1.1
Induction in turbulent conductors

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1. Introduction

In this brief review of what is now a well-established theory, I shall attempt to set the scene for some of the subsequent chapters.

Kinematic dynamo theory is governed by Faraday's law,

$$\partial \mathbf{B}/\partial t = -\nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0, \tag{1.1}$$

Ampere's law,

$$\mathbf{j} = \mu_0^{-1} \nabla \times \mathbf{B}, \tag{1.2}$$

and Ohm's law,

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \mathbf{j}/\sigma, \tag{1.3}$$

where, in standard notation, $\mathbf{E}$ and $\mathbf{B}$ represent electric and magnetic fields, $\mathbf{j}$ is electric current density, $\mathbf{u}$ is velocity field, and σ is the electrical conductivity of the fluid. Elimination of $\mathbf{j}$ and $\mathbf{E}$ from (1.1)–(1.3) gives the well-known induction equation

$$\partial \mathbf{B}/\partial t = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \tag{1.4}$$

where $\eta = (\mu_0 \sigma)^{-1}$ is the magnetic diffusivity of the fluid (assumed uniform). In the context of planetary interiors, it is certainly reasonable to suppose that we are dealing with an incompressible fluid, i.e.

$$\nabla \cdot \mathbf{u} = 0. \tag{1.5}$$

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In the solar context (and in stellar contexts), the incompressibility condition (1.5) is reasonable only for scales of motion small compared with a scale height $(d\ln\rho/dz)^{-1}$ where $\rho(z)$ is the density distribution as a function of height z.

The first term on the right of (1.4), $\nabla \times (\mathbf{u} \times \mathbf{B})$, represents a tendency to convect and distort magnetic lines of force (**B**-lines). In a turbulent flow, any two fluid particles close together tend to move apart; **B**-lines therefore tend to increase in length and the field strength tends to increase in proportion. The diffusion term in (1.4), $\eta\nabla^2\mathbf{B}$, describes diffusion of field relative to fluid, and generally tends to eliminate high field gradients. The relative importance of these two terms is measured by the magnetic Reynolds number R_m,

$$R_m = u_0 l_0/\eta \approx |\nabla \times (\mathbf{u} \times \mathbf{B})| / \eta |\nabla^2\mathbf{B}|, \quad (1.6)$$

where u_0 and l_0 are typical scales of velocity and length; if $R_m \gg 1$, diffusion is weak, and the convection effect dominates.

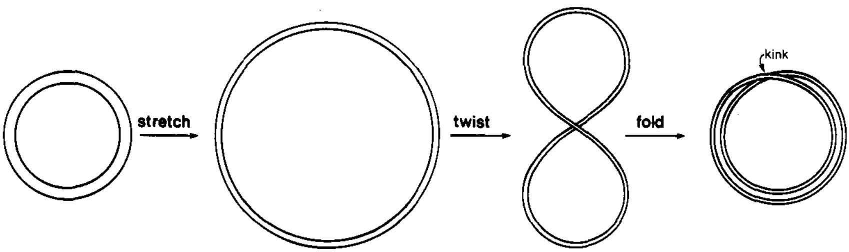


FIGURE 1 The Alfvén dynamo mechanism whereby field intensity is doubled by the stretch–twist–fold cycle; the kink in the final double loop can be eliminated only by the molecular diffusion process.

The dynamo process is essentially a process of systematic amplification of a magnetic field, without ultimate change in its structure. This process has to be three-dimensional since two-dimensional and axisymmetric dynamo processes are excluded by Cowling's theorem and its various generalisations. The simplest three-dimensional process leading to field amplification was conceived by Alfvén (1950), and is illustrated in Figure 1. We start with a circular flux tube which is then distorted in three stages — stretch, twist and fold — to give a doubled tube as illustrated; a little diffusion ($\eta \neq 0$ in (1.4)) will then (presumably) be sufficient to "iron out" the kink and to give a field with exactly the original structure, but double the intensity. Repetition of this cycle then implies "unlimited" field amplification — in practice the limit is ultimately set by dynamic (or energetic) considerations.

