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SIMPLE TOPOLOGICAL ASPECTS OF TURBULENT VORTICITY DYNAMICS

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The process by which turbulent vorticity is intensified is discussed, in relation to the dimensionality of the subspace in which vorticity is concentrated. For a single, randomly twisted vortex filament, no vortex stretching can occur and the enstrophy remains bounded for all time. If the vorticity is distributed on a randomly convoluted surface, the total helicity of the flow being non-zero, then the enstrophy can, and in general will, increase without limit due to local instabilities of Kelvin-Helmholtz type, leading to spiral singularities of the vortex sheet. The relation between the structure of these spiral singularities and the spectral exponent in the inertial range of wavenumbers is demonstrated.

INTRODUCTION

One of the essential features of the process of energy transfer in turbulent flow from large to small scales is the associated tendency for enstrophy (i.e. mean-square vorticity) to increase. This increase is ultimately balanced (in a statistically steady state) by viscous dissipation of enstrophy; but it is legitimate to enquire what happens in the inviscid limit $\nu \to 0$, and in particular to investigate the structure of the vorticity field that develops in this limit, as the enstrophy becomes unbounded. This structure (in physical space) presumably has a bearing on the energy spectrum that is established on scales at which viscous effects are negligible.

The equation for mean-square vorticity in homogeneous turbulence is

$$\frac{d}{dt}\,\tfrac{1}{2}\langle\boldsymbol{\omega}^2\rangle = \langle\boldsymbol{\omega}.(\boldsymbol{\omega}.\nabla)\mathbf{u}\rangle - \nu\langle(\nabla\wedge\boldsymbol{\omega})^2\rangle\,, \qquad (1)$$

where $\boldsymbol{\omega} = \text{curl}\,\mathbf{u}$. The most naïve closure enabling integration of this equation is

$$\langle\boldsymbol{\omega}.(\boldsymbol{\omega}.\nabla)\mathbf{u}\rangle = A\langle\boldsymbol{\omega}^2\rangle^{3/2}\,, \qquad (2)$$

where A is a positive dimensionless constant;with $\nu = 0$, (1) then integrates to give

$$\langle\boldsymbol{\omega}^2\rangle = \langle\boldsymbol{\omega}^2\rangle_0\,(1 - t/t_0)^{-2}\,, \qquad (3)$$

where $\langle\boldsymbol{\omega}^2\rangle_0$ represents the enstrophy at $t = 0$, and $t_0 = A^{-1}\langle\boldsymbol{\omega}^2\rangle_0^{-\frac{1}{2}}$. This solution 'blows up' at time $t = t_0$, a behaviour that is probably faithful to the real dynamics of turbulence (see, for example, the recent discussion of Brachet et al. 1983) even although the closure (2) is known to overestimate the vorticity intensification process.

The same explosion of enstrophy at finite time occurs for a slightly more sophisticated closure, in which we first decompose the vorticity field into two ingredients, $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$, the idea being that each ingredient provides the rate-of-strain field that is responsible for the intensification of the other ingredient. The inviscid enstrophy equations, with a closure analogous to (2), which encapsulate this process of mutual interaction, are

$$\left.\begin{aligned} \frac{d}{dt}\tfrac{1}{2}\langle\boldsymbol{\omega}_1{}^2\rangle &= A\langle\boldsymbol{\omega}_1{}^2\rangle\langle\boldsymbol{\omega}_2{}^2\rangle^{\frac{1}{2}} \\ \frac{d}{dt}\tfrac{1}{2}\langle\boldsymbol{\omega}_2{}^2\rangle &= A\langle\boldsymbol{\omega}_2{}^2\rangle\langle\boldsymbol{\omega}_1{}^2\rangle^{\frac{1}{2}} \end{aligned}\right\}. \tag{4}$$

These equations have a first integral

$$\langle\boldsymbol{\omega}_1{}^2\rangle^{\frac{1}{2}} - \langle\boldsymbol{\omega}_2{}^2\rangle^{\frac{1}{2}} = -C, \tag{5}$$

and the solution is then given by

$$\langle\boldsymbol{\omega}_1{}^2\rangle = C^2\left[\left(1 + \frac{C}{\langle\boldsymbol{\omega}_1{}^2\rangle_0^{\frac{1}{2}}}\right)\exp(-CAt) - 1\right]^{-2}, \tag{6}$$

which (like (3)) blows up at finite time (provided $A > 0$).

