Transport effects associated with turbulence with particular attention to the influence of helicity

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Transport effects associated with turbulence with particular attention to the influence of helicity

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Abstract

The action of turbulence on a passive convected scalar field (e.g. temperature) or vector field (e.g. the magnetic field in an electrically conducting fluid) is reviewed, with particular attention paid to anomalous effects that can arise through the influence of Coriolis forces in a rotating system on the statistics of the turbulence. The simplest such effect (which corresponds to a breaking of the Onsager symmetry relations) is a 'skew-diffusion' effect, i.e. the appearance of a component of turbulent heat flux perpendicular to the local mean temperature gradient. The famous α effect of magnetohydrodynamic dynamo theory is also in this category, as is the more subtle Rädler effect (the appearance of a mean electromotive force perpendicular to the mean current in a plasma). These effects are all associated with the *helicity* of a turbulent flow, i.e. the correlation between the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and the vorticity field $\boldsymbol{\omega}(\boldsymbol{x}, t) = \operatorname{curl} \boldsymbol{u}$.

Sections 1–4 are introductory in nature, and discuss the problem of heat spot dispersion, the interaction of molecular and turbulent convective effects, the spectral description of random scalar and vector fields, and the spectral properties of a passive scalar field which is subject to both influences. In § 5, the mean-field (or double-length scale) approach is presented for the scalar field problem, and the general theory of eddy diffusivity and skew diffusivity is developed. Sections 6 and 7 are devoted to the Lagrangian approach and the 'first-order smoothing' approach—complementary approaches which have some interesting points of contact and which provide some useful insights. Sections 8–10 describe the corresponding theory for the vector field problem. Here the general theory of flux expulsion is shown to emerge in a natural way from the mean-field approach, provided effects of *inhomogeneity* of the turbulence are included; the associated flux transport velocity is invariably *down* the gradient of local turbulence intensity.

Section 11 extends the double-length scale approach to incorporate many length scales; the effect of successive averaging over increasing scales leads to equations (in the vector field context) which are invariant in form, and which permit the evaluation of the dynamo coefficient α and the turbulent diffusivity β in the astrophysically interesting limit of large magnetic Reynolds number R_{m}.

Finally, in § 12, a model problem is analysed, showing how, when the molecular diffusivity η is extremely small, an intermittent structure can develop in a passive convected field. In this situation, the simple concept of an eddy diffusivity is inadequate, and attempts to relate the statistical properties of the convected field to low-order spectral properties of the convecting velocity field are unlikely to succeed.

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1. Introduction

Fluid flow in large-scale geophysical and astrophysical contexts is generally turbulent, i.e. there is a random velocity fluctuation superposed on the mean velocity, and this random fluctuation plays an important part, not only in the overall dynamics of the flow field, but also in the transport of any fluid property (e.g. temperature) or of any contaminant (e.g. radioactive waste in the oceans). The random convective action of the turbulent eddies is roughly analogous to the molecular agitation that is responsible for molecular diffusion effects, and the action of turbulence may in the simplest circumstances be represented by 'turbulent diffusivities' (sometimes known as eddy diffusivities), e.g. turbulent viscosity ν_{T}, turbulent thermal diffusivity κ_{T}, etc, quantities which are generally much larger than their molecular counterparts ν, κ, ... These turbulent diffusivities all have the physical dimensions of velocity × length, and a crude estimate of their magnitude is generally given as

$$\nu_T, \kappa_T, \ldots = \mathrm{O}(u_0 l)$$

where u_0 is the root mean square of the velocity fluctuation and l is (in some sense) a dominant scale of the turbulence. More generally, the transport coefficients ν_{T}, κ_{T}, etc, are determined in principle by the statistical properties of the turbulence; and turbulent transport theory is concerned with establishing the precise relationship between the transport coefficients and these statistical properties.

In large-scale geophysical and astrophysical contexts, Coriolis forces associated with global rotation have an effect on the statistical properties of the turbulence and hence on the transport coefficients. Firstly, it is obvious that the global rotation (e.g. rotation of the Earth in meteorological and oceanographic contexts, rotation of the Sun in the solar convection zone context) introduces a preferred direction in any turbulent region, so that the turbulence may be severely anisotropic. Secondly, and more subtly, a combination of energy flux parallel to the rotation vector and Coriolis effects leads to a correlation between turbulent velocity $\boldsymbol{u}$ and turbulent vorticity $\boldsymbol{\omega}$, known as the mean *helicity* of the turbulence $\mathscr{H} = \langle \boldsymbol{u} \cdot \boldsymbol{\omega} \rangle$. The existence of this mean helicity not only modifies the values of the standard transport coefficients, but also introduces new effects, qualitatively distinct from the traditional transport effects, which might well be described as 'pseudo-transport effects', in view of the pseudo-scalar character of the quantity $\mathscr{H}$. The most famous of these new effects is the 'α effect' (Steenbeck *et al* 1966) which is responsible for the generation of magnetic field (spontaneous dynamo action) in an electrically conducting fluid in turbulent motion. There is, however, an analogous effect in the thermal diffusion context; the presence of helicity can lead to a component of heat flux which is perpendicular to the local mean temperature gradient (no matter what the direction of the latter may be!) (This effect, which may be described as 'skew diffusion', will be analysed by means of the double-length scale procedure in § 5 below.)

In this review, we shall consider the action of turbulence on dynamically passive convected fields which are also subject to the influence of a molecular diffusivity, which in all cases we shall denote by η. The convected field may be either a scalar field $\Theta(\boldsymbol{x}, t)$ satisfying the transport equation

$$\partial\Theta/\partial t + \boldsymbol{u} \cdot \nabla\Theta = \eta\nabla^2\Theta \tag{1.1}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is the turbulent velocity field, or a vector field $\boldsymbol{B}(\boldsymbol{x}, t)$ satisfying $\nabla \cdot \boldsymbol{B} = 0$ and

$$\partial\boldsymbol{B}/\partial t = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}) + \eta\nabla^2\boldsymbol{B}. \tag{1.2}$$

In either case, we shall suppose, unless otherwise stated, that $\boldsymbol{u}$ is solenoidal, i.e. $\nabla \cdot \boldsymbol{u} = 0$, and that $\boldsymbol{u}$ is a stationary random function of $\boldsymbol{x}$ and of t with zero mean, i.e. $\langle \boldsymbol{u} \rangle = 0$. (The angular brackets $\langle \ldots \rangle$ represent an ensemble average.)

The function Θ in (1.1) may represent either temperature fluctuations (or the temperature excess over some standard temperature), η then being the thermal diffusivity, or the concentration of a convected solute, e.g. salt in water. We shall adopt thermal terminology, but the wider applications may be borne in mind. The equation may equally be written

$$\partial\Theta/\partial t = -\nabla \cdot \boldsymbol{q} \qquad \text{where } \boldsymbol{q} = \boldsymbol{u}\Theta - \eta\nabla\Theta. \tag{1.3(a, b)}$$

$\boldsymbol{q}$ represents the flux of temperature due to convection by the velocity field and diffusion down the gradient of Θ. If l_0 is the scale of the temperature fluctuations and u_0 is the root mean square velocity ($u_0 = \langle \boldsymbol{u}^2 \rangle^{1/2}$), the ratio of these two contributions has order of magnitude

$$\frac{|\boldsymbol{u}\Theta|}{|\eta\nabla\Theta|} \sim \frac{u_0 l_0}{\eta} = Pe. \tag{1.4}$$

Pe is the Péclet number, and it is generally large in situations of practical importance when l_0 is taken to be of the same order as the scale of the most energetic ingredients of the turbulence (the 'energy-containing eddies'—see Batchelor (1953)). This means that the turbulence causes a very severe distortion of the 'isoscalar surfaces' $\Theta =$ constant and high gradients of Θ develop, before molecular diffusivity effects are significant. It is this strong distortion of the temperature field which constitutes the main difficulty in the analysis of equation (1.1).

The function $\boldsymbol{B}$ in (1.2) may represent the magnetic field in an electrically conducting fluid; and indeed this is perhaps the *only* example of any importance. (The vorticity field in a non-conducting fluid satisfies (1.2) also, but in general it can hardly be described as dynamically passive, since the velocity field instantaneously adapts itself to satisfy $\nabla \wedge \boldsymbol{u} = \boldsymbol{\omega}$, $\nabla \cdot \boldsymbol{u} = 0$.) In the magnetic context, (1.2) is known as the induction equation, and it is obtained by eliminating $\boldsymbol{E}$ and $\boldsymbol{j}$ from the equations

$$\partial\boldsymbol{B}/\partial t = -\nabla \wedge \boldsymbol{E} \qquad \mu_0\boldsymbol{j} = \nabla \wedge \boldsymbol{B} \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{1.5}$$

and

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \wedge \boldsymbol{B}). \tag{1.6}$$

Here, $\boldsymbol{E}$ is the electric field, $\boldsymbol{j}$ is the electric current and σ is the electrical conductivity of the fluid ($\eta = (\mu_0\sigma)^{-1}$). Displacement current is omitted from the second of (1.5) (the magnetohydrodynamic approximation). Equation (1.6) is Ohm's law adapted to take account of the motion of the medium; it may be written equally in the form

$$\boldsymbol{E} = -\boldsymbol{u} \wedge \boldsymbol{B} + \eta\nabla \wedge \boldsymbol{B}$$

(cf 1.3(b)). If l_0 is now the scale of variation of $\boldsymbol{B}$, the ratio of the two contributions to $\boldsymbol{E}$ is

$$\frac{|\boldsymbol{u} \wedge \boldsymbol{B}|}{|\eta \nabla \wedge \boldsymbol{B}|} \sim \frac{u_0 l_0}{\eta} \tag{1.7}$$

as in (1.4). In this context, the ratio $u_0 l_0/\eta$ is known as the magnetic Reynolds number R_{m}. The problem of convection and distortion of $\boldsymbol{B}$ by turbulence is of acute interest in astrophysical contexts, particularly in the context of the magnetohydrodynamics of turbulent convection zones of stars like the Sun, and in these contexts R_{m} is generally large also (typically of the order of 10^6 or greater). Again this implies that a magnetic field will be very strongly distorted by turbulence before the weak effects of molecular diffusivity can have any significant influence on the field evolution.

If we simply put $\eta = 0$ in (1.1), then the Lagrangian derivative of Θ vanishes:

$$\frac{\mathrm{D}\Theta}{\mathrm{D}t} \equiv \frac{\partial \Theta}{\partial t} + \boldsymbol{u} \cdot \nabla \Theta = 0 \tag{1.8}$$

and so Θ is constant for any fluid element. Let $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t)$ represent the path of the fluid particle which passes through $\boldsymbol{a}$ at time $t = 0$ (figure 1(a)); then the solution of (1.8) is simply

$$\Theta(\boldsymbol{x}, t) = \Theta(\boldsymbol{a}, 0). \tag{1.9}$$

Similarly, if we put $\eta = 0$ in (1.2) and expand the right-hand side, using $\nabla \cdot \boldsymbol{u} = 0$, we obtain

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} \equiv \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{u}. \tag{1.10}$$

The solution of (1.10) analogous to (1.9) is

$$B_i(\boldsymbol{x}, t) = B_j(\boldsymbol{a}, 0)\partial x_i/\partial a_j. \tag{1.11}$$

Equation (1.10) implies that magnetic lines of force are 'frozen in the fluid', the flux of $\boldsymbol{B}$ across any material element being conserved. Equation (1.11) expresses this conservation in a different way: the tensor $\partial x_i/\partial a_j$ is the strain tensor for the fluid element initially at $\boldsymbol{a}$; its antisymmetric part represents the rotation of the element, and the symmetric part represents the stretching, and the initial field threading the fluid element at $\boldsymbol{a}$ is subject to both rotation and stretching (figure 1(b)).

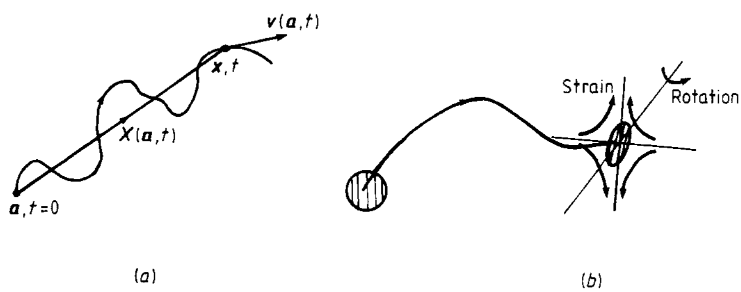

Figure 1. (a) Lagrangian path of a fluid particle. (b) Distortion of a fluid element; an initially spherical element becomes ellipsoidal under the action of irrotational strain plus rigid body rotation.

The generality of equations (1.8) and (1.10) were discussed in an early paper of Batchelor (1952), who also derived some of their important consequences. Since any line element tends to increase in length due to the action of turbulence, it follows that a magnetic line of force (or '$\boldsymbol{B}$ line') will also increase in length as it evolves according to (1.10); the field intensity increases in proportion to the length of elements of the $\boldsymbol{B}$ line, and so $\langle \boldsymbol{B}^2\rangle$ inevitably increases as a result of the action of turbulence. At the same time, however, the length scale of $\boldsymbol{B}$ decreases and ultimately molecular diffusion must intervene in the subsequent evolution of $\boldsymbol{B}$; whether $\langle \boldsymbol{B}^2\rangle$ then continues to increase or not is a question of great delicacy (Kraichnan and Nagarajan 1967) to which an absolutely definitive answer is not yet available.

Similarly, as argued by Batchelor (1952), the area of a material surface Θ = constant will increase exponentially under the action of turbulence (when $\eta = 0$), and so, since the volume of any fluid element is conserved when $\nabla \cdot \boldsymbol{u} = 0$, the normal distance between adjacent surfaces Θ = constant must decrease as time increases. The gradient of Θ, $\boldsymbol{G} = \nabla\Theta$, therefore increases nearly everywhere, and $\langle \boldsymbol{G}^2\rangle$ will certainly increase until the two contributions to $\boldsymbol{q}$ in (1.3) are (in the mean) of the same order of magnitude. A more detailed discussion of the effects of molecular diffusivity is given in § 2.2.

2. Heat spot dispersion

2.1. Lagrangian analysis of particle dispersion

If the sole source of temperature variation is a heat spot introduced at $\boldsymbol{x} = 0$ at time $t = 0$, then the initial condition for (1.1) is

$$\Theta(\boldsymbol{x}, 0) = \Theta_0\delta(\boldsymbol{x}). \tag{2.1}$$

If molecular diffusivity effects are negligible (i.e. $\eta = 0$) then the heat spot is simply carried with the fluid, i.e. it behaves like a marked fluid particle. The analysis of this problem (Taylor 1921) provides the natural starting point for all studies of turbulent diffusion.

Let $\boldsymbol{X}(t)$ be the position of the heat spot (i.e. of the marked particle) at time t, so that

$$\mathrm{d}\boldsymbol{X}/\mathrm{d}t = \boldsymbol{u}(\boldsymbol{X}(t), t) = \boldsymbol{v}(t). \tag{2.2}$$

$\boldsymbol{v}(t)$ is the Lagrangian particle velocity, and it is a stationary random function of t provided $\boldsymbol{u}(\boldsymbol{x}, t)$ is statistically both homogeneous in $\boldsymbol{x}$ and stationary in t†. We may then define a Lagrangian correlation tensor $R_{ij}^{(L)}(\tau)$ by

$$u_0^2 R_{ij}^{(L)}(\tau) = \langle v_i(t)v_j(t-\tau)\rangle \tag{2.3}$$

where $u_0^2 = \langle \boldsymbol{u}^2\rangle$. This may be expected to decrease rapidly as $\tau \to \infty$, and we may define a Lagrangian correlation time t_c by

$$t_c = \int_0^\infty R_{ii}^{(L)}(\tau)\,\mathrm{d}\tau \tag{2.4}$$

on the reasonable assumption that the integral converges. This is the time over which the particle velocity remains strongly correlated with its initial value; for $t \gg t_c$, the

† An appropriate modification for decaying homogeneous turbulence is described by Batchelor and Townsend (1956).

influence of initial conditions is 'forgotten', at least as far as the particle velocity is concerned.

The stationarity of $\boldsymbol{v}(t)$ implies, from (2.3), that

$$R_{ji}^{(L)}(\tau) = R_{ij}^{(L)}(-\tau) \tag{2.5}$$

and so $R_{ij}^{(L)}(\tau)$ is not symmetric unless it satisfies the 'time-reversibility' condition

$$R_{ij}^{(L)}(\tau) = R_{ij}^{(L)}(-\tau). \tag{2.6}$$

This type of condition is well known in statistical mechanics, where it is a particular consequence of the 'principle of microscopic reversibility' (Onsager 1931a, b). However, as recognised by Onsager, this principle is not universally valid, and in particular it fails to hold in a rotating system when Coriolis forces have a significant dynamical effect at the microscopic level. In the turbulence context, the 'microscopic level' is the level of turbulent fluctuation, and it is well known (see, for example, Ibbetson and Tritton 1975) that Coriolis forces can and do affect the turbulence statistics in a rotating system. Coriolis forces are certainly important in geophysical contexts (meteorological and oceanographic) on scales of the order of 10^2 km and greater, and even more so in the larger-scale contexts of planetary interiors and convection zones of rotating stars, towards which this review is particularly oriented. We shall therefore give particular emphasis to situations in which the condition (2.6) is *not* satisfied, and to consequences of this lack of symmetry.