It is important to recognise the role of diffusion in the above process. It is of course the erosive effect of diffusion which implies field decay in the absence of motion, and which makes us seek to explain field maintenance in terms of a dynamo process in the first place. The same non-zero diffusion turns out to be of crucial importance also in making a dynamo (or regenerative) process a possibility. Without the effect of diffusion in the above cycle, to iron out the small-scale kink, the field would become more and more complex in structure as the cycle is repeated; the ultimate structure would *not* be the same as the initial structure, and we would not have a genuine dynamo process.

Modern dynamo theory has been dominated by three overlapping approaches. The first is that of Parker (1955, 1970, 1979), which is based on the idea of representing turbulence as a field of "random cyclonic events". The second is that of Braginsky (1964a, b) (subsequently elaborated and clarified by Soward, 1972); this approach was developed primarily in the geophysical context, and is based on the idea that the velocity field in the core of the Earth is probably *nearly* axisymmetric about the axis of rotation — so that a theory based on *weak* departures from axisymmetry is indicated. The third is the theory of mean-field electrodynamics initiated by Steenbeck, Krause and Rädler (1966; see also Krause and Rädler, 1980) which provides (in my view) the simplest and yet most general formulation of the problem.

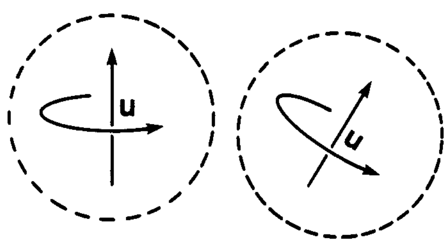


FIGURE 2 Parker's (1955) random cyclonic events, within each of which velocity and vorticity are correlated.

2. Some comments on the kinematical approaches of Parker (1955) and Braginsky (1964a, b)

Parker's (1955) theory rests, as mentioned above, on the concept of the "random cyclonic event" which is a motion localised in both $\mathbf{x}$ and t, in which velocity $\mathbf{u}$ and vorticity $\boldsymbol{\omega}$ are strongly correlated (Figure 2). The helicity of such an event is

$$H(t) = \int \mathbf{u} \cdot \boldsymbol{\omega} \, dV . \tag{2.1}$$

H is a pseudo-scalar (it changes sign under a change from right-handed to left-handed frame of reference). If $H > 0$ in a right-handed frame, then the streamlines have a right-handed helical structure. If $H < 0$, they have a left-handed structure. If the duration of an event is t_e, then

$$\left| \int H(t) \, dt \right| \sim H_0 t_e , \tag{2.2}$$

where H_0 is (say) the maximum value of $H(t)$.

Parker conceived of a random distribution of such events, well separated in space and in time, so that the net effect of each event on an ambient magnetic field could be treated in isolation. A long period of stasis ($\mathbf{u} = 0$) was imagined between events, during which diffusion eliminated "unwanted" field perturbations (like the kink in the Alfvén cycle). This elimination takes place on the stasis time-scale

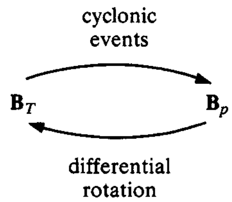
$$t_s \sim l_0^2 / \eta , \tag{2.3}$$

where l_0 is the length-scale of a typical event.

The major achievement of the analysis lay in the conception and construction of a process by which poloidal field could be generated from toroidal field in a spherical geometry. Parker found that the net effect of a random superposition of cyclonic events was to provide a toroidal electromotive force $\mathscr{E}_\varphi$ proportional to the toroidal field B_φ:

$$\mathscr{E}_\varphi = \Gamma B_\varphi , \tag{2.4}$$

(a forerunner of the α-effect of Steenbeck, Krause and Rädler, 1966). $\mathscr{E}_\varphi$ drives a toroidal current J_φ which acts as the source of a poloidal field $\mathbf{B}_p$. The toroidal field $\mathbf{B}_T = B_\varphi \hat{\mathbf{e}}_\varphi$ can be regenerated from $\mathbf{B}_p$ by the very well-known mechanism of differential rotation. Hence we have the possibility of a closed dynamo cycle:



(2.5)

or what is now described as an "αω-dynamo".