This behaviour is suggestive in relation to the toroidal vortex sheet model which will be discussed below.

RANDOMLY TWISTED VORTEX FILAMENT

Consider first the simplest possible three-dimensional example of a random vorticity field, i.e. a single randomly twisted vortex filament of strength Γ. Suppose (for simplicity) that the cross-section of the vortex filament is circular, with radius ε, which is everywhere small compared with the local radius of curvature of the filament. As shown by Da Rios (1906)[1] and Betchov (1965), the motion of the vortex filament is determined by the curvature $\kappa(s)$ and torsion $\tau(s)$, where s is a parameter along the filament. In this approximation, the velocity of the vortex filament is given by

$$\mathbf{V}(s) \sim \frac{\Gamma}{2\pi}\kappa(s)\left(\ln\left(\frac{1}{\kappa(s)\varepsilon}\right) + \text{cst.}\right)\hat{\mathbf{b}}(s), \tag{7}$$

where $\hat{\mathbf{b}}$ is the unit binormal to the filament. Since $\hat{\mathbf{s}}.\nabla\mathbf{V} = 0$ ($\hat{\mathbf{s}}$ being the unit tangent vector) the filament behaves like an inextensible thread. If it is closed on itself, and its length is L, then the associated enstrophy is

$$\int\boldsymbol{\omega}^2 dV = \Gamma^2 L^2/V, \tag{8}$$

where $V = \pi\varepsilon^2 L$ is the (constant) volume occupied by the vortex filament. Clearly this enstrophy is also constant.

Consider now the further integral invariants associated with this type of motion. Firstly, the kinetic energy is a conserved quantity. We may estimate this as follows. For points near the vortex filament, the velocity field is approximately that associated with an infinite line vortex, i.e.

$$|\mathbf{u}(\mathbf{x})| \sim \frac{\Gamma}{2\pi r} \qquad \text{for} \quad |r\kappa(s)| \ll 1 \tag{9}$$

where r is the distance from the vortex. The corresponding contribution to the kinetic energy of the flow is

$$T_1 \sim \rho\Gamma^2 L\left\langle\ln\left(\frac{1}{\varepsilon|\kappa(s)|}\right)\right\rangle, \tag{10}$$

where the average is over the length of the vortex filament. There is a second contribution (of order $\rho\Gamma^2 L$) to the kinetic energy of the flow from points that are more distant from the filament, but it is the contribution (10) which dominates. Since L is constant, it then follows that $\langle \ln|\epsilon\kappa(s)|^{-1}\rangle$ is constant also; invariance of enstrophy implies that, in this sense, energy cannot be transferred to smaller scales, and progressive 'crinkling' of the vortex filament is impossible.

There is one further invariant associated with the flow, and this is the helicity

$$\mathcal{H} = \int \underset{\sim}{u}\cdot\underset{\sim}{\omega}\,dV, \tag{11}$$

which, in general, has a topological interpretation associated with the degree of linkage of vortex lines (Moreau 1961, Moffatt 1969, Arnol'd 1974). This invariant takes a rather interesting form when the vorticity is concentrated in a single closed filament. Using the Biot-Savart law to express $\underset{\sim}{u}$ in terms of $\underset{\sim}{\omega}$, we have the alternative expression

$$\mathcal{H} = \frac{1}{4\pi}\iint \frac{\underset{\sim}{R}\cdot\underset{\sim}{\omega}(\underset{\sim}{x})\wedge\underset{\sim}{\omega}(\underset{\sim}{x}')}{R^3}\,dVdV', \tag{12}$$

where $\underset{\sim}{R} = \underset{\sim}{x}-\underset{\sim}{x}'$. In going to the limit of a vortex filament of vanishingly small cross-section, this becomes

$$\mathcal{H} \sim \Gamma^2\left(I + \frac{1}{2\pi}\int\tau(s)\,ds\right), \tag{13}$$

where

$$I = \frac{1}{4\pi}\iint \frac{\underset{\sim}{R}\cdot d\underset{\sim}{x}\wedge d\underset{\sim}{x}'}{R^3} \tag{14}$$

(the Gauss integral). The second contribution in (13), involving the torsion $\tau(s)$, comes from pairs of points $\underset{\sim}{x},\underset{\sim}{x}'$ for which $R = O(\epsilon)$. The invariant expression (13) was obtained in a purely topological context by Calugareanu (1959). (Note that the integral I does converge, despite the apparent singularity in the integrand at $R = 0$.) Invariance of $\mathcal{H}$ thus (in a sense) places a constraint on the development of torsion in the filament, just as invariance of energy places a constraint on the development of curvature.

