 = 0.293a$ as the width of the region in which instability is to be expected, Richardson's criterion for instability was

$$T_1 = a\Omega_1^2\delta_1^3/\nu^2 > 3344. \tag{46}$$

The structure of the eigenfunctions (see the discussion of Moffatt, 1978) confirms that the instability arises in the form of spontaneous Taylor vortices in the region $r/a \gtrsim 0.7$ while the motion in $r/a \lesssim 0.7$ consists of weak vortices, driven by viscous drag (Fig. 6).

The velocity field (30) also has decreasing circulation near to $r = a$, and may therefore be expected to be unstable if the modified Taylor number $T_2 = a\Omega^2\delta^3/\nu^2$ is sufficiently large. This speculation (Moffatt, 1965) does not appear yet to have been subjected to detailed analysis.

Flows generated by rotating fields may be expected to be turbulent, at least in a region near the container boundary, in most circumstances of practical interest. Measurements in the fully turbulent regime for the case of a circular cylinder have been reported by Robinson (1973), but theoretical ideas to explain the observed patterns of mean flow and turbulent intensity are as yet extremely rudimentary.

4. Cylinder of arbitrary cross section, $\omega a^2/\lambda \gg 1$

The case of a cylinder of arbitrary cross section, in the high frequency limit, has recently been considered by Sneyd (1979), using a boundary-layer technique. The essence of the

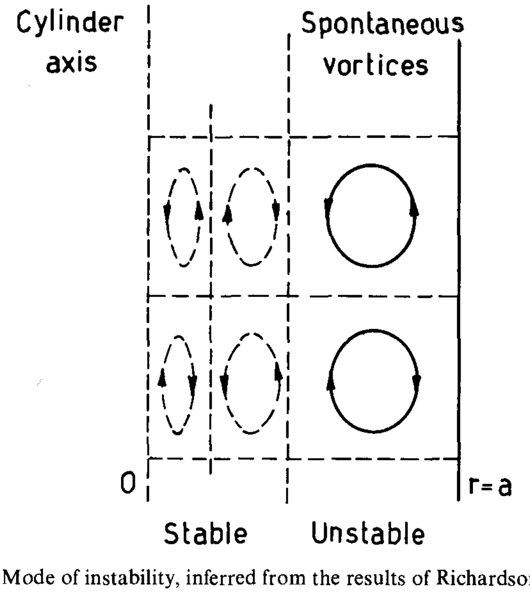

FIG. 6. Mode of instability, inferred from the results of Richardson (1974).

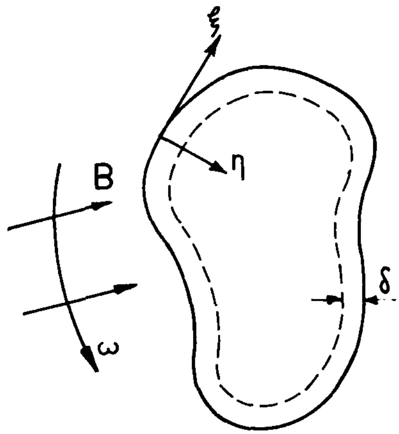

FIG. 7. Cylinder of arbitrary cross section in rotating field (high frequency limit). The Reynolds number of the resulting flow is assumed large and it is assumed that the flow does not separate; there is then a core region in which the vorticity is uniform.

argument is that, when $\omega a^2/\lambda \gg 1$, the rotating field is effectively excluded from the conductor (Fig. 7), penetrating only into the skin, thickness $O(\delta)$. In fact $\mathbf{A} = O(\delta)$ on the surface S of the cylinder, which is almost a line of force of the rotating field. If (ξ, η) are local tangential and normal coordinates, and if

$$\mathbf{A} = \operatorname{Re} f(\mathbf{x})\, e^{-i\omega t}\mathbf{k}, \tag{47}$$

then f satisfies the equation

$$-i\omega f = \lambda \nabla^2 f \approx \lambda \partial^2 f/\partial\eta^2, \tag{48}$$

the relevant solution in V being

$$f(\xi, \eta) = -\tfrac{1}{2}\delta\beta(\xi)(1 + i)\exp\{-(1 - i)\eta/\delta\}, \tag{49}$$

where $\beta(\xi) = \partial f/\partial\eta|_{\eta=0}$ is effectively the tangential component of $\mathbf{B}$ on S. The curl of the Lorentz force may then easily be calculated and is given by

$$\nabla \wedge \mathbf{F} = T(\xi)e^{-2\eta/\delta}\mathbf{k}, \tag{50}$$

where

$$T(\xi) = -\frac{\omega\delta}{2\mu_0\lambda}[\operatorname{Re}\beta'\beta^* + \operatorname{Im}\beta'\beta^*]. \tag{51}$$