The coefficient Γ in (2.4) is clearly of central importance. It has the dimensions of a velocity, and Parker's usual estimate, backed up by formal calculations (Parker, 1970, 1979; see also Moffatt, 1978, §7.10) is

$$\Gamma \sim u_c\,, \tag{2.6}$$

where u_c is a typical velocity in a cyclonic event (and one would then expect $u_c \sim u_0 = \langle \mathbf{u}^2 \rangle^{1/2}$ in maximally helical turbulence). The estimate (2.6) may well be correct, (when $R_m \gg 1$), although in this limit there are difficulties that are to some extent concealed in Parker's analysis. One difficulty is this: that, the smaller η is, the larger is the time t_s, according to (2.3), to eliminate small-scale field — or to restore the *status quo* as regards ambient field structure. The regenerative process is then effective in any region only during a fraction

$$t_e/t_s \sim t_e\eta/l_0^2 \tag{2.7}$$

of the total time available; as $\eta \to 0$ this fraction obviously goes to zero, and one would expect the generation coefficient to have this property also; i.e. one would expect that

$$\Gamma \to 0 \text{ as } \eta \to 0\,, \tag{2.8}$$

in contrast to the estimate (2.6), which does not depend on η.

Braginsky's (1964a, b) theory, by contrast, *does* lead to an expression for Γ which is proportional to η for small η. In this approach, it is supposed that the magnetic Reynolds number R_m based on the (dominant) axisymmetric toroidal velocity field is large, and perturbation fields $\mathbf{u}'$ and $\mathbf{b}$ are expanded in powers of $R_m^{-1/2}$. When the field $\mathbf{u}'$ is helical in character (i.e. $\langle \mathbf{u}' \cdot \nabla \times \mathbf{u}' \rangle = 0$, the average now being over the azimuthal angle φ) it turns out that

$$\mathscr{E}_\varphi = \langle \mathbf{u}' \times \mathbf{b} \rangle_\varphi = \Gamma B_\varphi\,, \quad \Gamma \sim R_m^{-1}\,. \tag{2.9}$$

Braginsky's analysis required not only that $\mathbf{u}'$ be small (in fact precisely of order $R_m^{-1/2}$), but that the displacement ζ of a fluid particle from its mean azimuthal circle remain small for all time. There are then (implicitly) two scales in the theory — (i) the scale l typical of the displacement field ζ and (ii) the radius of the mean circle L (in general of the same order as the radius R of the sphere of fluid). The distinct scale separation $l \ll L$ means that $\Gamma(r, \theta)$ is determined *locally* by $U(r, \theta)$ and the azimuthally averaged properties of $\mathbf{u}$. It is this local property which makes the resulting azimuthally averaged equations tractable.

Braginsky's theory is well-adapted to dynamical (as opposed to merely kinematical) development. Soward (1982) outlines some of these dynamical developments in Chapter 5.1 (see also Childress (1982) in Chapter 5.2). I shall say no more about the theory here.

3. Mean field-electrodynamics (Steenbeck, Krause and Rädler, 1966)

The concept of two scales is fundamental to the approach of Steenbeck, Krause and Rädler (1966). We may formalise the notion that $\mathbf{u}$ varies on the small (turbulent) scale l_0 and on the large (macro) scale L by introducing two space variables $\mathbf{x}$ and $\mathbf{X} = \varepsilon\mathbf{x}$ where $\varepsilon = l_0/L$, and by regarding $\mathbf{u}$ as a function of $\mathbf{x}$ and $\mathbf{X}$ (independently):

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathbf{X}), \tag{3.1}$$

(the time-dependence being implicit). Let $<\ldots>$ now represent an average over $\mathbf{x}$ keeping $\mathbf{X}$ constant;† (physically, this may be thought of as a space average over a volume large compared with l_0 but small compared with L). Thus, let

$$\mathbf{U}(\mathbf{X}) = <\mathbf{u}(\mathbf{x}, \mathbf{X})>, \quad \mathbf{u}'(\mathbf{x}, \mathbf{X}) = \mathbf{u} - \mathbf{U}, \tag{3.2}$$

(so that $<\mathbf{u}'> = 0$). Similarly, we may decompose $\mathbf{B}$ in the form

$$\mathbf{B} = \mathbf{B}_0(\mathbf{X}) + \mathbf{b}(\mathbf{x}, \mathbf{X}), \quad <\mathbf{b}> = 0. \tag{3.3}$$