A RANDOMLY CONVOLUTED VORTEX SHEET

The situation is rather different if the vorticity field is confined to a neighbourhood of a closed surface S. Suppose first that S is a torus (the simplest possibility if the vortex lines are not to be closed). We use the symbol S also for the area of S; the volume of its 'ϵ-neighbourhood' is then

$$V_\epsilon = \epsilon S, \tag{15}$$

and we suppose that $\underset{\sim}{\omega}$ is non-zero only in V_ϵ. The strength of the vortex sheet is then

$$\Delta U \sim \epsilon|\underset{\sim}{\omega}|, \tag{16}$$

and this represents the jump in fluid velocity as we cross from inside to outside the torus. Let V_{in} represent the volume inside the torus. Then in the inviscid dynamical evolution of the vorticity field, both V_{in} and V_ϵ are constant. The kinetic energy associated with the flow is clearly

$$T \sim \rho(\Delta U)^2 V_{in}, \tag{17}$$

and so ΔU is also conserved (in order of magnitude). The enstrophy is given by

$$\Omega = \int \underset{\sim}{\omega}^2 dV \sim \frac{(\Delta U)^2}{\varepsilon^2} V_\varepsilon = (\Delta U)^2 S^2 / V_\varepsilon . \tag{18}$$

Now, the constraint T = cst. places no restriction on Ω which can increase through increase of S. An enstrophy explosion ($\Omega \to \infty$) can in principle occur if and only if $S \to \infty$.

What can occur under the single constraint of energy conservation is not of course necessarily the same as what does occur under the full constraints implied by the Euler equations. The above argument does however suggest that the toroidal vortex sheet is a better candidate than the closed vortex line as the prototype structure which may be associated with the process of enstrophy explosion in turbulent flow. We pursue this suggestion in the following sections.

KELVIN-HELMHOLZ INSTABILITY AND ENSTROPHY INCREASE

A plane vortex sheet, as is well-known, is absolutely unstable to sinusoidal disturbances, the growth rate of weak disturbances of wave-number k being

$$\sigma \sim k \, \Delta U . \tag{19}$$

This type of dispersion relationship, together with non-linear interaction mechanisms, leads to the appearance after a finite time of weak singularities (e.g. discontinuities of curvature) in the surface sheet geometry (Moore 1979, Meiron et al. 1982).

Experimental observations (e.g. Thorpe 1971) indicate that an initially sinusoidal perturbation of a plane vortex sheet leads to an overturning process (as indicated in figure 1) and hence quite rapidly to the formation of spiral singularities distributed along the sheet. A vortex concentration then appears at the 'eye' of each spiral.

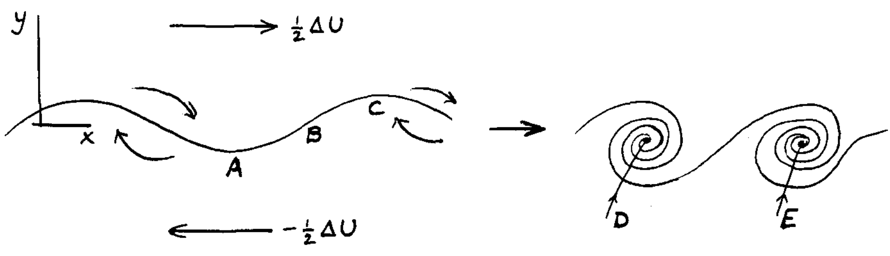

Figure 1
Development of spiral singularities on a vortex sheet due to Kelvin-Helmholtz instability

Clearly the area of the sheet increases greatly during the process of spiral formation. In a purely two-dimensional situation, the enstrophy is conserved, despite this increase of area. To understand this, we should take account of the initial sheet thickness, say ε. As time progresses, the sheet gets thinner over the section ABC of figure 1, where it is being most stretched. Here therefore the sheet strength decreases and this compensates the build-up of vortex concentrations near the spiral singularities D and E.

