CHAPTER 7

ASPECTS OF DYNAMO THEORY

H.K. Moffatt
Department of Applied Mathematics and Theoretical Physics
University of Cambridge
England CB3 9EW

The elements of dynamo theory are discussed, with particular attention to the particular problems that arise when, as in the solar context, the magnetic diffusivity is very small. The growth of the dipole moment of a localised current system is essentially diffusive in character; in the limit of vanishing diffusivity, the spatial structure of any dynamo must become increasingly complex; this is the 'fast dynamo' limit.

When convective eddies are persistent, the phenomena of flux expulsion and topological pumping play an important part in the dynamo process. These effects appear in the 'mean-field' theory of the turbulent dynamo via an 'effective velocity' of transport of the mean magnetic field <u>relative</u> to the fluid.

These effects are all discussed in the context of the solar dynamo, regarded as a dynamo of $\alpha\omega$-type, with magnetic buoyancy providing an equilibration mechanism.

7.1 THE HOMOPOLAR DISC DYNAMO

Some peculiarities of dynamo theory are very well illustrated by the prototype example of self-exciting dynamo action, viz. the homopolar disc dynamo sketched in figure 7.1. The conducting disc rotates about its axis under the action of an applied torque G. A wire, twisted about the axis in the manner shown, makes sliding contact with the disc at A, and with the axis at B, and carries a current $I(t)$. The magnetic field $\underline{B}$ associated with this current has a flux $\Phi = MI$ across the disc, where M is the mutual inductance between the wire and the rim of the disc. The rotation of the disc in the presence of this flux provides a radial electromotive force $\frac{\Omega}{2\pi}\Phi = \frac{\Omega}{2\pi}MI$ which drives the current I. On this simplistic description, the equation for I is

$$L\frac{dI}{dt} + RI = \frac{M}{2\pi}\Omega I \,, \tag{7.1}$$

where R is the total resistance of the circuit and L its self-inductance.

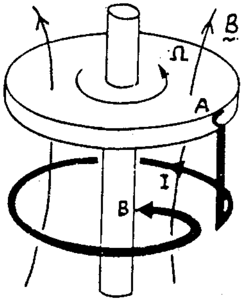

Figure 7.1

Suppose that Ω is maintained constant by suitable adjustment of the driving torque. Then (7.1) has exponential solution $I(t) = I(0)e^{pt}$ where

$$p = L^{-1}\left[\frac{M}{2\pi}\Omega - R\right] \,, \tag{7.2}$$

and we have exponential growth of $I(t)$ and so of the magnetic field to which it gives rise (i.e. we have dynamo action) provided $M\Omega > 2\pi R$, i.e. provided the disc rotates rapidly enough.

Appealing though this description is in its simplicity, it cannot be correct (although it will be found in many texts and review articles!). For consider the limiting situation of a perfectly conducting disc and wire, in which case $R = 0$. Then, on the one hand, (7.2) gives $p = M\Omega/2\pi L$ so that we still have dynamo action. But on the other hand, the rim of the disc is a closed circuit moving with a perfect conductor, and Alfven's theorem (the most basic theorem in magnetohydrodynamics) tells us that the flux Φ through this circuit must be constant. There is an obvious contradiction. What has gone wrong?

The answer is that we have neglected the currents that flow azimuthally in the disc - i.e. the very currents that are associated with the diffusion of flux across the rim of the disc. These currents become particularly important in the limit $R \to 0$, and they completely invalidate the above description. The paradox can be resolved by supposing that the azimuthal current $J(t)$ is constrained to flow round the rim of the disc (by a suitable distribution of radial insulating strips). Then the fluxes through the I and J circuits are given by

$$\Phi_1 = LI + MJ \tag{7.3}$$

$$\Phi_2 = MI + L'J$$

and the equations governing the current flow are

$$\frac{d\Phi_1}{dt} = \frac{\Omega}{2\pi}\Phi_2 - RI \tag{7.4}$$

$$\frac{d\Phi_2}{dt} = -R'J$$

where R', L' refer to the J-circuit. This system still admits exponential solutions, $(I,J) \propto e^{pt}$, and the criterion for dynamo action is still $M\Omega > 2\pi R$. Now however, $p \to 0$ as $R' \to 0$, and so the description is consistent with Alfven's theorem. Details may be found in Moffatt (1979) where the nonlinear dynamical system (including the equation for $\Omega(t)$ for constant torque G is considered. As shown by Knobloch (1981), a rescaling of the variables for this problem yields the Lorenz system with the now familiar chaotic characteristics. It is noteworthy that this simplest prototype dynamo system already contains the seeds of chaos (provided the formulation is self-consistent).

It is important to note that, while dynamo action requires that the resistance of the circuit R be low, i.e. that the conductivity σ of disc and wire be high, we lose the dynamo if we go to the limit $\sigma \to \infty$, because then the field cannot diffuse into the region in which induction is operative. An efficient dynamo requires a conductivity that is large but not too large.

7.2 THE STRETCH-TWIST-FOLD DYNAMO

The magnetic field $\underline{B}(\underline{x},t)$ evolves in a conducting fluid of diffusivity η moving with velocity $\underline{u}(\underline{x},t)$ according to the induction equation

$$\frac{\partial \underline{B}}{\partial t} = curl\,(\underline{u} \times \underline{B}) + \eta \nabla^2 \underline{B}\ . \tag{7.5}$$

In the perfectly conducting limit ($\eta \to 0$), the magnetic lines of force ('$\underline{B}$-lines') are frozen in the fluid, and if the motion is incompressible ($\nabla . \underline{u} = 0$), then stretching of $\underline{B}$-lines implies proportionate intensification. The simplest 'heuristic' dynamo is based on this effect: a magnetic tube of force can be doubled in intensity by the stretch-twist-fold cycle indicated in figure 7.2 (Vainshtein & Zel'dovich 1982).