Note that

$$T \sim \frac{\omega\delta}{\mu_0\lambda}\frac{B_0^2}{a} \tag{52}$$

in order of magnitude. As for the case of a circular cylinder, the Lorentz force is concentrated in a thin layer near the fluid boundary, and its variation $T(\xi)$ with the tangential coordinate is determined by the variation $\beta(\xi)$ of the tangential component of $\mathbf{B}$ on S.

We are now faced with the problem of determining the steady velocity field $\mathbf{u}(\mathbf{x})$ inside S that results from what may be regarded as a known steady rotational force field $\mathbf{F}(\mathbf{x})$. The Navier-Stokes equation may be written in the form

$$-\mathbf{u} \wedge \boldsymbol{\omega} = -\frac{1}{\rho}\nabla(p + \tfrac{1}{2}\rho u^2) + \mathbf{F} - \nu\nabla \wedge \boldsymbol{\omega}, \tag{53}$$

and, integrating this round any closed streamline C, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = \nu \oint_C (\nabla \wedge \boldsymbol{\omega}) \cdot d\mathbf{x}. \tag{54}$$

This indicates that inertial forces are impotent to control the circulation generated (although they presumably do have an influence on the streamline geometry). Viscous stresses would appear to be of crucial importance (no matter how small ν may be) in determining the level of circulation developed.

This circulation may be represented in terms of the vorticity field $\boldsymbol{\omega} = \omega(x, y)\,\mathbf{k}$. Suppose that the Reynolds number Re of the flow generated is *large*. Then in the core region (where $\mathbf{F} \approx 0$), provided the streamlines are closed within this region, the vorticity is uniform (Batchelor, 1956), i.e. $\omega(x, y) = \omega_0$ (Fig. 7). This region of uniform vorticity extends over nearly the whole cross section, of area $\mathscr{A}$ say, and so the circulation around a closed curve C_0 bounding this region of uniform vorticity is given by

$$K_0 = \oint_{C_0} \mathbf{u} \cdot d\mathbf{x} \approx \omega_0 \mathscr{A}. \tag{55}$$

If K_0 can be determined from a boundary layer analysis, then the vorticity (and so the flow) in the core is also determined.

The following argument determines the order of magnitude of K_0. Let $\psi(\xi, \eta)$ be the stream function of the flow in the skin. Given that viscosity is the controlling influence in the skin, we have

$$\rho\nu \frac{\partial^2 u}{\partial \eta^2} \approx -F_\xi \approx -\frac{\delta}{2} T(\xi)\, e^{-2\eta/\delta}, \tag{56}$$

where $u = \partial\psi/\partial\eta$ is the ξ-component of velocity. Hence

$$u(\xi, \eta) \approx \frac{\delta^3}{8\rho\nu} T(\xi)\,(1 - e^{-2\eta/\delta}) \tag{57}$$

$$\sim \frac{\delta^3}{8\rho\nu} T(\xi) \quad \text{as} \quad \eta/\delta \to \infty, \tag{58}$$

and so

$$K_0 = \frac{\delta^3}{8\rho\nu} \oint_C T(\xi)\, d\xi \sim \frac{\omega\delta^4 B_0^2}{\rho\nu\mu_0\lambda}. \tag{59}$$

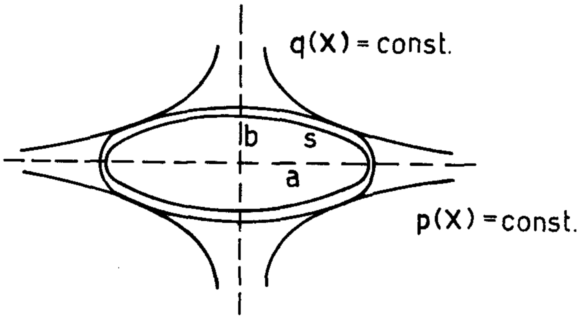

FIG. 8. The elliptic cylinder problem (after Sneyd, 1979).