Consider now the particle displacement $\boldsymbol{X}(t)$ which (like $\boldsymbol{v}(t)$) satisfies $\langle \boldsymbol{X} \rangle = 0$, but which is *not* in general a stationary random function, since the particle wanders from its initial position with excursions which increase on average with time. The tensor dispersion $\langle X_i X_j \rangle$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle X_i X_j \rangle &= \langle v_i X_j + v_j X_i \rangle \\ &= \int_0^t \langle v_i(t) v_j(t_1) + v_j(t) v_i(t_1) \rangle \,\mathrm{d}t_1 \\ &= u_0^2 \int_0^t (R_{ij}^{(L)}(\tau) + R_{ji}^{(L)}(\tau))\,\mathrm{d}\tau \end{aligned} \tag{2.7}$$

involving only the symmetric part of $R_{ij}^{(L)}(\tau)$. It follows from (2.7) that, as $t \to \infty$,

$$\langle X_i X_j \rangle \sim u_0^2 t \int_0^\infty (R_{ij}^{(L)}(\tau) + R_{ji}^{(L)}(\tau))\,\mathrm{d}\tau \tag{2.8}$$

provided the integral converges, and in particular

$$\langle X^2 \rangle \sim 6 D_0 t \qquad \text{where } D_0 = \tfrac{1}{3} u_0^2 t_c \tag{2.9}$$

a result characteristic of a diffusion process in three dimensions with diffusivity D_0. The probability distribution of $\boldsymbol{X}(t)$ then spreads as an ellipsoidal Gaussian 'cloud', with principal axes determined by the symmetric tensor on the right of (2.8), and length scale increasing like $t^{1/2}$.

The mean angular momentum (per unit mass) of the particle at time t is given by

$$\boldsymbol{h}(t) = \langle \boldsymbol{X}(t) \wedge \boldsymbol{v}(t) \rangle \tag{2.10}$$

and this can be non-zero if $R_{ij}^{(L)}(t)$ is not symmetric. In fact we find that, for $t \gg t_c$,

$$h_i \sim \epsilon_{ijk} \int_0^\infty R_{kj}^{(L)}(\tau)\, d\tau \tag{2.11}$$

so that the mean angular momentum is asymptotically constant. Thus, if $R_{ij}^{(L)}(\tau)$ is not symmetric, due to the dynamical influence of Coriolis forces, the particle may be expected to follow a random path which nevertheless exhibits a preferred sense of turning about the direction defined by (2.11) (see figure 2).

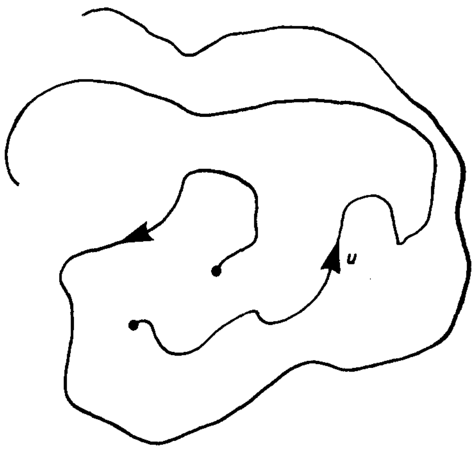

Figure 2. Particle paths for a random velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ with zero mean ($\langle \boldsymbol{u} \rangle = 0$) but preferred sense of turning ($\langle \boldsymbol{v}(\boldsymbol{a}, t) \wedge \boldsymbol{v}(\boldsymbol{a}, 0)\rangle \neq 0$).

Results analogous to (2.8) and (2.11) may be obtained similarly for higher-order statistical quantities; for example, the triple moment $\langle X_i X_j X_k \rangle$ is given asymptotically by

$$\langle X_i X_j X_k \rangle \sim u_0^3 t \int_0^\infty \int_0^\infty (R_{ijk}^{(L)}(\tau_1, \tau_2) + R_{jki}^{(L)}(\tau_1, \tau_2) + R_{kij}^{(L)}(\tau_1, \tau_2))\, d\tau_1\, d\tau_2 \tag{2.12}$$

where

$$u_0^3 R_{ijk}^{(L)}(\tau_1, \tau_2) = \langle v_i(t) v_j(t-\tau_1) v_k(t-\tau_2) \rangle. \tag{2.13}$$

This triple correlation may be expected to be small if either τ_1 or τ_2 is much greater than t_c, and the integral in (2.12) will be generally convergent. It is evident from (2.12) that any skewness in the probability distribution of $\boldsymbol{v}$ leads, as might be expected, to a corresponding skewness in the distribution of $\boldsymbol{X}$.

2.2. *Effects of weak molecular diffusivity*

Molecular diffusion causes the convected heat spot to spread, the initial spread being spherically symmetric and given by

$$\Theta(\boldsymbol{x}, t) = \frac{\Theta_0}{(2\pi\eta t)^{3/2}} \exp\left(-\frac{(\boldsymbol{x} - \boldsymbol{X}(t))^2}{2\eta t}\right) \tag{2.14}$$

the elementary solution of the three-dimensional diffusion equation. This diffused spot then interacts with the local velocity gradient $\partial u_i/\partial x_j$ at the moving point $\boldsymbol{x} = \boldsymbol{X}(t)$,

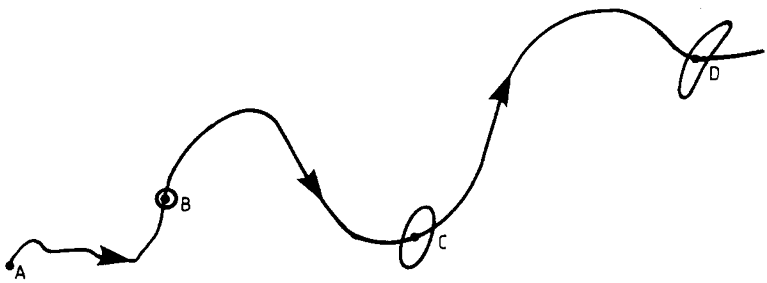

Figure 3. Combined action of convection and molecular diffusion on a heat spot released at A at time $t=0$. At B, the spot is still spherical under the action of diffusion alone; at C, the local velocity gradient combined with diffusion has distorted the spot to ellipsoidal shape; at D, the spot has spread outside the region of locally uniform distortion and its shape is no longer ellipsoidal.

which distorts it into approximately ellipsoidal shape (figure 3). This leads to 'accelerated' diffusion (Townsend 1951) in the sense that the dispersion of the spot relative to the position $\boldsymbol{x}=\boldsymbol{X}(t)$ is greater than it would be under the influence of diffusion alone. However, as shown by Saffman (1960), the same interaction between the local strain field and molecular diffusion leads to a *decrease* in the dispersion of the spot relative to its initial point of release. This dispersion is given by

$$S^2=\int x^2\Theta(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\left(\int\Theta\,\mathrm{d}\boldsymbol{x}\right)^{-1} \tag{2.15}$$

and Saffman showed that, for small times t after the instant of release of the spot,

$$S^2=6(D_0+\eta)t-\tfrac{1}{3}\eta(\partial u_i/\partial x_j)^2t^3+\mathrm{O}(t^4) \tag{2.16}$$

the interaction between the velocity gradient $\partial u_i/\partial x_j$ and the molecular diffusivity η being apparent in the form of the $\mathrm{O}(t^3)$ term. This at first sight surprising reduction in dispersion arises because the mean velocity over the distorted spot is generally *less* in magnitude than $|\boldsymbol{u}(\boldsymbol{X},t)|$ (since the velocity averages to zero over the whole fluid); the effective velocity convecting the spot away from its point of release is therefore reduced when its spread and distortion become significant.

The same type of reduction is apparent, and perhaps more easily appreciated, if we consider the dispersion of a 'blob' of dyed fluid that is initially *large* compared with the most energetic ingredients of the turbulence; e.g. we may consider the case of an initially spherical blob,

$$\Theta(\boldsymbol{x},0)=\begin{cases}\Theta_0 & (|\boldsymbol{x}|<a)\\ 0 & (|\boldsymbol{x}|>a).\end{cases} \tag{2.17}$$

Consider first what happens if molecular diffusion is neglected. Turbulent eddies on scales $l<a$ cause progressive distortion of the spherical boundary (figure 4). If the Reynolds number of the turbulence is sufficiently large for the existence of a well-established Kolmogorov energy spectrum (Batchelor 1953, chap VI), then the time scale of distortion on a scale l is given by

$$t_l\sim\epsilon^{-1/3}l^{2/3} \tag{2.18}$$

where ϵ is the rate of energy dissipation per unit mass, so that small ripples appear first on the surface, then larger-scale distortions, as indicated in the figure. During

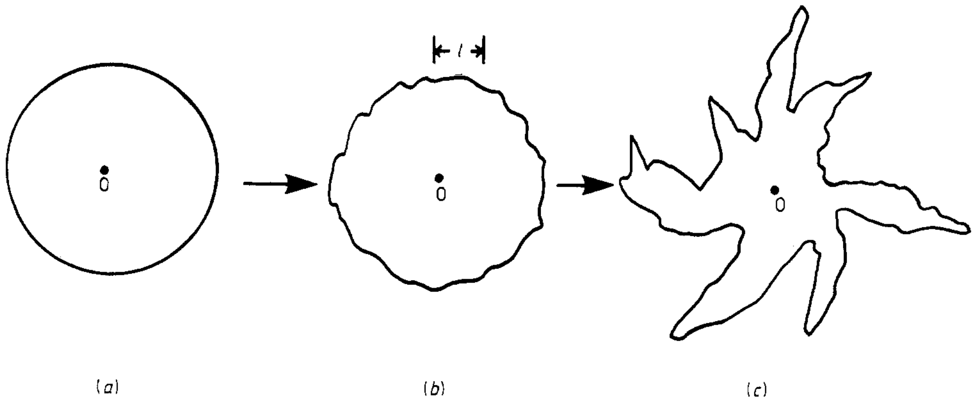

Figure 4. Distortion of an initially spherical blob of dye which is larger than the integral scale of the turbulence. In (*b*), eddies on the scale *l* have had time to distort the surface of the blob; in (*c*) larger eddies with longer characteristic time scales (see equation (2.18)) have had time to take effect. As the dyed fluid moves outwards (on average) from the centroid 0, the undyed fluid moves inwards to take its place.

this process, the volume of dyed fluid remains constant, a consequence of incompressibility. Thus the evident outward spread (or flux) of dyed fluid from its centre is accompanied by an inward flux of undyed fluid.

What then is the effect of weak molecular diffusion? Clearly this simply smooths out the initial discontinuity in dye concentration—and it does this by transferring dye *from* the fluid which is on average moving outwards *to* the fluid which is on average moving inwards; the *reduction* in the dispersion of the cloud of dye is an inevitable consequence.

3. Spectral terminology; energy and helicity spectrum functions

Although we shall make minimal appeal to the rather heavy mathematical apparatus of turbulence in the discussion that follows, some simple definitions and relationships are unavoidable, and are gathered together in this section, for future ease of reference. The random velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ admits the Fourier representation

$$\boldsymbol{u}(\boldsymbol{x}, t) = \int \hat{\boldsymbol{u}}(\boldsymbol{k}, \omega) \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)] \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\omega \tag{3.1}$$

with inverse

$$\hat{\boldsymbol{u}}(\boldsymbol{k}, \omega) = (2\pi)^{-4} \int \boldsymbol{u}(\boldsymbol{x}, t) \exp[-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t. \tag{3.2}$$

Here, $\hat{\boldsymbol{u}}(\boldsymbol{k}, \omega)$ must be interpreted as a generalised function; the reality of $\boldsymbol{u}$ and the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ imply that $\boldsymbol{u}$ satisfies

$$\hat{\boldsymbol{u}}(-\boldsymbol{k}, -\omega) = \hat{\boldsymbol{u}}^*(\boldsymbol{k}, \omega) \qquad \boldsymbol{k} \cdot \hat{\boldsymbol{u}}(\boldsymbol{k}, \omega) = 0 \tag{3.3}$$

for all $(\boldsymbol{k}, \omega)$ (the star denotes the complex conjugate).

The Eulerian correlation tensor $R_{ij}(\boldsymbol{r}, \tau)$ is defined by

$$R_{ij}(\boldsymbol{r}, \tau) = \langle u_i(\boldsymbol{x}, t) u_j(\boldsymbol{x} + \boldsymbol{r}, t + \tau) \rangle \tag{3.4}$$

a function only of the separations $\boldsymbol{r}$ and τ provided $\boldsymbol{u}$ is homogeneous and stationary.

The spectrum tensor $\Phi_{ij}(k, \omega)$ is defined as the Fourier transform of R_{ij}, i.e.

$$\Phi_{ij}(\boldsymbol{k}, \omega) = (2\pi)^{-4} \int R_{ij}(\boldsymbol{r}, \tau) \exp\left[-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega\tau)\right] \mathrm{d}\boldsymbol{r}\, \mathrm{d}\tau \tag{3.5}$$

and this satisfies the well-known relation

$$\langle \hat{u}_i^*(\boldsymbol{k}, \omega)\hat{u}_j(\boldsymbol{k}', \omega')\rangle = \Phi_{ij}(\boldsymbol{k}, \omega)\delta(\boldsymbol{k} - \boldsymbol{k}')\delta(\omega - \omega'). \tag{3.6}$$

The conditions (3.3) may then be used to demonstrate that $\Phi_{ij}(\boldsymbol{k}, \omega)$ satisfies the Hermitian symmetry conditions

$$\Phi_{ij}^*(\boldsymbol{k}, \omega) = \Phi_{ij}(-\boldsymbol{k}, -\omega) = \Phi_{ji}(\boldsymbol{k}, \omega) \tag{3.7}$$

and the incompressibility conditions

$$k_i\Phi_{ij}(\boldsymbol{k}, \omega) = 0 \qquad k_j\Phi_{ij}(\boldsymbol{k}, \omega) = 0 \tag{3.8}$$

for all $(\boldsymbol{k}, \omega)$.

In the context of homogeneous *decaying* turbulence, it is customary to deal only with instantaneous correlations

$$R_{ij}(\boldsymbol{r}, t) = \langle u_i(\boldsymbol{x}, t)u_j(\boldsymbol{x} + \boldsymbol{r}, t)\rangle \tag{3.9}$$

and to take Fourier transforms with respect to space variables only, e.g.

$$\Phi_{ij}(\boldsymbol{k}, t) = (2\pi)^{-3} \int R_{ij}(\boldsymbol{r}, t) \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r})\, \mathrm{d}\boldsymbol{r}. \tag{3.10}$$

In a field of stationary (maintained) turbulence these functions may, of course, still be defined, and they are then independent of t; when the notation $R_{ij}(\boldsymbol{r})$, $\Phi_{ij}(\boldsymbol{k})$ is used, it is this type of instantaneous correlation that is implied; the relation between $\Phi_{ij}(\boldsymbol{k})$ and $\Phi_{ij}(\boldsymbol{k}, \omega)$ is then simply

$$\Phi_{ij}(\boldsymbol{k}) = \int \Phi_{ij}(\boldsymbol{k}, \omega)\, \mathrm{d}\omega. \tag{3.11}$$

The kinetic energy density of the turbulence (per unit mass) is

$$\tfrac{1}{2}\langle \boldsymbol{u}^2\rangle = \tfrac{1}{2}R_{ii}(0, 0) = \int E(\boldsymbol{k}, \omega)\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega \tag{3.12}$$

where

$$E(\boldsymbol{k}, \omega) = \tfrac{1}{2}\Phi_{ii}(\boldsymbol{k}, \omega). \tag{3.13}$$

We may define also the *energy spectrum function* $E(k)$ by

$$E(k) = \int \mathrm{d}\omega \int_{S(k)} E(\boldsymbol{k}, \omega)\, \mathrm{d}S \tag{3.14}$$

where $S(k)$ is the surface of a sphere of radius k in $\boldsymbol{k}$ space. Then (3.12) becomes

$$\tfrac{1}{2}\langle \boldsymbol{u}^2\rangle = \int_0^\infty E(k)\, \mathrm{d}k \tag{3.15}$$

and $E(k)\,\mathrm{d}k$ may be thought of as the contribution to $\tfrac{1}{2}\langle \boldsymbol{u}^2\rangle$ from wavenumber magnitudes in the interval $(k, k + \mathrm{d}k)$.