Substitution in (1.4) and averaging leads to the mean-field equation

† Here we use a "local space average"; it is also possible to adopt a "local time average" or an "ensemble average" (Moffatt, 1978, §7.1).

$$\partial \mathbf{B}_0/\partial t = \varepsilon \nabla \times (\mathbf{U} \times \mathbf{B}_0) + \varepsilon \nabla \times \mathscr{E} + \eta \varepsilon^2 \nabla^2 \mathbf{B}_0 , \qquad (3.4)$$

where $\mathscr{E} = \langle \mathbf{u}' \times \mathbf{b} \rangle$.

Here we have included the multipliers ε, ε^2, because ∇ here represents differentiation with respect to $\mathbf{X}$ and $\varepsilon\partial/\partial\mathbf{X} = \partial/\partial\mathbf{x}$. The potentially dominant importance of the term $\varepsilon\nabla \times \mathscr{E}$ when ε is sufficently small is immediately apparent.

Subtraction of (3.4) from (1.4) gives the fluctuation equation

$$\partial \mathbf{b}/\partial t - \nabla \times (\mathbf{U} \times \mathbf{b}) = \nabla \times (\mathbf{u}' \times \mathbf{B}_0) + \nabla \times \mathbf{G} + \eta \nabla^2 \mathbf{b} , \qquad (3.5)$$

where

$$\mathbf{G} = \mathbf{u}' \times \mathbf{b} - \langle \mathbf{u}' \times \mathbf{b} \rangle , \quad \langle \mathbf{G} \rangle = 0 . \qquad (3.6)$$

If we suppose that $\mathbf{b}(\mathbf{x}, 0) = 0$, equation (3.5) establishes a linear relationship between $\mathbf{b}$ and $\mathbf{B}_0$, and so between $\mathscr{E}$ and $\mathbf{B}_0$. If $\mathbf{B}_0$ were strictly uniform (instead of slowly varying) this relationship could only take the form

$$\mathscr{E}_i = \alpha_{ij}(\mathbf{X})\, B_{0j} , \qquad (3.7)$$

where $\alpha_{ij}(\mathbf{X})$ is determined (in principle) by the statistical (i.e. $\mathbf{x}$-averaged) properties of the velocity field, and the value of η, which intervenes in the solution of (3.5). Allowing for the weak spatial variation of B_{0j}, (3.7) must be regarded as the leading term of a series

$$\mathscr{E}_i(\mathbf{X}) = \alpha_{ij}(\mathbf{X})\, B_{0j}(\mathbf{X}) + \varepsilon \beta_{ijk}(\mathbf{X})\, \partial B_{0j}/\partial X_k + \dots , \qquad (3.8)$$

any time derivatives in this series being replaced by space derivatives by means of (3.4). Rapid convergence of the series (3.8) is to be expected when $\varepsilon \ll 1$.

Two important observations may be made at this point. First, the argument leading to (3.8) is very general and does *not* require any assumption of the form $|\mathbf{b}| \ll |\mathbf{B}_0|$. The perturbation field may be as large as, or even much larger than, the mean field, without invalidating the argument.

Secondly, the argument leading to (3.8) is still valid even in the "dynamic" regime when the Lorentz force $\mathbf{j} \times \mathbf{B}$ is significant (or even dominant) in the dynamical equation for $\mathbf{u}$. However the coefficients α_{ij}, β_{ijk}, . . . in (3.8) will then depend (via their dependence on the mean properties of $\mathbf{u}$) on the local value of $\mathbf{B}_0$. A mean electromotive force of the form (3.8) does in fact appear (in somewhat disguised form) in dynamical dynamo theories, such as those of Soward (1974) and Busse (1975) where space-averaging in one form or another is involved.

4. Symmetric and antisymmetric parts of $\alpha_{ij}(\mathbf{X})$

We may always decompose $\alpha_{ij}(\mathbf{X})$ into its symmetric and antisymmetric parts:

$$\alpha_{ij}(\mathbf{X}) = \alpha_{ij}^{(s)}(\mathbf{X}) - \varepsilon_{ijk}\gamma_k(\mathbf{X}) , \tag{4.1}$$

where $\alpha_{ij}^{(s)}(\mathbf{X})$ is a pseudo-tensor which is non-zero only if the turbulence lacks reflexional symmetry. On the other hand, γ is a pure (polar) vector, which is in general non-zero even for reflectionally symmetric turbulence.