Suppose now that, with the coordinate system of figure 1, there is a superposed velocity component w in the z-direction which is also discontinuous across the vortex sheet. This velocity component is simply convected by the flow in the (x,y) plane and so the jump Δw in w across the sheet remains constant. The associated vorticity component in the (x,y) plane parallel to the sheet then increases at points where the sheet thickness ε decreases; and the enstrophy then increases also as the sheet area increases. This behaviour, although conceivable, is somewhat artificial, because the most unstable disturbance of a plane vortex sheet is always oriented so that its wavevector $\underset{\sim}{k}$ is parallel to the discontinuity of the vector tangential velocity $\Delta\underset{\sim}{U}$ across the sheet. For this most unstable disturbance, the above enstrophy increase cannot occur. The situation is however different when we consider instabilities to which a vortex sheet wrapped on a torus may be subject.

KELVIN-HELMHOLZ INSTABILITY OF A TOROIDAL VORTEX SHEET

We now return to the toroidal configuration considered above and we suppose that the vortex lines cover the set of toruses nested in the neighbourhood V_ε of S, i.e. each vortex line follows a helical path round a torus, with irrational pitch. If θ and ϕ are angular coordinates on the torus, then we may anticipate unstable disturbances proportional to $\exp[i(m\theta + n\phi)]$ where m and n are integers - the wave crests of such disturbances are helices which do close on the toroidal surface. Hence the wave-crests cannot now be oriented parallel to the vortex lines in V_ε; and by the argument of the previous paragraph, any instability which leads to explosive growth of the toroidal surface area S must at the same time lead to explosive growth of the total enstrophy Ω of the flow.

The most unstable mode (m,n) will, as we have seen, lead to intensification of the component of vorticity in the (m,n) direction on toroidal surfaces. The corresponding velocity jump is progressively oriented in the perpendicular direction (-n,m), and a secondary mode of instability with wave-vector in this direction then appears likely. The vorticity field $\underset{\sim}{\omega}$ may be decomposed into two fields $\underset{\sim}{\omega} = \underset{\sim}{\omega}_1 + \underset{\sim}{\omega}_2$ with ω_2 parallel to (m,n) and $\underset{\sim}{\omega}_1$ parallel to (-n,m). The primary instability associated with the field $\underset{\sim}{\omega}_1$ leads to intensification of $\underset{\sim}{\omega}_2$; and the secondary instability associated with $\underset{\sim}{\omega}_2$ leads to intensification of $\underset{\sim}{\omega}_1$. Equations (4) provide the simplest idealisation of this mutual intensification process.

It is an essential feature of this process that the vortex lines are not closed, but do actually cover toroidal surfaces. The helicity (see equation (11)) associated with such a flow is undoubtedly nonzero, and it might be thought that this vorticity distribution is therefore untypical of those examples of turbulent flow (e.g. grid turbulence) for which $\langle\underset{\sim}{u}.\underset{\sim}{\omega}\rangle = 0$. However, in any turbulent flow, $\underset{\sim}{u}.\underset{\sim}{\omega}$ is never identically zero, and a plausible model of the flow may be provided by a random superposition of fundamental toroidal units of the above kind, some with positive helicity and some with negative helicity, each unit subject to its own intrinsic instabilities and at the same time subject to convection and distortion by the velocity fields associated with the other units. This is reminiscent of attempts to portray turbulence as a random superposition of Hill's spherical vortices; but the non-zero helicity of the toroidal vortices is here an essential concomitant of the process of rapid increase of enstrophy; the units may moreover be linked and tangled with each other in a manner not available to Hill's vortices.

It is of course well known that helicity plays a crucial role in magnetohydrodynamic turbulence, in that, when the mean helicity is non-zero, the medium is subject to magnetic instabilities on all scales large compared with the energy-containing scales of the turbulence (for full accounts of this important problem, see Moffatt 1978, Krause & Rädler 1980). It is also known that helicity can have a strong influence on the effective turbulent diffusivity of a convected passive scalar field (Kraichnan 1976, Drummond, Duane & Horgan 1983). The assertion made here is that helicity may be of profound importance also in the non-linear dynamics of turbulence; and that even when the mean helicity is zero, local helicity fluctuations associated with toroidal vorticity structures (or possibly with structures that are topologically much more complex) are associated in an essential way with the fundamental process of increase of enstrophy. **A similar concept is developed in the recent paper of Levich & Tsinober (1983).**

SPECTRAL ANALYSIS OF SPIRAL SINGULARITIES

A random superposition of plane vortex sheets (figure 2) provides a primitive model of turbulence (Townsend 1951) for which the energy spectrum is

$$E(k) \propto k^{-2} f(k\ell_v) \qquad (20)$$

where ℓ_v is the mean thickness of the sheets (controlled by viscosity) and $f(x)$ is a function representing an exponential-type viscous cut-off ($f(0) = 1$). The k^{-2} factor in (20) is associated with the fact that any straight line cuts a finite number of vortex sheets per unit length; the velocity field on this transversal then has a finite number of discontinuities per unit length, and the k^{-2} factor in (20) is a simple consequence of this fact.