Clearly, the function $E(\boldsymbol{k}, \omega)$ involves only the symmetric part of $\Phi_{ij}(\boldsymbol{k}, \omega)$. We shall find that, in certain contexts, and particularly in the magnetic-field context (§ 8 below), the antisymmetric part

$$\Phi_{ij}^{(a)}(\boldsymbol{k}, \omega) = \tfrac{1}{2}[\Phi_{ij}(\boldsymbol{k}, \omega) - \Phi_{ji}(\boldsymbol{k}, \omega)] \tag{3.16}$$

can play an important, and indeed a dominant, role. Like the antisymmetric part of $R_{ij}^{(L)}(\tau)$ discussed in § 2, a non-zero antisymmetric part of $\Phi_{ij}(\boldsymbol{k}, \omega)$ is generally associated with the action of Coriolis forces in a rotating system. This antisymmetric part can be expressed in terms of a single real scalar function $H(\boldsymbol{k}, \omega)$, the helicity spectrum function, defined by

$$H(\boldsymbol{k}, \omega) = -\mathrm{i}k_m \epsilon_{ijm} \Phi_{ij}(\boldsymbol{k}, \omega) \tag{3.17}$$

(see Moffatt and Proctor 1982); we then have

$$\Phi_{ij}^{(a)}(\boldsymbol{k}, \omega) = \tfrac{1}{2}\mathrm{i}\epsilon_{ijk} k_k H(\boldsymbol{k}, \omega)/k^2. \tag{3.18}$$

The major qualitative difference between $H(\boldsymbol{k}, \omega)$ and $E(\boldsymbol{k}, \omega)$ is that $H(\boldsymbol{k}, \omega)$ is a *pseudo-scalar*, i.e. it changes sign under any change from a right-handed to a left-handed frame of reference (parity transformations). A field of turbulence which is statistically invariant under such transformations is said to have the property of *reflexional symmetry*. Clearly $H(\boldsymbol{k}, \omega)$ can be non-zero only if the turbulence *lacks* reflexional symmetry, i.e. only if, in some sense, 'right-handedness' and 'left-handedness' are distinguishable in the statistics of the velocity field.

This statement acquires precision from the fact that

$$H(\boldsymbol{k}, \omega) = (2\pi)^{-4} \int \langle \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{\omega}(\boldsymbol{x} + \boldsymbol{r}, t + \tau) \rangle \exp[-\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega\tau)] \, \mathrm{d}\boldsymbol{r} \, \mathrm{d}\tau \tag{3.19}$$

(where $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$, the vorticity field) as may easily be deduced from (3.5) and (3.17). Equivalently,

$$\langle \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{\omega}(\boldsymbol{x} + \boldsymbol{r}, t + \tau) \rangle = \int H(\boldsymbol{k}, \omega) \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega\tau)] \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\tau \tag{3.20}$$

and, in particular,

$$\langle \boldsymbol{u} \cdot \boldsymbol{\omega} \rangle = \int H(\boldsymbol{k}, \omega) \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\omega. \tag{3.21}$$

Clearly $\boldsymbol{u} \cdot \boldsymbol{\omega} > 0$ locally (in a right-handed frame of reference) if the streamlines have (locally) a right-handed helical structure. The function $H(\boldsymbol{k}, \omega)$ defined by (3.19) is evidently a sensitive measure of the correlation between velocity and vorticity field. H may, of course, be positive in one part of the $(\boldsymbol{k}, \omega)$ space and negative in another part. It is, however, bounded in magnitude by the inequality

$$|H(\boldsymbol{k}, \omega)| \leqslant 2kE(\boldsymbol{k}, \omega) \tag{3.22}$$

a consequence of Cramer's theorem (see Moffatt and Proctor 1982). Note that the conditions (3.7) imply that E and H both satisfy the same symmetry conditions:

$$E(-\boldsymbol{k}, -\omega) = E(\boldsymbol{k}, \omega) \qquad H(-\boldsymbol{k}, -\omega) = H(\boldsymbol{k}, \omega). \tag{3.23}$$

The manner in which Coriolis forces in a rotating system are responsible for the generation of a non-zero helicity spectrum in a field of turbulence has been discussed at length in the literature of dynamo theory (see particularly Moffatt 1978 chap 10–12, Parker 1979 chap 18, Krause and Rädler 1980 chap 9) and this discussion need not be repeated here. As regards the *dynamics* of helicity, we simply draw attention to the fact that the mean helicity $\langle \boldsymbol{u} \cdot \boldsymbol{\omega} \rangle$ is an inviscid invariant of the dynamical equations of motion (Moreau 1961), this invariance reflecting the invariance of the topological structure of the vorticity field when viscous diffusion is neglected (Moffatt 1969).

4. Spectral theory for convected scalar field

Suppose now that there is some source of temperature fluctuations $S(\boldsymbol{x}, t)$ on a scale larger than the scale l_0 of the energy-containing eddies of the turbulence, and that $\langle S \rangle = 0$, so that the turbulence generates a statistically homogeneous and stationary temperature fluctuation field $\theta(\boldsymbol{x}, t)$ satisfying $\langle \theta \rangle = 0$. Equation (1.1) is then replaced by

$$\partial\theta/\partial t + \boldsymbol{u} \cdot \nabla\theta = \eta\nabla^2\theta + S(\boldsymbol{x}, t) \tag{4.1}$$

and the problem that presents itself is to determine the statistical properties of θ in terms of those of the velocity field $\boldsymbol{u}$. This problem was initially considered by Obukhov (1949), Corrsin (1951) and Batchelor (1952) and was further developed in the far-reaching studies of Batchelor (1959) and Batchelor *et al* (1959), whose results we summarise in this section.

The argument is essentially an elaboration of the Kolmogorov picture of the dynamics of turbulence (see figure 5). In this, the all-important parameter is ϵ, the

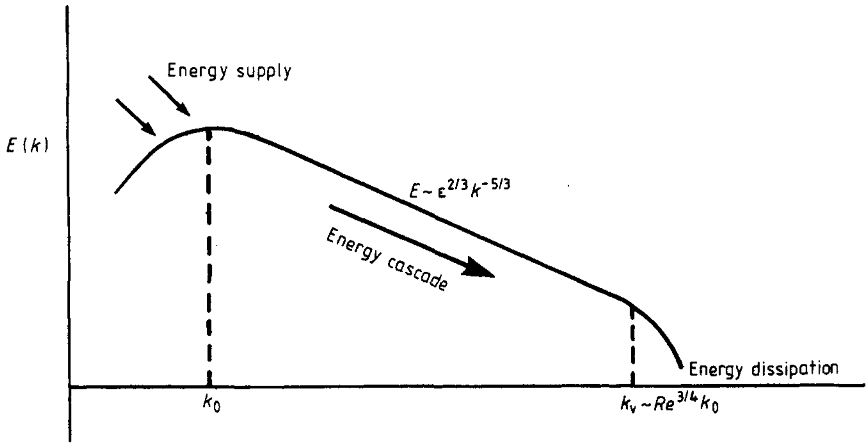

Figure 5. The Kolmogorov picture of the dynamics of turbulence; energy cascades through the spectrum at a rate ϵ, and is dissipated at wavenumbers $\geq O(k_v)$; (logarithmic scale).

rate of dissipation of energy per unit mass by viscosity. In a statistically steady state, this is also the rate of supply of energy to the turbulence on scales of the order of l_0 and greater. This energy cascades through non-linear interactions towards smaller and smaller length scales and ultimately to the 'viscous' length scale l_v at which molecular dissipation becomes effective; l_v is determined in order of magnitude by ϵ and ν only, and hence, on dimensional grounds,

$$l_v \sim (\nu^3/\epsilon)^{1/4}. \tag{4.2}$$

Moreover, ϵ satisfies the important semi-empirical relationship

$$\epsilon \sim u_0^3/l_0 \tag{4.3}$$

where $u_0 = \langle \boldsymbol{u}^2 \rangle^{1/2}$, and so (4.2) becomes

$$l_v \sim Re^{-3/4} l_0 \tag{4.4}$$

where $Re = u_0 l_0/\nu$ is the Reynolds number of the turbulence, assumed large.

In spectral terminology, we introduce wavenumbers

$$k_0 = l_0^{-1} \qquad k_v = l_v^{-1} \sim Re^{3/4} k_0 \tag{4.5}$$

and define the *inertial range*

$$k_0 \ll k \ll k_v \tag{4.6}$$

within which the statistical properties of the turbulence are determined solely by ϵ, the rate of cascade of energy, and the local wavenumber k. In particular, the energy spectrum function $E(k)$ is given by

$$E(k) = C\epsilon^{2/3} k^{-5/3} \qquad (k_0 \ll k \ll k_v) \tag{4.7}$$

where C is a universal constant of the order of unity. This is the famous Kolmogorov spectrum, for which there is much experimental support, particularly from atmospheric and oceanographic turbulence (see Monin and Yaglom 1975 chap 8).

Returning now to equation (4.1), we first multiply by θ and average, giving

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \theta^2 \rangle = -\eta \langle (\nabla\theta)^2 \rangle + \chi \tag{4.8}$$

where

$$\chi = \langle S\theta \rangle \tag{4.9}$$

the rate of input of contributions to the mean square θ field. Under statistically steady conditions,

$$\chi = \eta \langle (\nabla\theta)^2 \rangle \tag{4.10}$$

reflecting the balance between input of θ fluctuations on scales of the order of l_0 and greater, and destruction of θ fluctuations on (presumably) much smaller scales. χ plays a role analogous to that of ϵ in the Kolmogorov theory; and there is a semi-empirical relationship between χ and $\theta_0^2 = \langle \theta^2 \rangle$ analogous to (4.3), viz

$$\chi \sim \theta_0^2 u_0 / l_0. \tag{4.11}$$

The length scale l_c below which molecular diffusion is efficient in destroying θ fluctuations is larger or smaller than l_v according to whether η is greater or less than ν. If $\eta > \nu$, then $l_c > l_v$ and l_c is then determined solely by ϵ and η, i.e.

$$l_c \sim (\eta^3/\epsilon)^{1/4} \sim Pe^{-3/4} l_0. \tag{4.12}$$

The picture is, of course, only self-consistent if $Pe \gg 1$. If $\eta < \nu$, on the other hand, then as shown by Batchelor (1959) it is the root mean square strain rate $\zeta \sim (\epsilon/\nu)^{1/2}$ which is the 'relevant' property of the velocity field for determining the small-scale features of the θ field; hence, dimensionally,

$$l_c \sim (\eta/\zeta)^{1/2} \sim (\eta^2 \nu/\epsilon)^{1/4} \sim Pe^{-1/2} Re^{-1/4} l_0. \tag{4.13}$$

The corresponding wavenumber $k_c = l_c^{-1}$ is described as the 'conduction cut-off'.

Now let $\Gamma(k)$ be the spectrum function of θ, defined in a manner analogous to $E(k)$, and with the property

$$\langle\theta^2\rangle = \int_0^\infty \Gamma(k)\,\mathrm{d}k. \tag{4.14}$$

In the spectral range

$$k_0 \ll k \ll \min(k_c, k_v) \tag{4.15}$$

θ fluctuations are neither created nor destroyed, but are transferred through the influence of the term $\boldsymbol{u}\cdot\nabla\theta$ in (4.1) from the source of fluctuations at small wavenumbers to the sink at high wavenumbers. In this range, $\Gamma(k)$ is determined only by ϵ, χ and k, and the dependence on χ is a simple proportionality, in view of the linearity (in θ) of (4.1). Hence in the range (4.15), on dimensional grounds (Corrsin 1951)

$$\Gamma(k) = C_\theta \chi \epsilon^{-1/3} k^{-5/3} \tag{4.16}$$

where C_θ (like C) is a universal constant of order unity.

Experimental support for the result (4.16) is summarised by Monin and Yaglom (1975, § 23.5); the $k^{-5/3}$ law is tolerably well confirmed by the experimental data, but the value of C_θ shows a disturbing variability from one experimental context to another (the value preferred by Monin and Yaglom being $C_\theta \approx 1.2$).

When $Pr = \nu/\eta \ll 1$ (so that $k_c \ll k_v$) the spectrum function $\Gamma(k)$ in the 'conduction sub-range':

$$k_c \ll k \ll k_v \tag{4.17}$$

is determined (Batchelor *et al* 1959) essentially by a balance between the production of small-scale temperature fluctuations by distortion of the mean local gradient $\nabla\Theta$ by the turbulence, and diffusive suppression of these fluctuations. The equation representing this balance is

$$\boldsymbol{u}\cdot\nabla\Theta \approx \eta\nabla^2\theta' \tag{4.18}$$

where $\theta'(x, t)$ represents the fluctuation field on scales corresponding to (4.17). In (4.18), $\nabla\Theta$ can be treated as locally uniform and constant. The corresponding relationship between the spectra Γ and E is then

$$\Gamma(k) \approx \frac{2\langle(\nabla\Theta)^2\rangle}{3\eta^2 k^4} E(k) = \frac{2\chi}{3\eta^3 k^4} E(k) \tag{4.19}$$

(using (4.10)), a formula which (necessarily) matches in order of magnitude with (4.16) when $k \approx k_c$.

When $Pr \gg 1$ (so that $k_c \gg k_v$), temperature fluctuations persist down to scales small compared with the Kolmogorov scale k_v^{-1}. We are now down on the minute scales on which the velocity gradient is uniform to the first order of approximation and analysis of the evolution of θ fluctuations for $k \gg k_v$ can be based on this approximation. On this basis, Batchelor (1959) found that

$$\Gamma(k) \approx A k^{-1} \exp[-\alpha(k/k_c)^2] \tag{4.20}$$

where α is a constant of order unity, and A is a constant determined in order of magnitude by smooth matching of (4.20) and (4.16) near $k = k_v$. Some experimental support for the k^{-1} behaviour in the range $k_v \ll k \ll k_c$ is available (see Monin and Yaglom 1975, pp 513–5).

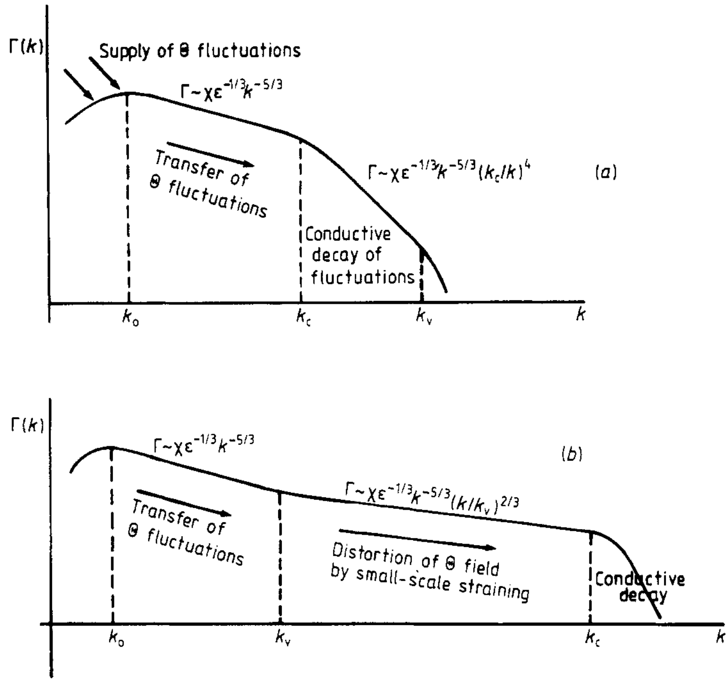

Figure 6. Qualitative sketches showing the form of the spectrum $\Gamma(k)$ of θ, (a) when $Pr \ll 1$ and (b) when $Pr \gg 1$ (from Batchelor 1959, Batchelor *et al* 1959) (logarithmic scale).

These spectral results are summarised by the qualitative sketches of figure 6. Note that from (4.14), when $Pr \ll 1$, the integral for $\langle\theta^2\rangle$ is dominated by contributions in the neighbourhood of k_0, i.e.

$$\langle\theta^2\rangle \sim \tfrac{3}{5} C_\theta \chi \epsilon^{-1/3} k_0^{-2/3} \sim \chi l_0/u_0 \tag{4.21}$$

in order-of-magnitude agreement with (4.11). When $Pr \gg 1$, however, there is a contribution to the integral (4.14) from the range $k_v \ll k \ll k_c$ which diverges logarithmically as $Pr \to \infty$; this contribution has order of magnitude

$$(\chi l_0/u_0) Re^{-1/2} \ln Pr \tag{4.22}$$

which is, however, in all practical circumstances, small compared with (4.21).

The approach of this section, which is based primarily on physical and dimensional reasoning, rather than on strict mathematical deduction, has been applied in a similar way to the problem of the convected vector field (Moffatt 1961, 1963; see also Knobloch and Rosner 1981). To some extent, however, the results of these investigations have been superseded by results based on the two-scale approach of Steenbeck *et al* (1966). We shall describe this approach first in the context of the scalar field (§§ 5–7) and then in the vector field context (§§ 8–10).

5. Double-length scale approach applied to the scalar field problem

The two-scale approach, which was pioneered in the magnetic context by Steenbeck *et al* (1966 and subsequent papers—see Roberts and Stix (1972)), is based on the assumption that the convected field—here Θ—is weakly inhomogeneous on a scale L much greater than the scale l_0 of the 'background' turbulence. Attention is then generally focused on the evolution of the ensemble-average field which evolves on the length scale L, and on a time scale large compared with the time scale $t_0 \sim l_0/u_0$ characteristic of the energy-containing eddies of the turbulence. The situation is depicted in figure 7 which shows the mean isoscalar surfaces $\langle\Theta\rangle = \text{constant}$, which

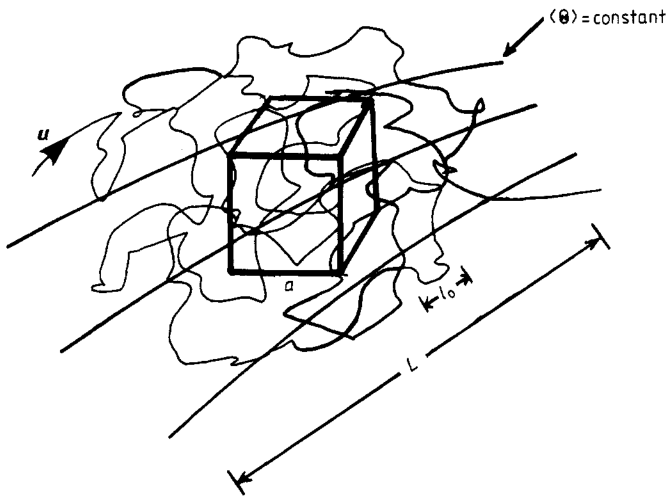

Figure 7. Double-length scale basis for mean-field theory; the scale l_0 of the turbulence is small compared with the scale L of the mean field. Averaging is carried out over a cube of side a where $l_0 \ll a \ll L$; within such a cube, $\boldsymbol{u}$ is statistically homogeneous, and $\nabla\Theta_0$ is approximately uniform.

are approximately planar on the scale l_0 of the turbulence. We may choose an intermediate scale a, e.g. $a = (l_0 L)^{1/2}$, satisfying

$$l_0 \ll a \ll L \tag{5.1}$$

and think of the average $\langle\ldots\rangle$ as a 'local average' over a cube of side a, averaged quantities varying only on larger scales of order L.