At any point $\mathbf{X}$, we may choose axes so that $\alpha_{ij}^{(s)}$ is diagonal:

$$\alpha_{ij}^{(s)} = \begin{pmatrix} \alpha^{(1)} & \cdot & \cdot \\ \cdot & \alpha^{(2)} & \cdot \\ \cdot & \cdot & \alpha^{(3)} \end{pmatrix} , \tag{4.2}$$

and the corresponding contribution to $\mathscr{E}$ is

$$\mathscr{E}^{(s)} = (\alpha^{(1)}B_{01}, \alpha^{(2)}B_{02}, \alpha^{(3)}B_{03}) . \tag{4.3}$$

Comparison with (2.7) and (2.9) shows that there is a meeting point with the Parker and Braginsky theories, if, say, the 3-direction is identified with the azimuth (φ-) direction, and $\alpha^{(3)} = \Gamma$. More generally, the relationship (4.3) can provide self-exciting fields (in other words, growing solutions of the mean field equation (3.4)).

Similarly, the antisymmetric part of (4.1) provides a contribution to $\mathscr{E}$

$$\mathscr{E}^{(a)} = \gamma \times \mathbf{B}_0 , \tag{4.4}$$

and a corresponding contribution $\nabla \times (\gamma \times \mathbf{B}_0)$ on the right-hand side of (3.4). This means that the effective mean velocity acting on $\mathbf{B}_0$ is not $\mathbf{U}$ but is instead

$$\mathbf{U}_e(\mathbf{X}) = \mathbf{U}(\mathbf{X}) + \gamma(\mathbf{X}) . \tag{4.5}$$

There is here an important qualitative effect, in that $\gamma(\mathbf{X})$ *is not* in general solenoidal even if the background velocity $\mathbf{u} = \mathbf{U} + \mathbf{u}'$ *is* strictly solenoidal. The mean magnetic field is nevertheless convected and distorted by this non-solenoidal field, as if it were a real part of the fluid motion.

This effect was first discovered by Zel'dovich (1956) in his study of the effect of two-dimensional turbulence on an initially uniform magnetic field in the plane of the motion. In this situation the vector potential of **B** has only one component, i.e. $\mathbf{B} = \nabla \times (0, 0, A(x, y))$, and A satisfies the heat conduction equation

$$\partial A/\partial t + \mathbf{u} \cdot \nabla A = \eta \nabla^2 A . \tag{4.6}$$

If we imagine that the fluid is contained in a large cylinder, $x^2 + y^2 < a^2$, then we have to solve (4.6) subject to a boundary condition specifying the variation of A on the cylinder. For the corresponding thermal problem, with a non-uniform temperature prescribed on $x^2 + y^2 = a^2$, one would expect the turbulence to establish a uniform temperature in a "core" region, all temperature variation being confined to a thermal boundary layer. Zel'dovich argued by analogy that, when $R_m \gg 1$, A will tend to a constant in the core region, and equivalently **B** must tend to zero. This is the *flux expulsion* effect, later studied in detail analytically by Parker (1963) and numerically by Weiss (1966). In the context of mean-field electrodynamics, this flux expulsion can be interpreted as due to a radial velocity field $\boldsymbol{\gamma}(\mathbf{X})$ which convects the magnetic field outwards from the turbulent region (Figure 3). Rädler (1968) has described this as "diamagnetic" behaviour.

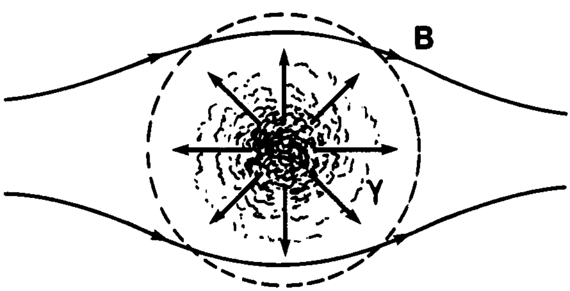


FIGURE 3 Flux expulsion due to a "γ-effect" associated with a radial gradient of turbulent intensity (Rädler, 1969).