There is, however, a difficulty* in making this argument more precise. This is that the tangential velocity distribution given by (57) is in general different from the tangential velocity distribution that would be obtained from solution of the 'core' problem

$$\nabla^2\psi = -\omega_0, \qquad \psi = 0 \text{ on } S. \tag{60}$$

Consequently a region of adjustment in which inertial and viscous forces both play a role, or alternatively a boundary layer of thickness $a/\mathrm{Re}^{1/2}$, embedded within the skin of thickness δ, is to be expected. (The same type of difficulty arises in the conceptually simpler problem of flow at high Reynolds number in a cylindrical container when the tangential velocity is a prescribed function of position on the boundary.)

The case of an elliptic cylinder (Fig. 8)

The case of a *nearly circular* elliptic cylinder has been completely solved by Sneyd (1979) and the difficulty referred to above has been overcome in this case. Here we consider instead the general elliptic cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{61}$$

the field being given by

$$\mathbf{A} = \big(p(x, y)\cos\omega t + q(x, y)\sin\omega t\big)\,\mathbf{k}, \tag{62}$$

where

$$p \sim B_0 r\cos\theta, \qquad q \sim B_0 r\sin\theta \quad \text{as} \quad r \to \infty. \tag{63}$$

The procedure is as follows:

(i) Solve the potential problems

$$\nabla^2 p = \nabla^2 q = 0, \qquad p = q = 0 \quad \text{on } S\colon \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \tag{64}$$

* I am indebted to Dr. A. D. Sneyd (private communication) who made me aware of this difficulty.

(ii) calculate $\partial p/\partial n$ and $\partial q/\partial n$ on S;
(iii) let $\beta(\xi) = \partial p/\partial n + i\,\partial q/\partial n$ on S, and calculate $T(\xi)$ as given by (51);
(iv) calculate ω_0 as given (at least in order of magnitude) by (55) and (59).

Steps (i) and (iii) are straightforward, and lead to the expression

$$T(\xi) = \frac{\omega\delta B_0^2(a+b)^2}{8\sqrt{2}\,\mu_0\lambda}\left[\frac{\sqrt{2}(a^2+b^2)-(a^2-b^2)\cos(2\phi+\pi/4)}{(a^2\sin^2\phi+b^2\cos^2\phi)^{5/2}}\right] \tag{65}$$

where $x = a\cos\phi$, $y = b\sin\phi$ on the ellipse; (ξ and ϕ are related by $d\xi = (a^2\sin^2\phi + b^2\cos^2\phi)^{1/2}\,d\phi$.) Vorticity production, as measured by $T(\xi)$, is clearly non-uniformly distributed around the cylinder boundary. When $b \ll a$, there are pronounced peaks in $T(\xi)$ near $\phi = 0$ and $\phi = \pi$, i.e. near the points of maximum curvature of the surface. In general, however, as pointed out by Sneyd (1979), the points of maximum $T(\xi)$ do *not* coincide with points of maximum curvature (contrary to a conjecture of Moffatt, 1965).

Step (iv) leads (using (55)) to the result

$$\omega_0 \approx \frac{\lambda B_0^2}{16\rho\nu\mu_0\omega a^2}\,\frac{(1+\varepsilon)^2(1+\varepsilon^2)^2}{\varepsilon^4} \tag{66}$$

for the core vorticity, where $\varepsilon = b/a$. As pointed out above, however, the method used to derive (59) is not altogether satisfactory, and the result (66) may be incorrect, although probably correct in order of magnitude. The singular behaviour as $\varepsilon \to 0$, keeping a fixed, is a consequence of the singularity in $T(\xi)$ when $b \ll a$, as noted above.

As in the case of the circular cylinder, inclusion of $\mathbf{u} \wedge \mathbf{B}$ effects may lead to the possibility of two steady solutions, but analysis of this situation would obviously present great difficulties.