Let $\langle\Theta\rangle = \Theta_0(\boldsymbol{x}, t)$, and let

$$\Theta = \Theta_0(\boldsymbol{x}, t) + \theta(\boldsymbol{x}, t) \tag{5.2}$$

so that θ now represents the fluctuation field, on scales of the order of l_0 and less; by definition $\langle\theta\rangle = 0$. The average of (1.3) then gives

$$\frac{\partial\Theta_0}{\partial t} = -\nabla \cdot \boldsymbol{F} + \eta\nabla^2\Theta_0 \tag{5.3}$$

where

$$\boldsymbol{F} = \langle\boldsymbol{u}\theta\rangle \tag{5.4}$$

the heat flux associated with convection of temperature fluctuation by the turbulence. The central problem here is to obtain a functional relationship between $\boldsymbol{F}$ and Θ_0 (or rather $\nabla\Theta_0$ as it turns out) so that (5.3) can be integrated.

Subtracting (5.3) from (1.3(a)) gives an equation for the fluctuation field:

$$\frac{\partial\theta}{\partial t} = -\boldsymbol{u}\cdot\boldsymbol{G} - \boldsymbol{\nabla}\cdot(\boldsymbol{u}\theta - \boldsymbol{F}) + \eta\boldsymbol{\nabla}^2\theta \tag{5.5}$$

where $\boldsymbol{G} = \nabla\Theta_0$, the local mean-field gradient. This equation establishes a linear relationship between $\boldsymbol{G}$ and θ (assuming that any transient effects associated with initial conditions have had time to disappear), and so likewise between $\boldsymbol{G}$ and $\boldsymbol{F}$. If $\boldsymbol{G}$ were strictly uniform then this linear relationship could only be of the form

$$F_i = -D_{ij}G_j \tag{5.6}$$

where D_{ij} is a tensor determined (in principle) by the statistical properties of the turbulence and the parameter η, which intervenes in the solution of (5.5). The heat flux $\boldsymbol{F}$ is then also strictly uniform, and note that, from (5.3), $\Theta_0(\boldsymbol{x}) = \boldsymbol{G}\cdot\boldsymbol{x}$ then remains constant in time.

If $\boldsymbol{G}$ is not strictly uniform, then (5.6) must be regarded as the leading term of a series of the form

$$F_i = -D_{ij}G_j + E_{ijk}\frac{\partial G_j}{\partial x_k} - D_{ij}^{(1)}\frac{\partial G_j}{\partial t} + \dots \tag{5.7}$$

an expansion which may be expected to converge rapidly when l_0/L is sufficiently small. The tensors D_{ij}, E_{ijk}, $D_{ij}^{(1)}, \dots$, are all determined by the statistical properties of the turbulence and η; indeed, they may themselves be regarded as statistical properties of the turbulence dependent on the auxiliary parameter η, and they are uniform (in $\boldsymbol{x}$) and constant (in t) if the turbulence is homogeneous and stationary. The time derivatives in (5.7) may be replaced by space derivatives by using the space derivative of (5.3), viz

$$\frac{\partial G_i}{\partial t} = -\frac{\partial^2}{\partial x_i \partial x_j}F_j + \eta\nabla^2 G_i. \tag{5.8}$$

Substitution in (5.7) gives

$$F_i = -D_{ij}G_j + E_{ijk}\frac{\partial G_j}{\partial x_k} - D_{ij}^{(1)}\frac{\partial^2}{\partial x_j \partial x_k}(D_{kl}G_l + \dots) + \dots. \tag{5.9}$$

For homogeneous turbulence, the two leading terms are not affected by this substitution. (Note, however, that if the turbulence is inhomogeneous, so that D_{kl} is a function of $\boldsymbol{x}$, then (5.9) becomes

$$F_i = -D_{ij}^{(M)}G_j + E_{ijk}^{(M)}\partial G_j/\partial x_k + \dots \tag{5.10}$$

where

$$D_{ij}^{(M)} = D_{ij} + D_{im}^{(1)}\frac{\partial^2 D_{kj}}{\partial x_m \partial x_k} \tag{5.11}$$

and

$$E_{ijk}^{(M)} = E_{ijk} - D_{ik}^{(1)}\frac{\partial}{\partial x_m}D_{mj} - D_{im}^{(1)}\frac{\partial}{\partial x_m}D_{kj} \tag{5.12}$$

and the leading terms are evidently both modified by virtue of the inhomogeneity.)

Since the coefficient D_{ij} does not depend on $\boldsymbol{G}$, it may be calculated on the simplifying assumption that $\boldsymbol{G}$ *is* strictly uniform, so that the relationship (5.7) reduces to (5.6). Similarly, in order to calculate E_{ijk}, it would be legitimate to assume that $\partial G_j/\partial x_k$ is uniform (and therefore constant in the case of homogeneous turbulence) so that only the first *two* terms of (5.7) survive; and so on.

Attention naturally focuses on the leading term given by (5.6). If the turbulence is isotropic, i.e. its statistical properties are invariant under rotations of the frame of reference, then D_{ij} is likewise isotropic, i.e. $D_{ij} = D\delta_{ij}$, and (5.6) becomes

$$\boldsymbol{F} = -D\boldsymbol{G} = -D\nabla\Theta_0 \tag{5.13}$$

a linear diffusive relationship, in which D must be interpreted as the turbulent, or 'eddy', diffusivity.

If the turbulence is not isotropic, then it is tempting to regard D_{ij} as simply an anisotropic diffusion tensor and indeed its symmetric part

$$D_{ij}^{(s)} \equiv \tfrac{1}{2}(D_{ij} + D_{ji}) \tag{5.14}$$

does fulfil this role. However, D_{ij} may also have an antisymmetric part

$$D_{ij}^{(a)} \equiv \tfrac{1}{2}(D_{ij} - D_{ji}) = \epsilon_{ijk} D_k^{(a)} \tag{5.15}$$

say, which makes a corresponding contribution

$$\boldsymbol{F}^{(a)} = \boldsymbol{D}^{(a)} \wedge \nabla\Theta_0 \tag{5.16}$$

to the heat flux in (5.6). Note that $\boldsymbol{D}^{(a)}$ is a pseudo-vector, so that $\boldsymbol{D}^{(a)} \cdot \boldsymbol{e}$ is a pseudo-scalar, where $\boldsymbol{e}$ is any fixed unit vector. Hence (cf the discussion concerning helicity in § 3), $\boldsymbol{D}^{(a)}$ can be non-zero only if the turbulence lacks reflexional symmetry.

It will be shown in the following section that the antisymmetric part of D_{ij} is intimately related to the antisymmetric part of the Lagrangian correlation function $R_{ij}^{(L)}(\tau)$ introduced in § 2, which is non-zero only if Onsager's (1931a, b) principle of microscopic reversibility is violated. As mentioned previously, when Coriolis forces are dynamically significant, this principle must be abandoned, and it is therefore not surprising that a contribution of the form (5.16) can then appear.

In homogeneous turbulence, D_{ij} is uniform, and therefore so is $\boldsymbol{D}^{(a)}$. From (5.16) we then have

$$\nabla \cdot \boldsymbol{F}^{(a)} = 0 \tag{5.17}$$

so that $\boldsymbol{F}^{(a)}$ makes no contribution to the mean-field equation (5.3); this is because (5.16) describes heat transfer parallel to surfaces of constant Θ_0. However, if the turbulence is *inhomogeneous*, then $\boldsymbol{D}^{(a)}$ (which should now be derived from the modified tensor $D_{ij}^{(M)}$ of (5.11)) will be a function of $\boldsymbol{x}$, and so

$$\nabla \cdot \boldsymbol{F}^{(a)} = \nabla \cdot (\boldsymbol{V}\Theta_0) \tag{5.18}$$

where

$$\boldsymbol{V} = \nabla \wedge \boldsymbol{D}^{(a)} \tag{5.19}$$

and this, substituted in (5.3), implies convection of Θ_0 by an 'effective' velocity $\boldsymbol{V}$, a phenomenon which has an important counterpart in the magnetic-field context to be considered later (see § 8). The velocity $\boldsymbol{V}$ is evidently solenoidal; the corresponding effective convection velocity for magnetic field is noteworthy in being, in general, non-solenoidal.

The above discussion is quite general and is based on the assertion that a linear relationship is established between the fluctuating field θ and the mean field gradient $\boldsymbol{G}$ (see equation (5.5)). It does not depend on any assumption concerning the 'smallness' of θ; and indeed the fluctuations in θ may be expected to be large when the Péclet number is large, because then fluid elements carry their Θ value over a great distance, and therefore over a large change in $\Theta_0 = \boldsymbol{G} \cdot \boldsymbol{x}$, before molecular diffusion begins to be effective.

The same linear relationship may, of course, be exploited in constructing other statistical properties of θ; for example, $\langle \theta^2 \rangle$ must be a quadratic functional of $\boldsymbol{G}$, i.e. at leading order in l_0/L,

$$\langle \theta^2 \rangle = \Psi_{ij} G_i G_j \tag{5.20}$$

where Ψ_{ij} is again a tensor determined exclusively by the statistical properties of the turbulence and by η. This sort of statement is entirely compatible with the results of § 4 (e.g. (4.13) and (4.18)); but it may be applied somewhat more generally, in a way that incorporates properties of the energy-containing eddies of the turbulence (on scales of the order of l_0) as well as properties on much smaller scales.

6. Lagrangian analysis of the fluctuation field

In this section, we examine the form of the tensor D_{ij}, on the assumption that diffusive effects are entirely negligible. The treatment here is closely analogous to that presented by Moffatt (1974) for the corresponding magnetic problem (see § 9 below).

We suppose that at time $t = 0$, Θ is non-random, i.e.

$$\Theta(\boldsymbol{x}, 0) = \Theta_0(\boldsymbol{x}, 0). \tag{6.1}$$

The Lagrangian solution (1.9) is then

$$\Theta(\boldsymbol{x}, t) = \Theta(\boldsymbol{a}, 0) = \Theta_0(\boldsymbol{a}, 0) \tag{6.2}$$

and so

$$\boldsymbol{F}(\boldsymbol{x}, t) = \langle \boldsymbol{u}(\boldsymbol{x}, t)\Theta(\boldsymbol{x}, t) \rangle = \langle \boldsymbol{v}(\boldsymbol{a}, t)\Theta_0(\boldsymbol{a}, 0) \rangle \tag{6.3}$$

where $\boldsymbol{a}(\boldsymbol{x}, t)$ is the initial position of the particle which passes through $\boldsymbol{x}$ at time t. For given $(\boldsymbol{x}, t)$, $\boldsymbol{a}$ varies from one realisation to another, and should be regarded as a random variable in (6.3).

On the assumption that the scale of Θ_0 is large, we now expand $\Theta(\boldsymbol{a}, 0)$ in Taylor series about $(\boldsymbol{x}, 0)$:

$$\Theta_0(\boldsymbol{a}, 0) = (1 - \boldsymbol{X} \cdot \nabla + \tfrac{1}{2}(\boldsymbol{X} \cdot \nabla)^2 - \ldots)\Theta_0(\boldsymbol{x}, 0) \tag{6.4}$$

where $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{a}$, and substitute in (6.3), giving

$$F_i(\boldsymbol{x}, t) = -D_{ij}(t) G_j(\boldsymbol{x}, 0) + E_{ijk}(t) G_{j,k}(\boldsymbol{x}, 0) - \ldots \tag{6.5}$$

where

$$D_{ij}(t) = \langle v_i(\boldsymbol{a}, t) X_j(\boldsymbol{a}, t) \rangle = \int_0^t \langle v_i(\boldsymbol{a}, t) v_j(\boldsymbol{a}, t_1) \rangle \, \mathrm{d}t_1 \tag{6.6}$$

and

$$E_{ijk}(t) = \langle v_i(\boldsymbol{a}, t)X_j(\boldsymbol{a}, t)X_k(\boldsymbol{a}, t)\rangle$$
$$= \int_0^t \int_0^t \langle v_i(\boldsymbol{a}, t)v_j(\boldsymbol{a}, t_1)v_k(\boldsymbol{a}, t_2)\rangle \, \mathrm{d}t_1 \, \mathrm{d}t_2. \tag{6.7}$$

We have already encountered symmetrised forms of the tensors $\boldsymbol{D}_{ij}$ and $\boldsymbol{E}_{ijk}$ (see equations (2.7) and (2.12)), and it is reassuring to find the same tensors appearing naturally in the two-scale treatment. As noted earlier, the integrals (6.6) and (6.7) approach their asymptotic ($t = \infty$) values rapidly for $t \gg t_c$, the Lagrangian correlation time, and it is these asymptotic values which may now be used in (6.5); these will be denoted simply by $\boldsymbol{D}_{ij}$ and $\boldsymbol{E}_{ijk}$.

The time scale on which the mean field Θ_0 changes is then given by

$$t_M \sim L^2/D$$

where $D \sim u_0^2 t_c$, i.e.

$$t_M \sim (L/u_0 t_c)^2 t_c. \tag{6.8}$$

Provided $t \ll t_M$, $\boldsymbol{G}(\boldsymbol{x}, 0)$ in (6.5) may be replaced by $\boldsymbol{G}(\boldsymbol{x}, t)$ (the correction affecting only higher terms in the expansion, as commented earlier) and (6.5) then takes the expected form

$$F_i(\boldsymbol{x}, t) = \left(-\boldsymbol{D}_{ij} + \boldsymbol{E}_{ijk} \frac{\partial}{\partial x_k} - \ldots\right) G_j(\boldsymbol{x}, t). \tag{6.9}$$

This derivation of the expansion (6.9) and of the tensors $\boldsymbol{D}_{ij}$ and $\boldsymbol{E}_{ijk}$, although making no appeal to the effects of molecular diffusivity, in fact depends on these effects in quite a subtle way. The reason is that we have assumed that there is an initial reference instant $t = 0$ at which the Θ field is non-random, and we have supposed that the elapsed time t from this instant is small compared with t_M, as given by (6.8). For $t = \mathrm{O}(t_M)$ and greater, the validity of the procedure leading to (6.9) must be in some doubt when $\eta = 0$. However, when η is non-zero but still extremely small (compared with $u_0 l_0$), molecular diffusion always acts in such a way as to eliminate θ fluctuations with a consequent 'loss of memory' of initial conditions. As shown by Batchelor (1952),

$$\langle (\nabla\Theta)^2 \rangle = (\nabla\langle\Theta\rangle)^2 + \langle(\nabla\theta)^2\rangle \sim \exp\left(\tfrac{1}{2}t\zeta\right) \tag{6.10}$$

where ζ is the mean logarithmic rate of extension of line elements, i.e.

$$\zeta = \lim_{t\to\infty} \frac{\partial}{\partial t} \ln \overline{\delta L(t)} \tag{6.11}$$

and this quantity, being determined by the smallest scales of the motion, is determined in order of magnitude by

$$\zeta \sim (\epsilon/\nu)^{1/2}. \tag{6.12}$$

The length scale of the θ field, $l(t)$, is therefore given in order of magnitude by

$$l^2 \sim L^2 \exp\left(-\tfrac{1}{2}t\zeta\right) \tag{6.13}$$

and molecular diffusivity effects become significant when

$$\eta \int_0^t l^{-2}\,\mathrm{d}t \sim \frac{2\eta}{L^2\zeta}[\exp(\tfrac{1}{2}t\zeta)-1] \tag{6.14}$$

becomes of the order of unity, i.e. when

$$t \sim \frac{2}{\zeta}\ln(L^2\zeta/2\eta). \tag{6.15}$$

Provided this time is small compared with t_M, the initial conditions should be irrelevant in the derivation of (6.9). Using (6.12) and (6.8), and putting $t_c \sim l_0/u_0$ (probably a low estimate), this criterion reduces to

$$\ln Pe \ll Re^{1/2}(L/l_0)^2 - \ln[Re^{1/2}(L/l_0)^2] \tag{6.16}$$

and it is evident that in practice this criterion will always be satisfied when $Re \gg 1$ and $L/l_0 \gg 1$, as has been assumed. This means that an extremely small value of η will in general be sufficient to achieve the required loss of memory of initial conditions, to make (6.9) valid for all t.