In some circumstances, however, a "paramagnetic" effect may be envisaged. As shown by Drobyshevski and Yuferev (1974), topological asymmetry in a convecting layer will generally lead to transport (or "pumping") of magnetic flux in the direction of "connected fluid flow". In the solar context, this direction in the convection zone is vertically downwards, since, on average, rising hot blobs of fluid are disconnected, whereas the falling fluid subsides in a connected network. We may therefore expect $\boldsymbol{\gamma}(\mathbf{X})$ to be radially inward, and magnetic flux will then be pumped downwards, a paramagnetic behaviour. In Chapter 1.4 Nightingale (1982) discusses the important influence that this may have on certain dynamo models.

5. Explicit calculation of $\alpha_{ij}(\mathbf{X})$

There are two limiting situations in which $\alpha_{ij}(\mathbf{X})$ may be explicitly calculated.

The weak turbulence limit

If $\mathbf{u}'$ is (in some sense) weak, then $\mathbf{b}$ will also be weak, and the term involving $\mathbf{G}$ in (3.5) may then be neglected. There appear to be two situations in which this important "first-order smoothing" approximation may be made:

$$(a)\ |\mathbf{u}'|\, l/\eta \ll 1, \quad \text{and}\ (b)\ |\mathbf{u}'|/\omega l \ll 1, \tag{5.1}$$

where ω is a typical frequency of the $\mathbf{u}'$ field. The condition (a) may be a reasonable one in planetary contexts, but it is certainly not satisfied in the solar context; the condition (b) is not satisfied by turbulence, as normally observed, for which $|\mathbf{u}'|/\omega l = O(1)$, but it may be applicable when $\mathbf{u}'$ represents a sea of weakly interacting waves — e.g. inertial waves or internal gravity waves — governed by a dispersion relation $\omega = \omega(\mathbf{k})$.

In either case, if $\mathbf{G}$ is neglected in (3.5) we have to solve a linear inhomogeneous equation for $\mathbf{b}$. Since $\mathbf{U}$ varies on a scale large compared with that of $\mathbf{u}'$, it is self-consistent to treat $\mathbf{U}$ as locally uniform, and to refer to a frame of reference moving with this velocity.† This has the effect of eliminating the term $\nabla \times (\mathbf{U} \times \mathbf{b})$ also. We may then solve for $\mathbf{b}$ using simple Fourier transform techniques, and then construct $\mathscr{E} = \langle \mathbf{u}' \times \mathbf{b} \rangle$

† The effect of the local gradient of U on the small-scale fields can be incorporated (see Krause and Rädler, 1980, Chapter 8).

The coefficient of $\mathbf{B}_0$ in this expression provides the tensor α_{ij}. It is found in this approximation that α_{ij} is symmetric (i.e. $\boldsymbol{\gamma} = 0$) and

$$\alpha = \tfrac{1}{3}\alpha_{ii} = -\tfrac{1}{3}\eta \iint \frac{k^2 F(k,\omega)}{\omega^2 + \eta^2 k^4}\, dk\, d\omega\,, \tag{5.2}$$

where $F(k, \omega)$ is related to the mean helicity $\langle \mathbf{u}' \cdot \nabla \times \mathbf{u}' \rangle$ by

$$\langle \mathbf{u}' \cdot \nabla \times \mathbf{u}' \rangle = \iint F(k,\omega)\, dk\, d\omega\,, \tag{5.3}$$

that is to say $F(k, \omega)$ is the helicity spectrum function. A very similar expression to (5.2) was found by Braginsky (1964a, b) for the coefficient Γ in (2.9); this significant point of contact in the two theories is almost miraculous in view of the (apparently) totally different initial assumptions!