5. The strong field situation

Let us return now to the case of the cylinder of circular cross section. As noted in §3, the flow with circular streamlines is not possible when the Hartmann number M is large. Even when M is small, small departures from circular streamlines are to be expected, for the following reason. We have argued that, when the $\mathbf{u} \wedge \mathbf{B}$ term is retained in Ohm's law, the expression (17) for $\mathbf{F}$ should be replaced by

$$\mathbf{F} = \frac{\tilde{\omega}(r)}{2\mu_0 r}\left[p\nabla q - q\nabla p + \nabla\{pq\cos 2\omega t + (p^2-q^2)\sin 2\omega t\}\right], \tag{67}$$

where $\tilde{\omega}(r) = \omega - v(r)/r$, $v(r)\,\hat{\mathbf{e}}_\theta$ being the 'zero-order' velocity field. Hence

$$\nabla \wedge \mathbf{F} = \frac{1}{2\mu_0\lambda}\left[2\tilde{\omega}\nabla p \wedge \nabla q + \tilde{\omega}'(r)\,\hat{\mathbf{e}}_r \wedge (p\nabla q - q\nabla p)\right.$$
$$\left. + \tilde{\omega}'(r)\,\hat{\mathbf{e}}_r \wedge \nabla\{pq\cos 2\omega t + (p^2-q^2)\sin 2\omega t\}\right]. \tag{68}$$

There is now a non-zero time-periodic ingredient, which clearly must drive a perturbation to the flow, related to the instantaneous configuration of the applied field.

In this situation, the problem is more easily analysed in a frame of reference rotating with the field. In this frame, the applied field $\mathbf{B}_0(\mathbf{x})$ is steady, but the cylinder rotates with angular velocity $-\omega\mathbf{k}$. The problem is to find $\mathbf{u}(\mathbf{x})$ in $r < a$ satisfying $\mathbf{u} = -\omega a \hat{\mathbf{e}}_\theta$ on $r = a$.

Normally, one might expect complications to arise from the Coriolis force in a rotating frame of reference. Here however the Coriolis force is

$$2\omega\mathbf{k} \wedge \mathbf{u} = -2\omega\mathbf{k} \wedge (\mathbf{k} \wedge \nabla\psi) = \nabla(2\omega\psi). \tag{69}$$

Since this is irrotational, it is compensated by a pressure adjustment, and does not affect the velocity field.

In the rotating frame with $A = \operatorname{Re} f(r)\, e^{i\phi}$ (corresponding to the case of a uniform applied field) the expression (68) becomes

$$\nabla \wedge \mathbf{F} = -\frac{1}{2\lambda\mu_0 r}\frac{d}{dr}\tilde{\omega}(r)\,|f|^2 - \frac{\tilde{\omega}'(r)}{2\lambda\mu_0 r}\{\operatorname{Re}(f^2)\cos 2\phi + \operatorname{Im}(f^2)\sin 2\phi\} \tag{70}$$

(where $\phi = \theta - \omega t$) and the second term will clearly drive a non-circular streamline pattern. (When we return to the laboratory frame, this of course becomes the time-periodic flow driven by (68).) This steady perturbation to the circular streamline pattern was considered by Smith (1964), in the low frequency limit $\omega a^2/\lambda \ll 1$, in a treatment based on power series expansions in the Hartmann number.

The strong field limit has been considered more recently by Alemany and Moreau (1977), again under the assumption $\omega a^2/\lambda \ll 1$. Under the additional assumption that

$$N = M^2/R \gg 1, \tag{71}$$

inertia effects may be neglected, and the equation for the stream function takes the well-known form

$$\nabla^4\psi = \frac{M^2}{a^2}\frac{\partial^2\psi}{\partial x^2}, \tag{72}$$

where the x-axis is in the direction of the applied field. When $M \gg 1$, there is a non-uniform Hartmann layer on the cylinder boundary (Fig. 9), within which (72) is approximated by

$$\frac{\partial^4\psi}{\partial r^4} \approx \frac{M^2}{a^2}\cos^2\phi\,\frac{\partial^2\psi}{\partial r^2}, \tag{73}$$

where ϕ is measured from the x-axis; (73) is valid only provided ϕ is not too near $\pi/2$ or $3\pi/2$. The relevant solution of (73) is

$$u_\phi = \frac{\partial\psi}{\partial r} = -\omega a e^{-M|\cos\phi|(1 - r/a)}, \tag{74}$$