It is natural now to focus attention on the leading term of the expansion (6.9). For $t \gg t_c$, we have

$$D_{ij} \sim u_0^2 \int_0^\infty R_{ji}^{(L)}(\tau)\,\mathrm{d}\tau \tag{6.17}$$

and in the case of isotropic turbulence, $D_{ij} = D_0\delta_{ij}$, where

$$D_0 = \tfrac{1}{3}D_{ii} = \tfrac{1}{3}u_0^2 \int_0^\infty R_{ii}^{(L)}(\tau)\,\mathrm{d}\tau \tag{6.18}$$

in agreement with the value of D_0 given by (2.4) and (2.9).

When $t \ll t_c$, $\boldsymbol{v}(\boldsymbol{a}, t)$ may be expanded in Taylor series about $t = 0$ (when $\boldsymbol{a} = \boldsymbol{x}$):

$$\boldsymbol{v}(\boldsymbol{a}, t) = \boldsymbol{u}\Big|_{t=0} + t\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\Big|_{t=0} + \tfrac{1}{2}t^2\frac{\mathrm{D}^2}{\mathrm{D}t^2}\boldsymbol{u}\Big|_{t=0} + \ldots \tag{6.19}$$

where $\mathrm{D}/\mathrm{D}t$ is the Lagrangian derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla. \tag{6.20}$$

$\boldsymbol{v}(\boldsymbol{a}, \tau)$ may of course be expanded similarly. Substituting in (6.6) and retaining only terms of order t^2, we have

$$D_{ij}(t) = t\langle u_i u_j\rangle_0 + t^2\left[\left\langle\frac{\mathrm{D}u_i}{\mathrm{D}t}u_j\right\rangle_0 + \frac{1}{2}\left\langle u_i\frac{\mathrm{D}u_j}{\mathrm{D}t}\right\rangle_0\right] + \mathrm{O}(t^3) \tag{6.21}$$

where the suffix 0 indicates evaluation at $t = 0$. Even at this order, we see the appearance of the cubic mean quantity

$$\langle u_j(\boldsymbol{u}\cdot\nabla)u_i\rangle. \tag{6.22}$$

The complete series for $D_{ij}(t)$ obviously involves Eulerian mean products of the velocity field of *all* orders.

6.1. The antisymmetric part of D_{ij}

The pseudo-vector $\boldsymbol{D}^{(a)}$ defined by (5.15) can be evaluated simply from (6.6) and is given by

$$\boldsymbol{D}^{(a)}(t)=\tfrac{1}{2}\int_0^t \langle \boldsymbol{v}(\boldsymbol{a},t)\wedge \boldsymbol{v}(\boldsymbol{a},\tau)\rangle\,\mathrm{d}\tau \tag{6.23}$$

and it is easy to see that this will be non-zero if there is any systematic tendency for particles to 'veer' to one side rather than the other in their otherwise random path (as in figure 2); note that a particle path that is 'clockwise' in a right-handed frame of reference is 'anticlockwise' in a left-handed frame. A preference for one 'sense' rather than the other is therefore a symptom of a lack of reflexional symmetry in the turbulence. This lack of reflexional symmetry as it appears in (6.23) may also be interpreted as a lack of symmetry with respect to time reversal; for, from (6.6) and (2.5),

$$\begin{aligned}D_{ij}-D_{ji}&=u_0^2\int_0^t (R_{ji}^{(L)}(\tau)-R_{ij}^{(L)}(\tau))\,\mathrm{d}\tau\\ &=u_0^2\int_0^t [R_{ji}^{(L)}(\tau)-R_{ji}^{(L)}(-\tau)]\,\mathrm{d}\tau.\end{aligned} \tag{6.24}$$

Hence, if $R_{ij}^{(L)}(\tau)$ is invariant under the time reversal $\tau\to-\tau$, then $D_{ij}=D_{ji}$ (cf Onsager 1931a, b). A motion like that sketched in figure 2 is clearly not statistically invariant under time reversal, since reversing the arrows on the streamlines converts a systematic clockwise motion into a systematic anticlockwise motion.

For $t\ll t_c$, the expansion for $\boldsymbol{D}^{(a)}(t)$ corresponding to (6.21) is

$$\boldsymbol{D}^{(a)}(t)=-\tfrac{1}{4}t^2\langle \boldsymbol{u}\wedge \mathrm{D}\boldsymbol{u}/\mathrm{D}t\rangle+\mathrm{O}(t^3) \tag{6.25}$$

which, using

$$\mathrm{D}\boldsymbol{u}/\mathrm{D}t=\partial\boldsymbol{u}/\partial t-\boldsymbol{u}\wedge\boldsymbol{\omega}+\tfrac{1}{2}\nabla(\boldsymbol{u}^2) \tag{6.26}$$

may be easily converted to the form

$$\boldsymbol{D}^{(a)}(t)=-\tfrac{1}{4}t^2[\langle \boldsymbol{u}\wedge\partial\boldsymbol{u}/\partial t\rangle+\tfrac{1}{2}\langle \boldsymbol{u}^2\boldsymbol{\omega}\rangle-\langle \boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{\omega})\rangle]+\mathrm{O}(t^3) \tag{6.27}$$

where $\boldsymbol{\omega}=\nabla\wedge\boldsymbol{u}$. Each term here is a pseudo-vector, non-zero only if the turbulence lacks reflexional symmetry.

The first term is of particular interest, being quadratic in the velocity field. Using (3.1) and (3.6) it is easy to show that

$$\langle \boldsymbol{u}\wedge\partial\boldsymbol{u}/\partial t\rangle_i=-\mathrm{i}\epsilon_{ijk}\int \omega\Phi_{jk}(\boldsymbol{k},\omega)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega \tag{6.28}$$

which clearly involves only the antisymmetric part of Φ_{jk}. Using (3.20), this reduces to

$$\langle \boldsymbol{u}\wedge\partial\boldsymbol{u}/\partial t\rangle=\int k^{-2}\omega\boldsymbol{k}H(\boldsymbol{k},\omega)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega \tag{6.29}$$

a weighted integral of the helicity spectrum function, which is in general non-zero.

Some interest attaches here to the special case of 'two-dimensional turbulence' which is of independent importance in geophysical contexts. Here,

$$\boldsymbol{u}=(\partial\psi/\partial y,-\partial\psi/\partial x,0) \tag{6.30}$$

where $\psi(x, y, t)$ is a stationary random function of its three arguments, and

$$\boldsymbol{\omega} = (0, 0, -\nabla^2\psi). \tag{6.31}$$

It is evident from (3.19) that $H(\boldsymbol{k}, \omega)$ is identically zero for such a flow (and of course $\boldsymbol{u} \cdot \boldsymbol{\omega} \equiv 0$ also). In this situation, only the second term on the right-hand side of (6.27) survives, and we have

$$\boldsymbol{D}^{(\mathrm{a})}(t) = -\tfrac{1}{8}t^2\langle \boldsymbol{u}^2\boldsymbol{\omega}\rangle + \mathrm{O}(t^3). \tag{6.32}$$

6.2. Limitations of the Lagrangian analysis

Although the above analysis provides an expression for the eddy diffusion tensor D_{ij} which appears altogether plausible, and consistent with the Taylor approach, it does not stand up to attempts to construct other statistical quantities; even the simplest of these, the mean square of the fluctuation $\theta \approx -\boldsymbol{X} \cdot \boldsymbol{G}$, is given by

$$\langle \theta^2\rangle = G_iG_j\langle X_iX_j\rangle \sim 2tD_{ij}G_iG_j \tag{6.33}$$

which apparently increases without limit. This divergence arises partly because molecular diffusion has been neglected, and partly because the gradient $\boldsymbol{G}$ has been supposed uniform over the full extent of a particle trajectory throughout a time t (even though this increases in root mean square like $t^{1/2}$). In fact, we have seen in § 4 that the equilibrium level of $\langle \theta^2\rangle$ is given by (4.21) (with possibly the logarithmic correction (4.22)); the process by which the transition occurs from the linear growth (6.33) to the asymptotic steady level (4.21) is clearly controlled by molecular diffusivity. Certain aspects of this process will be examined further by means of a simplified model problem, in § 12.

7. First-order smoothing theory

The Lagrangian approach described in the preceding section involves (in effect) neglect of the diffusion term $\eta\nabla^2\theta$ in the fluctuation equation (5.5). An alternative approach, which can provide some useful insights, is based (at lowest order) on neglect of the term $\nabla \cdot (\boldsymbol{u}\theta - \boldsymbol{F})$ which involves the product of fluctuating fields; more formally, we construct a solution of the form

$$\theta = \Sigma\theta^{(n)}(\boldsymbol{x}, t) \qquad \boldsymbol{F} = \Sigma\boldsymbol{F}^{(n)}(\boldsymbol{x}, t) \tag{7.1}$$

where

$$\partial\theta^{(0)}/\partial t - \eta\nabla^2\theta^{(0)} = -\boldsymbol{u} \cdot \boldsymbol{G} \tag{7.2}$$

$$\partial\theta^{(n+1)}/\partial t - \eta\nabla^2\theta^{(n+1)} = -\nabla \cdot (\boldsymbol{u}\theta^{(n)} - \boldsymbol{F}^{(n)}) \tag{7.3}$$

with

$$\boldsymbol{F}^{(n)} = \langle \boldsymbol{u}\theta^{(n)}\rangle. \tag{7.4}$$

At lowest order, this is known as the first-order smoothing approximation, or the quasi-linear approximation. The Fourier transform of (7.2) (treating $\boldsymbol{G}$ as uniform) is

$$(-\mathrm{i}\omega + \eta k^2)\hat{\theta}^{(0)} = -\hat{\boldsymbol{u}} \cdot \boldsymbol{G} \tag{7.5}$$

and so, using (3.6),

$$F_i^{(0)} = \langle u_i \theta^{(0)} \rangle = -\int (-\mathrm{i}\omega + \eta k^2)^{-1} \Phi_{ij}(\boldsymbol{k}, \omega) G_j \,\mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega. \tag{7.6}$$

Hence

$$F_i^{(0)} = -D_{ij}^{(0)} G_j \tag{7.7}$$

where

$$D_{ij}^{(0)} = \int \frac{\mathrm{i}\omega + \eta k^2}{\omega^2 + \eta^2 k^4} \Phi_{ij}(\boldsymbol{k}, \omega)\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega. \tag{7.8}$$

The symmetric part of this is given by

$$D_{ij}^{(0\mathrm{s})} = \int \frac{\eta k^2}{\omega^2 + \eta^2 k^4} \Phi_{ij}(\boldsymbol{k}, \omega)\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega \tag{7.9}$$

and is a positive-definite tensor (or at any rate non-negative-definite) of a genuinely diffusive character. Note that, as $\eta \to 0$, the integrand becomes singular at $\omega = 0$; but integration through the singularity gives

$$D_{ij}^{(0\mathrm{s})} \sim \pi \int \Phi_{ij}^{(\mathrm{s})}(\boldsymbol{k}, 0)\, \mathrm{d}\boldsymbol{k} \tag{7.10}$$

which is in fact identical with the linearised form of (6.17) (when the Lagrangian correlation degenerates to an Eulerian correlation). The result (7.10) may therefore be seen as the meeting point of the Lagrangian theory and the first-order smoothing theory, which of course must agree when *both* diffusive effects *and* fluctuation-interaction effects are negligible in (5.5).

The antisymmetric part of (7.8) is given by

$$D_{ij}^{(0\mathrm{a})} = \epsilon_{ijk} D_k^{(0)} \tag{7.11}$$

where

$$\boldsymbol{D}^{(0)} = -\tfrac{1}{2} \int \frac{\omega \boldsymbol{k} H(\boldsymbol{k}, \omega)}{k^2(\omega^2 + \eta^2 k^4)} \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega \tag{7.12}†$$

which admits comparison with (6.27) and (6.29). Again, we see that it is helicity that gives rise to an antisymmetric ingredient of D_{ij}.

The general treatment of equations (7.1)–(7.4) at higher orders ($n = 1, 2, \ldots$) leads to heavy notation, and we shall therefore limit attention here to two simple examples which shed light on some of the physical effects involved. It is well known in the magnetic context (Roberts 1970, 1972, Childress 1970, 1979) that there is a close parallel between the action of turbulence on a convected field and the action of a space-periodic velocity field. The thermal problem lends itself more readily to similar investigation, since even with a two-dimensional velocity field of the form (6.30), non-trivial results may be obtained. Suppose first that

$$\psi = \psi_0 \sin kx \sin ky \tag{7.13}$$

so that the velocity field consists of a square array of eddies (figure 8). As before, in

† I am indebted to Peter Watterson, who drew my attention to the fact that the antisymmetric part of D_{ij} can be non-zero even under the first-order smoothing approximation.

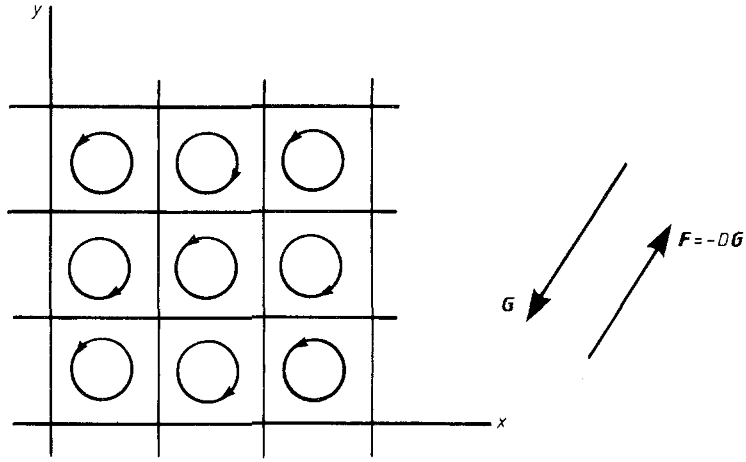

Figure 8. Sketch of streamline configuration for the streamfunction (7.13), for which adjacent eddies rotate in opposite senses. The heat flux is parallel to the mean temperature gradient.

calculating D_{ij}, we may suppose that $\boldsymbol{G}$ is uniform and steady, and the steady solution of (6.2) is then

$$\theta^{(0)} = -\frac{1}{2\eta k^2}\left(G_1\frac{\partial\psi}{\partial y} - G_2\frac{\partial\psi}{\partial x}\right). \tag{7.14}$$

Hence

$$\boldsymbol{F}^{(0)} = \langle \boldsymbol{u}\theta^{(0)}\rangle = -\frac{\psi_0^2}{8\eta}(G_1, G_2, 0) \tag{7.15}$$

the average being taken over a square cell. This, of course, is just the counterpart of the more general result (7.7) and (7.8). Now, however, we may continue the procedure to higher order in the relevant Péclet number $Pe = \psi_0/\eta$. We find that $\boldsymbol{F}^{(1)} = 0$ and

$$\boldsymbol{F}^{(2)} = \frac{\psi_0^4}{128\eta^3}(G_1, G_2, 0). \tag{7.16}$$

All the odd terms vanish, because changing the sign of ψ_0 in (7.13) is equivalent to a change of origin which clearly can have no effect on the mean diffusive properties of the flow. Evidently the diffusion tensor D_{ij} for which $F_i = -D_{ij}G_j$ has the form

$$D_{ij} = \eta F(Pe)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{7.17}$$

where, from (7.15) and (7.16),

$$F(Pe) = \tfrac{1}{8}Pe^2 - \tfrac{1}{128}Pe^4 + \mathrm{O}(Pe^6) \tag{7.18}$$

for small Pe. The radius of convergence of this series is probably close to $Pe = 4$. Incidentally the high Péclet number limit for this problem can be treated by boundary layer methods similar to those developed by Proctor (1975) in a related problem; this suggests that†

$$F(Pe) \sim Pe^{1/2} \qquad \text{as} \qquad Pe \to \infty \tag{7.19}$$

† I am indebted to Dr M R E Proctor for this observation.

so that $D_{ij} \propto \eta^{1/2}$ as $\eta \to 0$ (all other parameters being fixed). It should be emphasised however that the behaviour (7.18) and (7.19) is critically dependent on the assumed *steadiness* of the velocity field (7.13); unsteadiness is an essential feature of real turbulent flows, so that the results (7.18) and (7.19) are no more than suggestive in the turbulence context.

We have seen in § 6 that a motion which exhibits a preferred 'sense of turning' gives a non-symmetric D_{ij}. Let us now see whether this behaviour arises also for small Pe. To get an array of eddies all with the same sense, we need a ψ which is a periodic function of x and y and such that $u = \partial\psi/\partial y$ does not change sign as x varies (for fixed y) and $v = -\partial\psi/\partial x$ does not change sign as y varies (for fixed x). A suitable simple choice is

$$\psi = \psi_0 \sin^2 kx \sin^2 ky \tag{7.20}$$

the streamlines of which (ψ = constant) are sketched in figure 9. All the eddies rotate in the same sense, and a change of sign in ψ_0 reverses this sense and is not equivalent

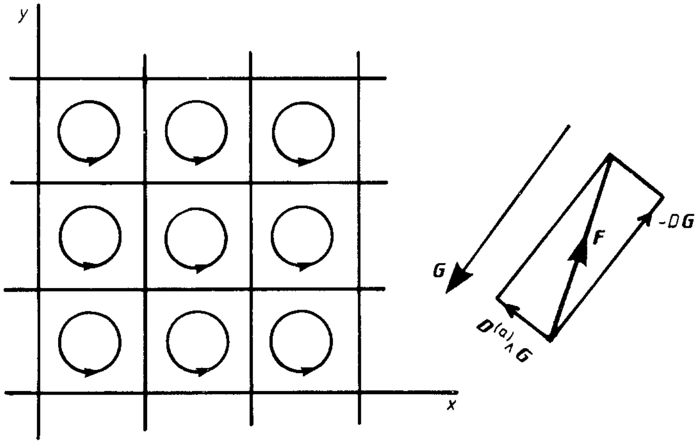

Figure 9. Sketch of streamline configuration for the streamfunction (7.20), for which all eddies rotate in the same sense. The heat flux has a component perpendicular to the mean temperature gradient.

to a mere change of origin; put another way, the motion is not invariant under time reversal (cf the discussion in § 6.1 above).