The order of magnitude of the α calculated above under the assumption (*a*) is

$$\alpha \sim R_m^2 \eta / l\,, \quad R_m = |\mathbf{u}'|\, l/\eta \ll 1\,, \tag{5.4}$$

and, as noted above, γ is zero at this order of approximation. Iteration does however give a γ-effect involving triple correlations, and in order of magnitude

$$\gamma \sim R_m^3 \eta / l \quad \text{when } R_m \ll 1\,. \tag{5.5}$$

The coefficient $\beta_{ijk}(\mathbf{X})$ in (3.8) may also be calculated by this method, and it is found that, when $R_m \ll 1$,

$$\beta = \tfrac{1}{6}\varepsilon_{ijk}\beta_{ijk} \sim R_m^2 \eta\,. \tag{5.6}$$

This particular combination of the elements of β_{ijk} acts as an "eddy diffusivity" on $\mathbf{B}_0$, but clearly in this limit ($R_m \ll 1$) the effect is swamped by the dominant molecular diffusion process. Rädler (1969) has emphasised the possible importance of other combinations of elements of β_{ijk} which may also be associated with regenerative (i.e. dynamo) action. My own view (Moffatt, 1978, §7.9) is that such effects are unlikely to provide more than a small perturbation of the α-effect; this is because these effects, like the α-effect, rely for their existence on a lack of reflexional symmetry in the background turbulence, and when this occurs the β-terms of the expansion (3.8) are generally smaller by a factor ε than the α-terms — but see Moffatt and Proctor (1982).

(iii) The high conductivity limit ($\eta \to 0$)

This limit is of course the relevant limit to consider in the solar context, where R_m (based on granulation scales) is of order 10^4. There are, however, peculiar difficulties in treating this limit, which have not as yet been fully resolved.

One can start with the rather drastic assumption that η can be simply put equal to zero. A Lagrangian analysis does then lead to an expression for α that looks at least plausible, viz. (Moffatt, 1974)

$$\alpha = -\tfrac{1}{3} \lim_{t\to\infty} \int_0^t \langle \mathbf{v}(\mathbf{a}, t) \cdot \nabla_{\mathbf{a}} \times \mathbf{v}(\mathbf{a}, \tau) \rangle \, d\tau \qquad (5.7)$$

where $\mathbf{v}(\mathbf{a}, t) = \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t)$ is the velocity in Lagrangian variables; and a similar, but more complicated, expression for β. The expression (5.7) is closely analogous to the expression for turbulent diffusivity obtained by Taylor (1921) in the scalar diffusion problem. It is also akin to the expression for Γ obtained by Parker (1970), on which his estimate (2.6) was based.

Total neglect of diffusion is however an admittedly dangerous procedure; for example, it leads also to the conclusion that the mean square field fluctuation $\langle \mathbf{b}^2 \rangle$ increases without limit through the frozen field distortion effect, but we know that the field gradient also increases without limit in this process so that ultimately the effects of molecular diffusion must exert some kind of controlling influence. Mathematically, the difficulty is that we are interested in asymptotic behaviour as $t \to \infty$, but the limiting processes $t \to \infty$ and $\eta \to 0$ may not commute. They certainly do not commute in the better understood problem of flux expulsion — see the discussion of Moffatt (1978, §3.8).

An alternative approach to the high conductivity limit recognizes that flux expulsion will occur from eddies that are sufficiently long-lived and that dynamo action will then have to be concentrated in thin layers and strands at the boundaries of such eddies. The α-effect associated with such processes were investigated by Childress (1979) and in Chapter 2.2 Childress (1982a) explores these "flux-rope dynamos" from a slightly different point of view.

A third approach, under active consideration (for a preliminary discussion, see Moffatt, 1981), involves successive averaging over a large number of "nested" length-scales — a procedure related to the renormalization-group technique, which has been developed in the context of turbulence by Forster, Nelson and Stephen (1977). This leads to coupled differential equations for functions $\alpha(k)$, $\beta(k)$ representing the α-effect and β-effect (i.e. eddy diffusivity) associated with all scales of motion less than $O(k^{-1})$. The approach works well for the simpler problem of a convected scalar field, and this provides some encouragement that the results for the magnetic field problem, although hard to justify rigorously, may nevertheless have some validity.

References

Alfvén, H., "Origin of solar magnetic fields," *Tellus* **2**, 74 (1950).

Braginsky, S.I., "Self-excitation of a magnetic field during the motion of a highly conducting fluid," JETP **47**, 1084–1098 (1964a), Engl. transl. *Sov. Phys.* JETP **20**, 726–735 (1965).