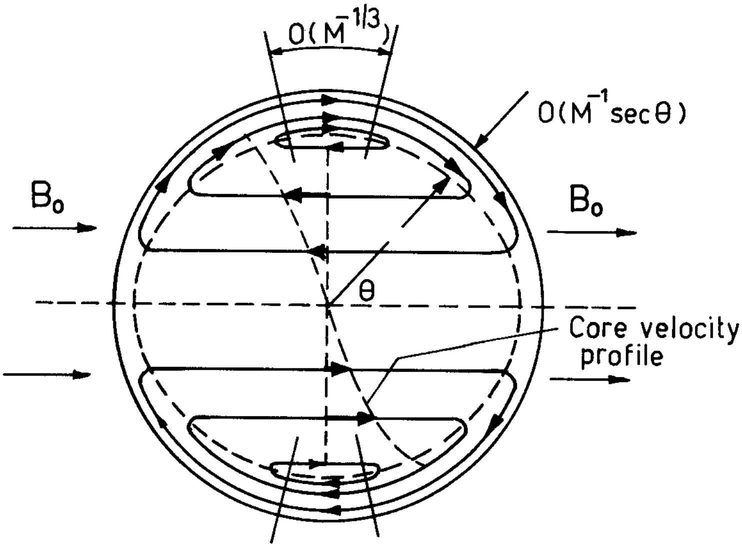

FIG. 9. The circular cylinder problem in the strong field limit (after Alemany and Moreau, 1977). The streamline pattern is steady in a frame of reference rotating with the field, and is characterised by a non-uniform Hartmann layer, and a weak return flow in the inviscid core.

i.e. a Hartmann layer of thickness

$$\delta(\phi) = \frac{a}{M|\cos\phi|}. \tag{75}$$

[Near $\phi = \pi/2, 3\pi/2$, it is not difficult to show that $\delta \sim aM^{-2/3}$.]

The total flux in the Hartmann layer is

$$Q(\phi) = \int_0^a u_\phi \, dr = -\frac{\omega a^2}{\lambda}|\sec\phi|, \tag{76}$$

and, since $dQ/d\phi \neq 0$, there is a weak compensating flow in the core. In fact, in the core $\mathbf{u} = v(y)\,\mathbf{i}$ and the mass-conservation equation in the form

$$v\,dy = dQ \tag{77}$$

immediately gives $v(y)$ in the form

$$v(y) = -\frac{\omega a^3}{M}\frac{y}{(a^2 - y^2)^{3/2}}, \tag{78}$$

as given by Alemany and Moreau. Note that $v(y) = O(\omega a)$ when

$$\frac{a - |y|}{a} = O(M^{-2/3}), \tag{79}$$

and this defines the extent of the singular regions near $\phi = \pi/2, 3\pi/2$.

It will be evident that the flow field in the strong field limit is very far removed from the circular streamline flow that results when $\mathbf{u} \wedge \mathbf{B}$ effects are negligible, and is controlled by a strong tendency to eliminate any component of motion perpendicular to the applied field in the core region. This tendency is evident also when the applied field is non-uniform in which case again the core flow is characterised by the condition $\mathbf{u} \wedge \mathbf{B}_0(\mathbf{x} = 0)$ (Alemany and Moreau, 1977).

6. Some comments on three-dimensional effects

If the finite length of the container is taken into account, then the current and field distributions are necessarily three-dimensional, and the curl of the Lorentz force then has a time-periodic part as well as a steady part. It is again natural to take axes rotating with the applied field; care must now be taken to include the Coriolis force which is no longer irrotational.

The only attempts to analyse such situations are those of Dremov and Kapusta (1970) who considered flow in a cylinder of finite length and Nigam (1969) who considered flow in a sphere; these authors considered only the steady ingredient of the Lorentz force, and furthermore restricted attention to a low Reynolds number situation in which case inertia forces (including Coriolis forces) provide at most a small perturbation. Nigam's assertion that the periodic part of $\nabla \wedge \mathbf{F}$ is zero for the case of a sphere seems to be incorrect, according to Sneyd (1979). The steady part of the Lorentz force drives both a swirling motion in the sense of the field rotation, and an axisymmetric motion in meridian planes through the axis of rotation of the field. These two ingredients of the flow field may be expected to interact if Reynolds number effects are taken into consideration.

7. Concluding remarks

It will be evident that, despite the potential importance of rotating field effects in molten metal technology, only the most idealised situations are at present at all well understood. Even the most idealised situation, viz the infinite cylinder of circular cross section, is understood only in rather restricted regions of parameter space, and even there, some of the assertions made (e.g. in §3 above) have a speculative flavour. It is to be hoped that significant progress will be made in analysing these magnetically driven flows over the next few years.

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