For this motion, following the above procedure we find

$$\boldsymbol{F}^{(0)} = -\frac{\eta}{64} Pe^2(G_1, G_2, 0) \tag{7.21}$$

where, as before, $Pe = \psi_0/\eta$. Now, however, the contribution $\boldsymbol{F}^{(1)}$ does not vanish; we find

$$\boldsymbol{F}^{(1)} = \frac{\eta}{256} Pe^3(G_2, -G_1, 0) = \boldsymbol{D}^{(1)} \wedge \boldsymbol{G} \tag{7.22}$$

where

$$\boldsymbol{D}^{(1)} = (0, 0, -\eta Pe^3/256). \tag{7.23}$$

The contribution $\boldsymbol{F}^{(1)}$ to the total heat flux is perpendicular to the local mean temperature gradient. Note that, for the motion described by the stream function (7.20)

$$\langle \boldsymbol{u}^2 \boldsymbol{\omega} \rangle = (0, 0, \psi_0^3 k^4/4) \tag{7.24}$$

so that

$$\boldsymbol{D}^{(1)} = \frac{-1}{4\eta^2 k^4} \langle \boldsymbol{u}^2 \boldsymbol{\omega} \rangle \tag{7.25}$$

which admits suggestive comparison with (6.32).

The physical reason for the appearance of this transverse component of heat flux is indicated by the sketch of figure 10: the isotherms are distorted antisymmetrically

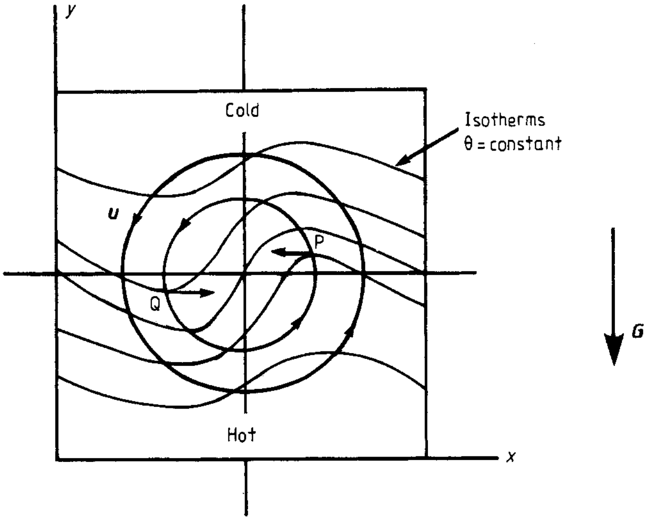

Figure 10. Sketch of the distorted isotherms when ψ is given by (7.20) and $\boldsymbol{G} = (0, G_2, 0)$. The temperature excess at P is convected to the left and the deficit at Q is convected to the right, implying a net component of heat flux in the negative x direction.

in each cell; hot fluid is convected to the left in region P and cold fluid to the right in region Q; this provides a net heat flux to the left which is not compensated by heat flux arising in any other parts of the cell.

8. Double-scale analysis for the convected vector field

We turn now to the case of the convected vector field $\boldsymbol{B}(\boldsymbol{x}, t)$ satisfying equation (1.2). This has been treated very thoroughly in many previous studies (e.g. Moffatt 1978, Krause and Rädler 1980) and so only the main points will be recapitulated here The field $\boldsymbol{B}$ is decomposed into its mean part $\langle \boldsymbol{B} \rangle = \boldsymbol{B}_0(\boldsymbol{x}, t)$ and its fluctuating part $\boldsymbol{b}$, and the mean and fluctuating parts of (1.2) are then

$$\partial \boldsymbol{B}_0/\partial t = \nabla \wedge \boldsymbol{\mathscr{E}} + \eta \nabla^2 \boldsymbol{B}_0 \tag{8.1}$$

$$\partial \boldsymbol{b}/\partial t = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}_0) + \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{b} - \langle \boldsymbol{u} \wedge \boldsymbol{b} \rangle) + \eta \nabla^2 \boldsymbol{b} \tag{8.2}$$

where $\boldsymbol{\mathscr{E}} = \langle \boldsymbol{u} \wedge \boldsymbol{b} \rangle$. Just as for equation (5.5), equation (8.2) establishes a linear relationship between $\boldsymbol{b}$ and $\boldsymbol{B}_0$, and thence between $\boldsymbol{\mathscr{E}}$ and $\boldsymbol{B}_0$; and the expansion analogous to (5.7) is

$$\mathscr{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk} \partial B_{0j}/\partial x_k + \alpha_{ij}^{(1)} \partial B_{0j}/\partial t + \dots. \tag{8.3}$$

Elimination of the time derivative using (8.1) gives the modified expansion, analogous to (5.9),

$$\mathscr{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk} \partial B_{0j}/\partial x_k + \alpha_{ij}^{(1)} \left[\epsilon_{jkm} \frac{\partial}{\partial x_k} \left(\alpha_{mp} B_{0p} + \beta_{mpq} \frac{\partial B_{0p}}{\partial x_q} \right) + \eta \nabla^2 B_j \right] + \dots. \tag{8.4}$$

For strictly homogeneous turbulence, this takes the form

$$\mathscr{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk}^{(M)} \partial B_{0j}/\partial x_k + \dots \tag{8.5}$$

where

$$\beta_{ijk}^{(M)} = \beta_{ijk} + \alpha_{ip}^{(1)} \epsilon_{pkm} \alpha_{mj} \tag{8.6}$$

so that here there *is* a non-trivial modification of the second term of the expansion. For inhomogeneous turbulence, the leading term is modified (cf (5.11)) thus:

$$\alpha_{ij}^{(M)} = \alpha_{ij} + \alpha_{ip}^{(1)} \epsilon_{pkm} \frac{\partial}{\partial x_k} \alpha_{mj} \tag{8.7}$$

and there is a similar further modification to β_{ijk}.

The coefficients $\alpha_{ij}, \beta_{ijk}, \dots$, in (8.3) are pseudo-tensors ('pseudo' because $\boldsymbol{B}_0$ is an axial vector whereas $\boldsymbol{\mathscr{E}}$ is a polar vector) which again are determined by the statistical properties of the turbulence and by the parameter η. Consider first the leading coefficient α_{ij} which may be determined on the simplifying assumption that $\boldsymbol{B}_0$ is uniform and steady. Again, this may be split into symmetric and antisymmetric parts:

$$\alpha_{ij} = \alpha_{ij}^{(s)} - \epsilon_{ijk} \gamma_k \tag{8.8}$$

so that, at this order,

$$\mathscr{E}_i = \alpha_{ij}^{(s)} B_{0j} + (\boldsymbol{\gamma} \wedge \boldsymbol{B}_0)_i. \tag{8.9}$$

Here $\alpha_{ij}^{(s)}$ is a pseudo-tensor so that

$$\alpha_{ij}^{(s)} e_i^{(1)} e_j^{(2)}$$

is a pseudo-scalar, for any unit vectors $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$. This obviously means that $\alpha_{ij}^{(s)}$ can be non-zero only if the turbulence lacks reflexional symmetry, and a relationship, direct or indirect, between $\alpha_{ij}^{(s)}$ and the helicity spectrum function $H(\boldsymbol{k}, \omega)$ may be anticipated. This dominant importance of helicity in determining the *leading* term of (8.3) is noteworthy; the appearance of the mean electromotive force $\alpha_{ij}^{(s)} B_{0j}$ is what is known as the 'α effect' in dynamo theory (Steenbeck and Krause 1966), and is the key to the understanding of how magnetic fields can be generated by turbulent fluid motion (see, for example, Moffatt 1978, Parker 1979, Krause and Rädler 1980).

The vector $\boldsymbol{\gamma}$ appearing in (8.9) is a polar vector, and as such should be determined by the reflectionally symmetric ingredients of the turbulence. It leads to a term $\nabla \wedge (\boldsymbol{\gamma} \wedge \boldsymbol{B}_0)$ in the mean-field equation (8.1), implying an apparent convection of the mean field $\boldsymbol{B}_0$ relative to the fluid with effective velocity $\boldsymbol{\gamma}$. Obviously this effect can arise only if the turbulence is non-isotropic, the direction of $\boldsymbol{\gamma}$ appearing as a 'preferred direction'.

The second term of (8.3) is also of great importance in dynamo theory, since it incorporates the effective eddy diffusivity acting on the mean field, as well as certain subsidiary effects. To see this, it is sufficient to consider the case of isotropic turbulence for which β_{ijk} must take the isotropic form

$$\beta_{ijk} = \beta\epsilon_{ijk} \tag{8.10}$$

where β is a pure scalar. The corresponding contribution to $\boldsymbol{\mathscr{E}}$ in (8.3) is then $-\beta(\nabla \wedge \boldsymbol{B}_0)$ and the corresponding contribution to $\nabla \wedge \boldsymbol{\mathscr{E}}$ in (8.1) (assuming β is uniform) is $\beta\nabla^2\boldsymbol{B}_0$, so that the total effective diffusivity acting on $\boldsymbol{B}_0$ is $\eta + \beta$.

9. Lagrangian analysis for the convected vector field ($\eta = 0$)

Following the procedure of § 6, let us now suppose that at time $t = 0$, $\boldsymbol{B}$ is non-random, i.e.

$$\boldsymbol{B}(\boldsymbol{x}, 0) = \boldsymbol{B}_0(\boldsymbol{x}, 0). \tag{9.1}$$

The Lagrangian solution (1.11) then gives

$$B_i(\boldsymbol{x}, t) = B_{0j}(\boldsymbol{a}, 0)\, \partial x_i/\partial a_j \tag{9.2}$$

from which we may construct

$$\mathscr{E}_i(\boldsymbol{x}, t) = \epsilon_{ijk}\langle v_j(\boldsymbol{a}, t) B_{0l}(\boldsymbol{a}, 0)\, \partial x_k/\partial a_l\rangle. \tag{9.3}$$

In order to calculate $\alpha_{ij}(t)$ (a function of t because of the special initial condition (9.1)), $\boldsymbol{B}_0$ in (9.3) may be regarded as uniform, and we then have

$$\alpha_{il}(t) = \epsilon_{ijk} \int_0^t \left\langle v_j(\boldsymbol{a}, t) \frac{\partial}{\partial a_l} v_k(\boldsymbol{a}, t_1)\right\rangle \mathrm{d}t_1. \tag{9.4}$$

Again it is illuminating to consider the initial development of this function. For t small, we may use the Taylor series (6.19) for $\boldsymbol{v}(\boldsymbol{a}, t)$; at leading order this gives

$$\alpha_{il}(t) = \epsilon_{ijk}\left\langle u_j(\boldsymbol{x}, 0) \frac{\partial}{\partial x_l} u_k(\boldsymbol{x}, 0)\right\rangle t + \mathrm{O}(t^2). \tag{9.5}$$

The symmetric part of this is characterised by

$$\alpha \equiv \tfrac{1}{3}\alpha_{ii} = -\tfrac{1}{3}\langle \boldsymbol{u} \cdot \nabla \wedge \boldsymbol{u}\rangle_{t=0} t + \mathrm{O}(t^2) \tag{9.6}$$

which shows up very clearly the relevance of helicity in this context. The antisymmetric part is given by

$$\gamma_p \equiv -\tfrac{1}{2}\epsilon_{ilp}\alpha_{il} = -\tfrac{1}{2}\nabla \cdot \langle \boldsymbol{u} u_p\rangle t + \mathrm{O}(t^2). \tag{9.7}$$

The leading term here is non-zero only if the turbulence is *inhomogeneous*; suppose, for example, that, with respect to Cartesian coordinates $0xyz$, $\boldsymbol{u} = (u, v, w)$, and that the turbulence is homogeneous in the (x, y) plane but inhomogeneous with respect to the z direction. Then, from (9.7),

$$\boldsymbol{\gamma} = (0, 0, \gamma(z, t)) \qquad \text{where} \qquad \gamma(z, t) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\langle w^2\rangle t + \mathrm{O}(t^2) \tag{9.8}$$

i.e. a flux convection velocity directed *down the gradient of turbulent intensity* (see figure 11). This is the *diamagnetic effect* first identified by Zel'dovich (1957); it is

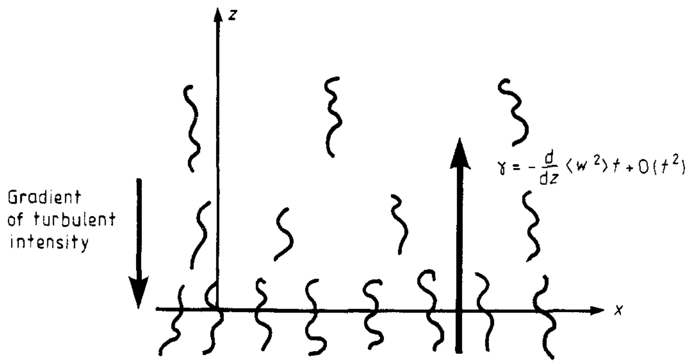

Figure 11. Schematic representation of the flux convection effect associated with inhomogeneous turbulence.

closely associated with the well-known phenomenon of *flux expulsion*, studied in the context of deterministic flows by Parker (1963), Weiss (1966) and others. The effect was recognised by Rädler (1968) and it has recently been incorporated in dynamo models by Nightingale (1983). Localised (inhomogeneous) turbulence tends to expel magnetic flux with an effective velocity $\boldsymbol{\gamma}$. It is important to note that, in general in this situation,

$$\nabla \cdot \boldsymbol{\gamma} \neq 0 \tag{9.9}$$

(even although the background turbulent field is strictly solenoidal).

The expression (9.7) is, of course, valid only for small times t; but, assuming that statistical equilibrium is established for $t \gg t_c$ where t_c is a Lagrangian correlation time, a reasonable approximation to the asymptotic form of $\boldsymbol{\gamma}$ should be given by

$$\boldsymbol{\gamma} = -\tfrac{1}{2}\nabla \cdot \langle \boldsymbol{uu}\rangle t_c \qquad (t \gg t_c). \tag{9.10}$$

It is interesting that it is the divergence of the Reynolds stress $\langle \boldsymbol{uu}\rangle$ that appears here, a quantity which also governs the evolution of the mean velocity field $\boldsymbol{U}(\boldsymbol{x}, t)$ in the *dynamics* of inhomogeneous turbulence (the mean pressure field being responsible for the maintenance of a solenoidal $\boldsymbol{U}$).

If the turbulence is strictly homogeneous, then the leading term of (9.7) vanishes; there is, however, an effect at order t^2, viz

$$\boldsymbol{\gamma} = \tfrac{1}{4}\langle \boldsymbol{uu} : \nabla\nabla \boldsymbol{u}\rangle t^2 + \mathrm{O}(t^3) \tag{9.11}$$

involving a *cubic* velocity correlation. This mean quantity vanishes if the turbulence is isotropic; but for non-isotropic turbulence it simply provides a uniform convection of mean magnetic field.

9.1. Effects associated with mean-field gradient

We now turn to the development of the tensor β_{ijk} appearing in the fundamental expansion (8.3). To analyse this, we suppose that, at $t = 0$,

$$B_{0i}(\boldsymbol{x}, 0) = B_{0i} + x_j(\partial B_{0i}/\partial x_j) \tag{9.12}$$

in which B_{0i} and $\partial B_{0i}/\partial x_j$ on the right-hand side will be taken as uniform. Then from (9.3),

$$\begin{aligned}\mathscr{E}_i(\boldsymbol{x}, t) &= \epsilon_{ijk}\left\langle v_j(\boldsymbol{a}, t)\frac{\partial x_m}{\partial a_l}\left(B_{0l} - X_m\frac{\partial B_{0l}}{\partial a_m} + x_m\frac{\partial B_{0l}}{\partial a_m}\right)\right\rangle \\ &= \alpha_{il}B_{0l}(\boldsymbol{x}, 0) + \beta_{ilm}\,\partial B_{0l}/\partial x_m\end{aligned}$$

where $\alpha_{il}(t)$ is as previously determined, and

$$\beta_{ilm}(t) = -\epsilon_{ijl}\langle v_j X_m\rangle - \epsilon_{ijk}\langle v_j X_m\,\partial X_k/\partial a_l\rangle. \tag{9.13}$$

(Here $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{a}$ is the Lagrangian particle displacement—see figure 1.)