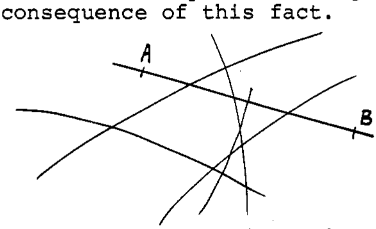

Figure 2
Random vortex sheets; the velocity field on a transversal has a finite number of discontinuities on any section AB, leading to a k^{-2} spectral power law.

The slower fall-off represented by the $k^{-5/3}$-Kolmogorov spectrum suggests that, at least on scales on which viscous effects are negligible, the vorticity structure must in some sense be worse than that associated with random vortex sheets. This worsening is conveniently provided by the spiral singularities which develop as a result of the Kelvin-Helmholz instability. If these occur more or less wherever the vortex sheets occur, then a straight line transversal will occasionally pass very near a spiral singularity. Let us analyse the effect that this will have on the resulting energy spectrum

Let x be a coordinate along the transversal, and let u(x) be a velocity component perpendicular to the transversal. Suppose this transversal passes near a spiral singularity situated on the plane x = O (figure 3).

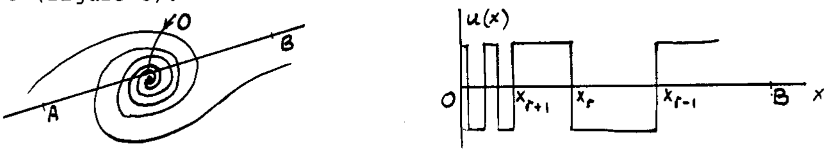

Figure 3
Transversal passing near a spiral singularity; there is then a large number of discontinuities of transverse velocity near the point O.

Then u(x) has discontinuities at a sequence of points x_o, x_1,...x_N, say, where N is large. The Fourier **cosine** coefficients of the function u(x) in the range $0 < x < \pi$ are given by

$$a_n = \frac{2}{\pi} \int_0^{\pi} u(x) \cos nx \, dx \tag{21}$$

and the behaviour for large n is given by

$$a_n \sim \frac{2}{\pi n} \sum_{r=1}^{N} \Big[u(x) \sin nx\Big]_{x_r}^{x_{r+1}} + o(\tfrac{1}{n}) . \tag{22}$$

If we now suppose n fixed, and consider the limit $N \to \infty$, we see that we get a significant contribution to the series (22) only from those terms for which $nx_r = O(1)$ or greater. Let M be the number of such terms. Due to the factor $\sin nx_r$, which in effect takes random values between -1 and +1, (22) gives in order of magnitude

$$a_n \sim \frac{2}{\pi n} M^{\frac{1}{2}} \Delta u \tag{23}$$

where Δu is an average value of the jump in $|u(x)|$ across the sheet.

Suppose, for example, that the jumps occur at $x_r = r^{-s}$ for some s>O. Then M is given by

$$n/M^s = O(1), \qquad \text{i.e. } M \sim n^{1/s}, \tag{24}$$

and so, from (23),

$$a_n \sim \frac{2}{\pi} n^{(-1+1/2s)} \Delta u . \tag{25}$$

The corresponding spectral power law is

$$E(k) \sim k^{-\lambda} \qquad \text{where } \lambda = 2 - \frac{1}{s} . \tag{26}$$

The Kolmogorov exponent $\lambda = 5/3$ corresponds to s = 3, i.e. to velocity jumps at $x = 2^{-3}, 3^{-3}, 4^{-3}$, etc.

This type of analysis is incomplete in that clearly the density and probability distribution of spiral singularities should also play a part in determining the resulting energy spectrum. It is nevertheless illuminating, in that it shows how the structure of the typical spiral singularity can have a determining influence on the spectral power law in the inertial range. Conversely, if this power law is known, from independent considerations, then this gives information about the structure and distribution of spiral singularities, assuming that these are indeed the dominant physical feature of turbulent flows.

FOOTNOTE

[1] I am indebted to Dr. M. Germano who drew my attention to this early, and little known, paper.

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