Braginsky, S.I., "Theory of the hydromagnetic dynamo," JETP **47**, 2178–2193 (1964b), Engl. transl. *Sov. Phys.* JETP **20**, 1462–1471 (1965).

Busse, F.H., "A model of the geodynamo," *Geophys. J. Roy. astr. Soc.* **42**, 437–459 (1975).

Childress, S., "Alpha-effect in flux ropes and sheets," *Phys. Earth Planet. Inter.* **20**, 172–180.

Childress, S., "Stationary induction by intermittent velocity fields," in these Proceedings, 81–90 (1982a).

Childress, S., "The macrodynamics of spherical dynamos," in these Proceedings, 245–257 (1982b).

Drobyshevski, E.M. and Yuferev, V.S., "Topological pumping of magnetic flux by three-dimensional convection," *J. Fluid Mech.* **65**, 33–44 (1974).

Forster, D., Nelson, D.R. and Stephen, M.J., "Large distance and long-time properties of a randomly stirred fluid," *Phys. Rev.* **A16**, 732–749 (1977).

Krause, F. and Rädler, K.-H., *Mean-field magnetohydrodynamics and dynamo theory*, Akademie-Verlag, Berlin and Pergamon Press, Oxford (1980).

Moffatt, H.K., "The mean electromotive force generated by turbulence in the limit of perfect conductivity," *J. Fluid Mech.* **65**, 1–10 (1974).

Moffatt, H.K., *Magnetic field generation in electrically conducting fluids*, Cambridge: University Press (1978).

Moffatt, H.K., "Some developments in the theory of turbulence," *J. Fluid Mech.* **106**, 27–47 (1981)

Moffatt, H.K. and Proctor, M.R.E., "The role of the helicity spectrum function in turbulent dynamo theory," *Geophys. Astrophys. Fluid Dynam.* **21**, 265–283 (1982).

Nightingale, S.J., "Topological pumping effects on the α^2-mechanism," in these Proceedings, 49–63 (1982).

Parker, E.N., "Hydromagnetic dynamo models," *Astrophys. J.* **122**, 293–314 (1955).

Parker, E.N., "Kinematical hydromagnetic theory and its applications to the low solar photosphere," *Astrophys. J.* **138**, 552–575 (1963).

Parker, E.N., "The generation of magnetic fields in astrophysical bodies, I. The dynamo equations," *Astrophys. J.* **162**, 665–673 (1970).

Parker, E.N., *Cosmical magnetic fields*, Clarendon Press, Oxford (1979).

Rädler, K.-H., "Zur Elektrodynamik turbulent Beusugler leitender Medien II. Turbulenzbedingte Leithähigkeits — und permeabilitätsänderungen," *Z. Naturforsch.* **23a**, 1851–1860 (1968).

Rädler, K.-H., "Zur Electrodynamik in turbulanten, Coriolis — Kräften unterworfenen lietenden Medien," *Mber. Dtsk. Akad. Wiss. Berlin* **11**, 194–201 (1969).

Soward, A.M., "A kinematic theory of large magnetic Reynolds number dynamos." *Phil. Trans. Roy. Soc.* **A272**, 431–462.

Soward, A.M., "A convection-driven dynamo I. The weak field case," *Phil. Trans. Roy. Soc.* **A275**, 611–651 (1979).

Soward, A.M., "Convection-driven dynamos," in these Proceedings, 237–244 (1982).

Steenbeck, M., Krause, F. and Rädler, K.-H., "Berechnung der mittleren Lorentz-

Feldstärke $\overline{\mathbf{v} \times \mathbf{B}}$ für ein elektrisch leitendes Medium in turbulenter, durch Coriolis – Kräfte beeinflußter Bewegung," *Z. Naturforsch.* **21a**, 369–376 (1966).
Taylor, G.I., "Diffusion by continuous movements," *Proc. Lond. Math. Soc.* **A20**, 196–211 (1921).
Weiss, N.O., "The expulsion of magnetic flux by eddies," *Proc. Roy. Soc.* **A293**, 310–328 (1966).
Zel'dovich, Ya.B., "The magnetic field in the two-dimensional motion of a conducting turbulent fluid," JETP **31**, 154–155 (1956), Engl. transl. *Sov. Phys.* JETP **4**, 460–462 (1957).

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