The effective eddy diffusivity is (from (8.10))

$$\begin{aligned}\beta &\equiv \tfrac{1}{6}\epsilon_{ilm}\beta_{ilm} \\ &= \tfrac{1}{3}\langle \boldsymbol{v}\cdot\boldsymbol{X}\rangle - \tfrac{1}{6}\langle(\boldsymbol{v}\cdot\nabla_{\boldsymbol{a}})(\tfrac{1}{2}\boldsymbol{X}^2) - (\boldsymbol{v}\cdot\boldsymbol{X})\nabla_{\boldsymbol{a}}\cdot\boldsymbol{X}\rangle.\end{aligned} \tag{9.14}$$

The first contribution here is the eddy diffusivity

$$D = \tfrac{1}{3}\langle \boldsymbol{v}\cdot\boldsymbol{X}\rangle \tag{9.15}$$

that acts upon a scalar field (see § 6). But we now have an additional contribution, cubic in the displacement $\boldsymbol{X}$, which acts only on the vector field. This distinction between turbulent diffusivities for scalar and vector fields was first identified by Moffatt (1974).

Again, we may examine the Taylor expansion of (9.14) for small t. Using (6.19), we find

$$D(t) = \tfrac{1}{3}\langle \boldsymbol{u}^2\rangle_0 t + \tfrac{3}{2}\langle \boldsymbol{u}\cdot \mathrm{D}\boldsymbol{u}/\mathrm{D}t\rangle_0 t^2 + \mathrm{O}(t^3) \tag{9.16}$$

and

$$\beta(t) = D(t) + \tfrac{1}{3}t^2\left\langle \boldsymbol{u}^2\frac{\partial u_m}{\partial x_j}\frac{\partial u_j}{\partial x_m}\right\rangle - \tfrac{1}{12}t^2\nabla\cdot\langle \boldsymbol{u}u^2\rangle + \mathrm{O}(t^3). \tag{9.17}$$

The distinction between $D(t)$ and $\beta(t)$ therefore appears in general at order t^2, the difference being cubic in $\boldsymbol{u}$ in inhomogeneous turbulence, and quartic in homogeneous turbulence.

The tensor (9.13) also contains a 'skew-diffusion' effect, somewhat analogous to the effect described by the antisymmetric part of D_{ij} (see § 6). This is here represented by the pseudo-vector

$$-\tfrac{1}{2}\beta_{imm} = D_i^{(\mathrm{a})} - \tfrac{1}{2}\langle(\boldsymbol{v}\wedge\nabla_{\boldsymbol{a}})\boldsymbol{X}^2\rangle \tag{9.18}$$

where $\boldsymbol{D}^{(\mathrm{a})}$ is given by (6.23). The corresponding contribution to $\mathscr{E}$ has the form $\nabla\wedge(\boldsymbol{s}\wedge\boldsymbol{B})$ where $s_i = \frac{1}{2}\beta_{imm}$. This effect is close to, although not identical with, the effect discovered by Rädler (1969) (and known as the Rädler effect), viz the appearance of a contribution to $\mathscr{E}$ perpendicular to $\nabla\wedge\boldsymbol{B}$ in turbulence that is statistically axisymmetric. Skew effects of this kind can appear only in turbulence that lacks reflexional symmetry.

10. First-order smoothing theory for a convected vector field

Following the procedure of § 7, we may expand $\boldsymbol{b}$ and $\boldsymbol{\mathscr{E}}$ in the form

$$\boldsymbol{b} = \Sigma \boldsymbol{b}^{(n)}(\boldsymbol{x}, t) \qquad \boldsymbol{\mathscr{E}} = \Sigma \boldsymbol{\mathscr{E}}^{(n)}(\boldsymbol{x}, t) \tag{10.1}$$

where

$$\partial \boldsymbol{b}^{(0)}/\partial t - \eta \nabla^2 \boldsymbol{b}^{(0)} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}_0) \tag{10.2}$$

$$\partial \boldsymbol{b}^{(n+1)}/\partial t - \eta \nabla^2 \boldsymbol{b}^{(n+1)} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{b}^{(n)} - \boldsymbol{\mathscr{E}}^{(n)}) \tag{10.3}$$

with $\boldsymbol{\mathscr{E}}^{(n)} = \langle \boldsymbol{u} \wedge \boldsymbol{b}^{(n)} \rangle$, and may then (in principle) solve iteratively for $\boldsymbol{b}^{(0)}, \boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)}, \ldots$. If the turbulence is homogeneous, the procedure is straightforward and, at lowest order, we obtain

$$\mathscr{E}_i^0 = \alpha_{ij}^{(0)} B_{0j} + \beta_{ijk}^{(0)} \, \partial B_{0j}/\partial x_k + \ldots \tag{10.4}$$

where

$$\alpha_{ij}^{(0)} = -\eta \int \frac{k_i k_j}{\omega^2 + \eta^2 k^4} H(\boldsymbol{k}, \omega) \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\omega \tag{10.5}$$

and

$$\beta_{ijk}^{(0)} = \eta \epsilon_{ijm} \int \frac{k^2}{\omega^2 + \eta^2 k^4} \Phi_{mk}^{(\mathrm{s})}(\boldsymbol{k}, \omega) \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\omega$$
$$+ \int \frac{\omega}{\omega^2 + \eta^2 k^4} \left(\frac{4\eta^2 k_i k_j k_k}{\omega^2 + \eta^2 k^4} + \frac{k_i \delta_{jk} - k_j \delta_{ik}}{2k^2} \right) H \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\omega. \tag{10.6}$$

The detailed derivation of these results may be found in Moffatt and Proctor (1982). They should be compared with the results (7.8)–(7.12) for the scalar field case. Note that the coefficient $\alpha_{ij}^{(0)}$ is symmetric in the suffixes (i, j) (i.e. there is no 'flux convection' effect at this level of approximation) and that it depends only on the helicity spectrum function $H(\boldsymbol{k}, \omega)$. The coefficient $\beta_{ijk}^{(0)}$, on the other hand, involves two parts, the first (antisymmetric in (i, j)) involving the symmetric part of Φ_{ij}, and the second involving again the helicity spectrum function. The former part provides a (generally anisotropic) eddy diffusion effect closely analogous to the effect represented by (7.9); and the latter provides the 'skew-diffusive' effects such as the Rädler effect referred to in the previous section.

If this approach is pursued to the next level then we obtain $\boldsymbol{\mathscr{E}}^{(1)}$ in the form

$$\mathscr{E}_i^{(1)} = \alpha_{ij}^{(1)} B_{0j} + \beta_{ijk}^{(1)} \, \partial B_{0j}/\partial x_k + \ldots \tag{10.7}$$

where $\alpha_{ij}^{(1)}$ and $\beta_{ijk}^{(1)}$ are *cubic* mean functionals of the velocity field. The main new qualitative effect that appears at this level is that, if the turbulence is non-isotropic, $\alpha_{ij}^{(1)}$ is in general non-symmetric, and there is therefore a flux convection effect represented by the velocity $\gamma_k^{(1)} \equiv -\frac{1}{2}\epsilon_{ijk}\alpha_{ij}^{(1)}$, which is again a cubic mean functional of $\boldsymbol{u}$ (cf (9.11)).

It may be anticipated, however, from § 9 that if the turbulence is *inhomogeneous*, then a stronger flux convection effect may be expected, directed down the gradient of turbulent intensity. This effect does indeed appear at the first-order level (represented by the coefficient $\alpha_{ij}^{(0)}$) and was so identified by Rädler (1968). It will be sufficient here to give a simple example, using again a two-dimensional model (cf the examples of § 7). Let $\boldsymbol{u} = (\partial\psi/\partial z, 0, -\partial\psi/\partial x)$ where (figure 12)

$$\psi = \psi_0 \sin kx \sin kz\, f(z) \tag{10.8}$$

where $f(z)$ is a slowly varying function of z (i.e. $|f'(z)/f(z)| \ll k$). Suppose that $\boldsymbol{B}_0 = (B_0, 0, 0)$ (uniform) and let us represent the perturbation field $\boldsymbol{b}^{(0)}$ in terms of

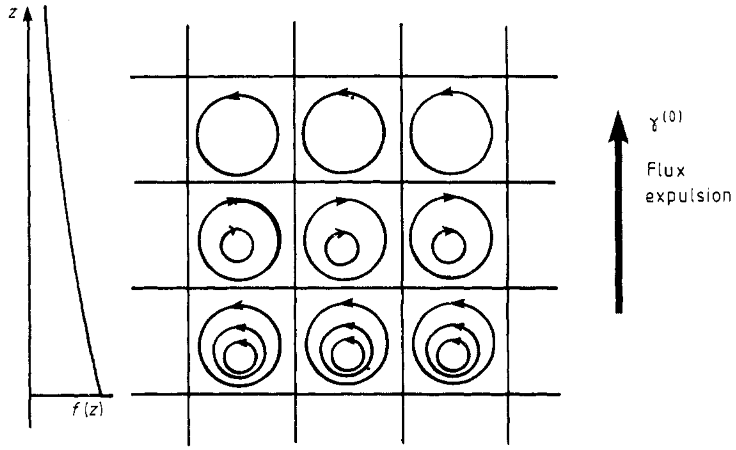

Figure 12. Streamline configuration for the weakly inhomogeneous streamfunction (10.8) with $f(z)$ as shown; the resulting flux convection velocity is given by equation (10.11).

its flux function $\chi^{(0)}(x, z)$, i.e. $\boldsymbol{b}^{(0)} = (\partial\chi^{(0)}/\partial z, 0, -\partial\chi^{(0)}/\partial x)$. Then under quasi-steady conditions,

$$\eta \nabla^2 \chi^{(0)} = -B_0\, \partial\psi/\partial x \tag{10.9}$$

with approximate solution

$$\chi^{(0)} = -\frac{B_0\psi_0}{\eta k} \cos kx \sin kz\, f(z). \tag{10.10}$$

Hence we may calculate the mean electromotive force $\mathscr{E}^{(0)} = \langle \boldsymbol{u} \wedge \boldsymbol{b}^{(0)} \rangle$. We find that

$$\mathscr{E}^{(0)} = \boldsymbol{\gamma}^{(0)} \wedge \boldsymbol{B}_0$$

where

$$\boldsymbol{\gamma}^{(0)} = -\frac{\psi_0^2}{8\eta} \nabla(f^2) = -\frac{1}{4\eta k^2} \nabla\langle \boldsymbol{u}^2 \rangle. \tag{10.11}$$

The similarity between this expression and (9.10) is noteworthy. Note that here the diffusive time scale $(\eta k^2)^{-1}$ appears rather than the Lagrangian time scale t_c.

In the case of isotropic turbulence, equations (10.5) and (10.6) may be greatly simplified:

$$\begin{aligned} \alpha_{ij}^{(0)} &= \alpha^{(0)}\delta_{ij} \qquad & \alpha^{(0)} &= -\frac{\eta}{3}\int \frac{k^2}{\omega^2 + \eta^2 k^4} H(\boldsymbol{k}, \omega)\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega \\ \beta_{ijk}^{(0)} &= \beta^{(0)}\epsilon_{ijk} \qquad & \beta^{(0)} &= \frac{2\eta}{3}\int \frac{k^2}{\omega^2 + \eta^2 k^4} E(\boldsymbol{k}, \omega)\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega. \end{aligned} \tag{10.12}$$

If moreover the energy and helicity of the turbulence are concentrated in the region of $(\boldsymbol{k}, \omega)$ space for which $\omega^2 \ll \eta^2 k^4$, then these expressions may be further simplified to the form

$$\alpha^{(0)} = -\frac{1}{3\eta}\int_0^\infty k^{-2}H(k)\,\mathrm{d}k$$
$$\beta^{(0)} = \frac{2}{3\eta}\int_0^\infty k^{-2}E(k)\,\mathrm{d}k. \tag{10.13}$$

These simplified expressions will be useful as a starting point for the following section.

11. Multiple-scale (or renormalisation group) determination of transport coefficients

In the foregoing sections, we have discussed two approaches to the determination of the transport coefficients D_{ij}, α_{ij}, $\beta_{ijk}, \ldots$. The first is a purely Lagrangian approach which gives reliable indications of the development of these coefficients for small times t from some instant at which the convected fields are non-random, but which runs into convergence problems for $t \gg t_c$ due to the development of unbounded fluctuations in the fluctuation fields or their spatial derivatives. The second is a perturbation approach whose convergence is presumably limited to values of Pe or R_m of the order of unity or less. What we need now is a technique whereby results valid for $Pe \leqslant 1$ may somehow be pushed into the interesting and practically important regime $Pe \gg 1$. One possible technique, which has been described in the scalar field context by Moffatt (1981), is that of successive averaging over a number of widely separated scales. This technique is akin to the 'renormalisation group' procedures which have achieved considerable success in the theory of phase transitions. The wide spectrum of scales present in any turbulent field suggest that successive averaging over regions of gradually increasing scale may, at the least, provide some valuable insights. We describe this procedure here for the magnetic context, the primary aim being to determine expressions for the transport coefficients $\alpha = \frac{1}{3}\alpha_{ii}$ and $\beta = \frac{1}{6}\epsilon_{ijk}\beta_{ijk}$ valid when $R_m \gg 1$.

We shall restrict attention to the case of isotropic (but not necessarily reflexionally symmetric) turbulence with energy and helicity spectra satisfying the realisability conditions

$$|H(k)| \leqslant 2kE(k) \qquad E(k) \geqslant 0 \qquad (\text{all } k). \tag{11.1}$$

We replace $E(k)$ and $H(k)$ by discrete spectra (see figure 13) concentrated around wavenumbers $k_1 = l_1^{-1}, k_2 = l_2^{-1}, \ldots, k_n = l_n^{-1}$ where

$$l_1 \ll l_2 \ll l_3 \ll \ldots \ll l_n. \tag{11.2}$$

Let us first average the exact induction equation (1.2) over the 'innermost' scale l_1. Denoting this average by $\langle \ldots \rangle_1$, let

$$\boldsymbol{U}_1 = \langle \boldsymbol{u} \rangle_1 \qquad \boldsymbol{u}_1 = \boldsymbol{u} - \boldsymbol{U}_1 \tag{11.3}$$

so that $\langle \boldsymbol{u}_1 \rangle_1 = 0$. We shall suppose that in a frame of reference moving with the local mean velocity $\boldsymbol{U}_1$, the field $\boldsymbol{u}_1$ is statistically isotropic, and we suppose further that

$$R_{m1} = \langle \boldsymbol{u}_1^2 \rangle_1^{1/2} l_1/\eta \ll 1 \tag{11.4}$$

(an inequality that is, of course, ensured provided l_1 is small enough). Then the

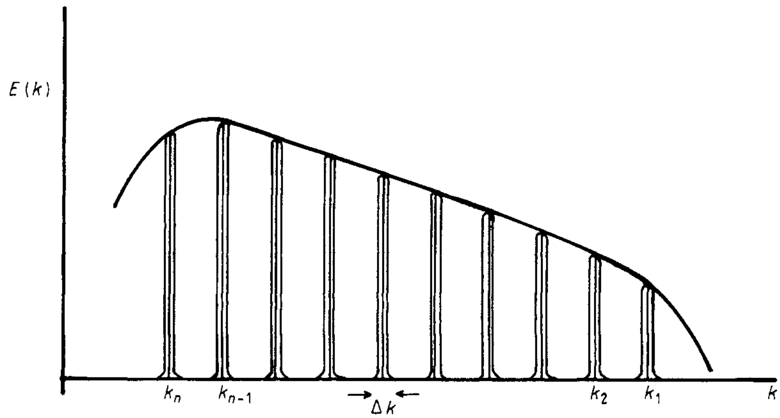

Figure 13. Replacement of the continuous spectrum $E(k)$ by a discrete spectrum centred on wavenumbers $k_1, k_2, \ldots, k_n$, with $k_1 \gg k_2 \gg \ldots$ (the scale is logarithmic).

first-order smoothing technique of § 10 is applicable, and the resulting mean-field equation for $\boldsymbol{B}_1 = \langle \boldsymbol{B} \rangle_1$ is

$$\partial \boldsymbol{B}_1/\partial t = \nabla \wedge (\boldsymbol{U}_1 \wedge \boldsymbol{B}_1) + \alpha_1 \nabla \wedge \boldsymbol{B}_1 + (\eta + \beta_1)\nabla^2 \boldsymbol{B}_1 \tag{11.5}$$

where, from (10.13),

$$\alpha_1 = -\frac{1}{3\eta}\int_{\Delta k_1} k^{-2} H(k)\,\mathrm{d}k \tag{11.6}$$

$$\beta_1 = \frac{2}{3\eta}\int_{\Delta k_1} k^{-2} E(k)\,\mathrm{d}k. \tag{11.7}$$

Note that by virtue of (11.1),

$$\beta_1 \geqslant 0 \qquad \text{and} \qquad |\alpha_1| \leqslant 2k_1\beta_1. \tag{11.8}$$

We now repeat the first-order smoothing procedure by averaging over the scale l_2, i.e. we let $\boldsymbol{U}_1 = \boldsymbol{U}_2 + \boldsymbol{u}_2$ where $\langle \boldsymbol{u}_2 \rangle_2 = 0$, and we assume that $\boldsymbol{u}_2$ is statistically isotropic in a frame of reference moving with the local velocity $\boldsymbol{U}_2$ and that

$$R_{m2} \equiv \langle \boldsymbol{u}_2^2 \rangle_2^{1/2} l_2 (\eta + \beta_1)^{-1} \ll 1. \tag{11.9}$$

Note that we now get a little help in satisfying this inequality through the appearance of the 'renormalised' diffusivity $\eta + \beta_1$. Applying first-order smoothing to (11.5) leads to the mean-field equation

$$\partial \boldsymbol{B}_2/\partial t = \nabla \wedge (\boldsymbol{U}_2 \wedge \boldsymbol{B}_2) + \alpha_2 \nabla \wedge \boldsymbol{B}_2 + (\eta + \beta_2)\nabla^2 \boldsymbol{B}_2 \tag{11.10}$$

where now (with $r = 1$)

$$\alpha_{r+1} = \alpha_r - \frac{1}{3}\int_{\Delta k_{r+1}} \frac{2\alpha_r E(k) + (\eta + \beta_r) H(k)}{k^2(\eta + \beta_r)^2 - \alpha_r^2}\,\mathrm{d}k \tag{11.11}$$

$$\beta_{r+1} = \beta_r + \frac{1}{3}\int_{\Delta k_{r+1}} \frac{2(\eta + \beta_r) E(k) + \alpha_r k^{-2} H(k)}{k^2(\eta + \beta_r)^2 - \alpha_r^2}\,\mathrm{d}k. \tag{11.12}$$

The important point here is that the *structure* of (11.10) is identical with that of (11.5); only the suffix 1 has been replaced by the suffix 2. It is evident therefore that we may (in principle) repeat the procedure any number of times, always obtaining a mean-field equation with the same structure. Moreover equations (11.11) and (11.12) remain valid for $r = 2, 3, 4, \ldots$.

We now take the bold step (which cannot be justified with any degree of rigour: the reasonableness of the end product must justify the means!) of returning to a continuous spectrum. Then it is clear that the difference equations (11.11) and (11.12) must be replaced by differential equations†

$$\frac{\mathrm{d}\alpha}{\mathrm{d}k} = \frac{1}{3}\frac{2\alpha E(k) + (\eta+\beta)H(k)}{k^2(\eta+\beta)^2 - \alpha^2} \tag{11.13}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}k} = -\frac{1}{3}\frac{2(\eta+\beta)E(k) + \alpha k^{-2}H(k)}{k^2(\eta+\beta)^2 - \alpha^2} \tag{11.14}$$

and $\alpha(k)$ and $\beta(k)$ must now be interpreted as the values of α and β associated with all scales $\leqslant k^{-1}$. For given $E(k)$ and $H(k)$, we may integrate these equations inwards from $k = \infty$, with initial conditions $\alpha(\infty) = 0$, $\beta(\infty) = 0$. We may easily see what must happen. Note first that, from (11.13) and (11.14),

$$\frac{\mathrm{d}}{\mathrm{d}k}\{\alpha(\eta+\beta)\} = \frac{1}{3k^2}H(k)$$

so that

$$\alpha(\eta+\beta) = -\frac{1}{3}\int_k^\infty \frac{H(k)}{k^2}\,\mathrm{d}k = -h(k). \tag{11.15}$$

Hence we may eliminate $(\eta+\beta)$ from (11.13), obtaining

$$\frac{\mathrm{d}}{\mathrm{d}k}\alpha^2 = \frac{2\alpha^2(2\alpha^2 E - hH)}{3(kh-\alpha^2)(kh+\alpha^2)}. \tag{11.16}$$

Figure 14 shows what happens if we integrate this inwards from $k = \infty$: α^2 increases until the value of k ($=k_m$, say) at which $\alpha^2 = kH/2E$; for $k < k_m$,

$$|kh| > \alpha^2 > hH/2E \tag{11.17}$$

and α^2 must decrease to zero as $k \to 0$. This decrease is in fact ultimately linear in k, so that $\alpha \sim k^{1/2}$, $\beta + \eta \sim k^{-1/2}$ (assuming $h(0) \neq 0$) as $k \to 0$. This unbounded behaviour of β as $k \to 0$ is associated with asymptotic vanishing of the denominator in (11.14); the condition

$$k^2(\eta+\beta)^2 \leqslant \alpha^2 \tag{11.18}$$

may be recognised as the condition for dynamo action, i.e. unbounded growth of magnetic perturbations of sufficiently large scale. The condition (11.18) is never in fact attained for $k > 0$ according to (11.13) and (11.14), but

$$\frac{k^2(\eta+\beta)^2}{\alpha^2} \to 1 \qquad \text{as} \qquad k \to 0 \tag{11.19}$$

and the associated magnetic fluctuation increase (without limit) as the length scale increases.

† Similar equations have been obtained by S I Vainshtein (private communication).

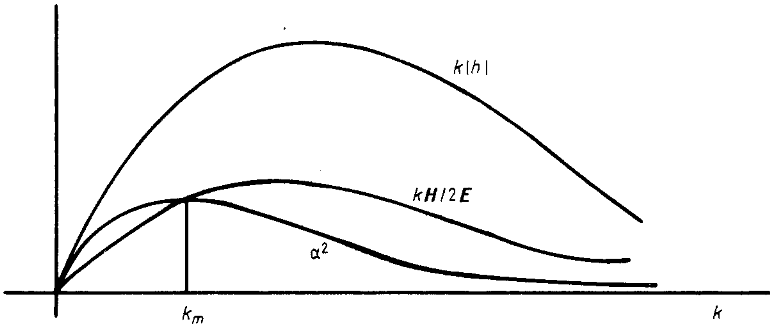

Figure 14. Form of the function $\alpha^2(k)$ as determined by integration of equation (11.16).

In any finite geometry (e.g. a stellar convection zone) the turbulence is homogeneous only on scales small compared with the external geometric scale (e.g. the stellar radius or the depth of the convection zone). Equations (11.13) and (11.14) can then be used to represent (and parametrise) the effects of turbulence on scales smaller than some scale $l_0 = k_0^{-1}$ satisfying this criterion. Ingredients of the motion on scales larger than l_0 can be described explicitly only in large-scale numerical computations such as those of Gilman and Miller (1981).

The importance of equations such as (11.13) and (11.14) is that, no matter how small η may be, they lead to values of α and β of the order of $\langle \boldsymbol{u}^2\rangle^{1/2}$ and $l_0\langle \boldsymbol{u}^2\rangle^{1/2}$, respectively, both independent of η, although ultimately the process by which α and β are 'generated' is diffusive in character, the first step in the averaging process requiring the strong diffusion condition (11.4).

12. Development of intermittency in convected fields

As we have remarked in § 6, expansions of the type (6.9) and (8.3) would appear to be reliable only if molecular diffusion effects acting on the convected field are operative before a fluid particle has moved out of a region in which the convected-field gradient is appreciably uniform. If this condition is not satisfied, then the series (6.9) and (8.3) will converge very slowly, or may even diverge, and the mean-field approach will not then be viable. In this final section, we shall analyse some of the effects that may arise.

We assume an initial θ field of the form

$$\theta(\boldsymbol{x}, 0) = \theta_0 \cos K_0 x \tag{12.1}$$

having a single length scale $L_0 \sim K_0^{-1}$, and we suppose that this field is subject to convection by the steady velocity field

$$\boldsymbol{u} = (u(y), 0, 0) \tag{12.2}$$

where $u(y)$ is a stationary random function of y with zero mean, $\langle u\rangle = 0$ (here $\langle \dots \rangle$ will indicate an average over y). This is clearly a highly idealised model problem; but in choosing a steady velocity field, we focus attention on the effects of *persistence* of a turbulent velocity field, in which case the escape of fluid particles from regions of uniform field gradient will occur most rapidly. Note that, in the two-dimensional situation considered here, the vector potential $(0, 0, A)$ of a magnetic field $\boldsymbol{B}$ satisfies the same equation as θ, so that results obtained for the scalar field problem may be immediately carried over to the vector field problem.

The root mean square vorticity

$$\omega_0 = \langle (\mathrm{d}u/\mathrm{d}y)^2 \rangle^{1/2} \tag{12.3}$$

plays an important part in this problem, and also the associated shear time scale, $t_s = \omega_0^{-1}$. The relevant Péclet number is

$$Pe = \omega_0 / K_0^2 \eta \tag{12.4}$$

and we shall suppose that $Pe \gg 1$. The isoscalar surfaces θ = constant are then strongly distorted before diffusion becomes effective (see figure 15). We aim to follow the

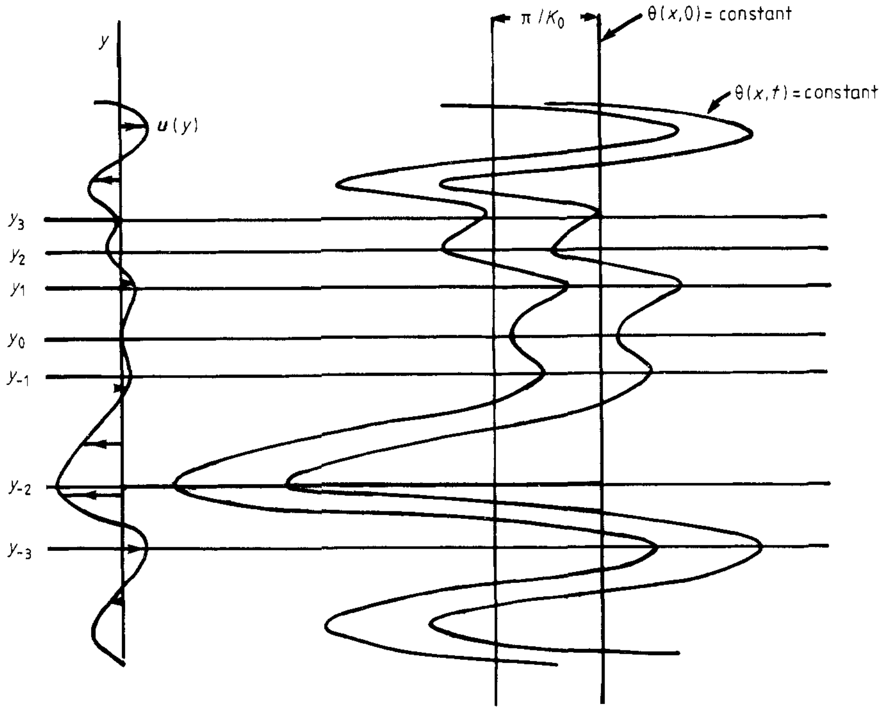

Figure 15. Distortion of the scalar field $\theta(\boldsymbol{x}, 0) = \theta_0 \cos K_0 x$ by the random velocity field $\boldsymbol{u} = (u(y), 0, 0)$. Note how high values of $\partial\theta/\partial y$ build up, except in the neighbourhood of the layers $y = y_r$ $(r = 0, \pm 1, \pm 2, \dots)$ where $u' = 0$. Diffusion eliminates the θ field except in these neighbourhoods.

detailed development of the field $\theta(x, y, t)$ during this initial process of distortion, and during the subsequent decay.

The relevant equation is

$$\frac{\partial \theta}{\partial t} + u(y) \frac{\partial \theta}{\partial x} = \eta \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \theta \tag{12.5}$$

and the solution evidently has the form

$$\theta(x, y, t) = \mathrm{Re}\{\theta_0 f(y, t) \exp(\mathrm{i} K_0 x)\} \tag{12.6}$$

where

$$\frac{\partial f}{\partial t}+\mathrm{i}K_0u(y)f=\eta\left(\frac{\partial^2}{\partial y^2}-K_0^2\right)f \tag{12.7}$$

and $f(y, 0)=1$. The initial phase of development is given by simply putting $\eta=0$, so that $f(y, t)=\exp(-\mathrm{i}K_0u(y)t)$, and correspondingly

$$\theta(x, y, t)=\mathrm{Re}\{\theta_0\exp[\mathrm{i}K_0(x-u(y)t)]\} \tag{12.8}$$

the Lagrangian solution (cf (6.2)). To improve on this, let

$$f(y, t)=\exp(-\mathrm{i}K_0u(y)t)F(y, t) \tag{12.9}$$

where we expect that $F(y, t)$ will be a slowly varying function of t (at least for some time interval). Substitution in (12.7) leads to the equation

$$\frac{\partial F}{\partial t}=\eta\left(-K_0^2t^2u'^2F-\mathrm{i}K_0tu''F+\frac{\partial^2 F}{\partial y^2}-2\mathrm{i}K_0tu'\frac{\partial F}{\partial y}-K_0^2F\right). \tag{12.10}$$

The first term on the right-hand side increases most rapidly with t, and if we retain only this term we may integrate the equation to obtain

$$F(y, t)=\exp(-\tfrac{1}{3}\eta K_0^2u'^2t^3) \tag{12.11}$$

indicating decay of the θ fluctuations on a time scale

$$t_1=(\eta K_0^2\omega_0^2)^{-1/3}=Pe^{1/3}t_s. \tag{12.12}$$

For $t \ll t_1$, the Lagrangian solution (12.8) is a good approximation to the exact solution.

The decay implied by (12.11) obviously occurs only at values of y where $u' \neq 0$. For $t \gg t_1$, the function $F(y, t)$ becomes very sharply peaked in the neighbourhood of points y_r $(r=0, \pm 1, \pm 2, \ldots)$ where $u'=0$. Equally, the temperature fluctuations become sharply concentrated in these neighbourhoods (figure 16). A more refined

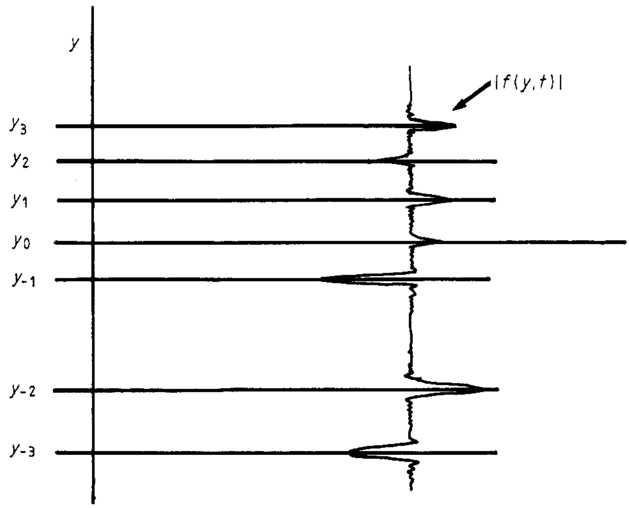

Figure 16. Development of intermittency in the amplitude $|f(y, t)|$ of the temperature field, for $t_s \ll t \ll Pe^{1/3}t_s$.

analysis of the behaviour in these neighbourhoods shows that ultimately the term $\eta\, \partial^2 F/\partial y^2$ in (12.10) limits the decrease of scale in the y direction; this occurs when $t \sim (K_0 L)^{1/2} Pe^{2/3}$ where L is a typical scale of variation of the shear function $u'(y)$. The solution (12.11) then requires modification.

Let us now return to the Lagrangian solution (12.8), valid for $t \ll Pe^{1/3} t_s$. The mean field $\Theta(x, t) = \langle\theta\rangle$ is given by

$$\Theta = \lim_{Y\to\infty} \frac{1}{2Y} \int_{-Y}^{Y} \theta_0 \cos K_0(x - u(y)t)\, dy. \tag{12.13}$$

For $t \gg t_s$, the dominant contribution to the integral comes from the neighbourhoods of the points y_r of stationary phase where $u'(y) = 0$; and evaluation of (12.13) by the method of stationary phase yields

$$\Theta(x, t) = \lim_{Y\to\infty} \frac{\theta_0}{2Y} \sum_r \left(\frac{2\pi}{|u_r''|t}\right)^{1/2} \cos\left[K_0(x - u_r t) \pm \pi/4\right] \tag{12.14}$$

where the + or − is chosen according as u_r'' is greater or less than 0. If N is the mean number of extrema of $u(y)$ per unit length in the y direction, then (12.14) may equally be written

$$\Theta = N\theta_0 \left(\frac{2\pi}{t}\right)^{1/2} \overline{|u_r''|^{-1/2} \cos\left[K_0(x - u_r t) \pm \pi/4\right]} \tag{12.15}$$

where the overbar represents an average with respect to the suffix r labelling the extrema. (If $u(y)$ is statistically invariant with respect to change of sign ($u \to -u$), then (12.15) simplifies to

$$\Theta = N\theta_0 \left(\frac{2\pi}{t}\right)^{1/2} \overline{|u_r''|^{-1/2} \cos\left(K_0 u_r t \mp \pi/4\right)} \cos K_0 x \tag{12.16}$$

but otherwise there may be a phase shift $\delta(t)$ of Θ in the x direction.) The result (12.15) provides an explicit representation of Θ in terms of the Eulerian statistics of the velocity field $u(y)$, and is valid for $t_s \ll t \ll Pe^{1/3} t_s$; but it is clear that these statistics are not representable in terms of low-order spectra of u (e.g. quadratic, cubic or even quartic); indeed, any attempt to express the mean quantity appearing in (12.15) in terms of spectra of $u(y)$ would necessarily involve spectra of all orders, and such a representation might not even exist. It is moreover apparent that $\Theta(x, t)$ as given by (12.15) certainly does *not* satisfy a diffusion equation, and that the flux $\langle u\theta\rangle$ does not therefore bear a simple relationship of the form (6.9) with the mean-field gradient $\partial\Theta/\partial x$. We end this discourse on this warning note: that although the mean-field approach has great appeal and has a number of outstanding achievements to its credit, yet it must be used with caution, particularly when there is a spikiness, or strong intermittency, in the distribution of the convected field, as in the example treated here.

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