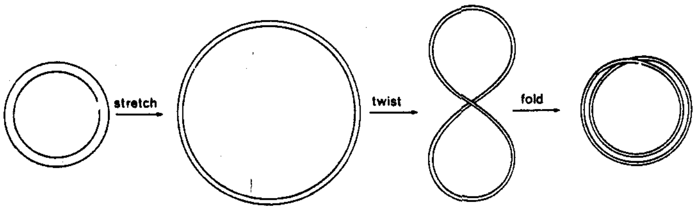

Figure 7.2

Clearly, as recognized by Vainshtein & Zel'dovich, a little diffusion is needed to 'get back to square one', but nevertheless the doubling time for the process does not apparently depend on diffusivity; in this sense the dynamo is a 'fast' dynamo.

Here again, however, there is a danger of over-simplification. When account is taken of the tube structure, and the way that this evolves under repeated application of the cycle of figure 7.2 (see Moffatt & Proctor 1984), a highly complex field structure emerges, and the indications are that the field $\underline{B}(\underline{x},t)$ develops increasingly fine-scale structure as the cycle continues, right down to the diffusive scale $O(\eta^{1/2})$. In the limit $\eta \to 0$, the field becomes non-differentiable everywhere. So here also, although the doubling process of figure 7.2 is non-diffusive in character, the fast dynamo, if it exists, depends in a subtle way on the action of diffusion even in the limit $\eta \to 0$.

7.3 BEHAVIOUR OF THE DIPOLE MOMENT IN A CONFINED SYSTEM

This vital influence of diffusivity in <u>permitting</u> dynamo action is evident also from the classical results of Bondi & Gold (1950) concerning the dipole moment $\underline{\mu}(t)$ associated with electric currents confined to a sphere of radius R of conducting fluid. If $\underline{B}$ is the resulting magnetic field, then two equivalent expressions for $\underline{\mu}(t)$ are

$$\underline{\mu}(t) = \frac{3}{8\pi}\int_{r<R} \underline{B}\, dV = \frac{3}{8\pi}\int_{r=R} (\underline{B}\,.\,\underline{n})\,\underline{x}\, d\underline{S}\ . \tag{7.6}$$

From the second of these expressions, it is easy to obtain an upper bound on $|\mu|$, viz

$$|\mu| \leq \frac{3}{4\pi} R\,\Phi \tag{7.7}$$

where Φ is the total flux of $\underline{B}$ entering the sphere, i.e. the integral of $\underline{B}\,.\,\underline{n}$ over that part of S on which $B\,.\,\underline{n} > 0$. If $\eta = 0$, then $\Phi = cst.$ (Alfven's theorem again) and so exponential increase of $\underline{\mu}$ is certainly impossible; no matter what the velocity field $\underline{\mu}(\underline{x}\,.\,t)$ may be, the inequality (7.7) controls the situation.

Diffusivity however may release this control. Using (7.6), and some elementary manipulation, we have

$$\frac{d\underline{\mu}}{dt} = \frac{3}{8\pi}\int_{r=R} \underline{u}(\underline{n}\,.\,\underline{B})\, dS - \eta\frac{3}{8\pi}\int_{r=R} \underline{n}\times(\nabla\times\underline{B})\, dS\ . \tag{7.8}$$

When $\eta = 0$, the first term redistributes the flux on $r = R$, but respects the inequality (7.7). When $\eta \neq 0$, provided the velocity field is such as to maintain a predominantly positive value of $[-\underline{\mu}\,.\,\underline{n}\times(\nabla\times\underline{B})]$ over the surface $r = R$, diffusion will provide a sustained (and potentially unbounded) increase of $|\underline{\mu}|$. Here therefore the primary mechanism for dynamo action is diffusion, and the growth rate p may be expected to depend on η, with $p \to 0$ as $\eta \to 0$. This is a 'slow' dynamo in the terminology of Vainshtein & Zel'dovich (1982). In fact all known dynamos that have been rigorously established are of the 'slow' variety. Frequently $p = O(\eta^q)$ with $0 < q < 1$, as $\eta \to 0$.

7.4 THE PROS AND CONS OF DYNAMO ACTION

As mentioned in §7.1, dynamo action can occur only if the fluid conductivity is 'sufficiently large', i.e. only if $\eta = (\mu_0\sigma)^{-1}$ is sufficiently small. How small is sufficient? A partial answer is provided by two classical results obtained by manipulation of the equation for magnetic energy associated with electric currents in a sphere $r < R$:

necessary conditions for dynamo action are

$$\eta < e_m R^2 / \pi^2 \qquad \text{(Backus 1958)} \tag{7.9}$$

$$\eta < U_m R / \pi \qquad \text{(Childress 1969)} \tag{7.10}$$

where U_m is the maximum value of $|\underline{u}|$ in $r < R$, and e_m is the maximum of the largest principle rate of strain in $r < R$. Frequently $e_m R = O(U_m)$, so that (7.9) and (7.10) are comparable, though not the same. It may happen however that $e_m R \ll U_m$ (if the velocity gradients are everywhere high as in a turbulent flow), and then (7.10) is a much stronger results.

It must be emphasised that (7.9) and (7.10) are necessary for dynamo action, but by no means sufficient. A simple sufficient condition can be formulated only for turbulent flow (see §7.6 below).

The results (7.9) and (7.10), which have been strengthened by Proctor (1977), are the 'pros' of dynamo action. The 'cons' are provided by the various anti-dynamo theorems, mainly variants and generalisations of Cowling's (1934) theorem which states that "steady axisymmetric dynamo action is impossible". A systematic treatment of this class of theorems is provided by the recent work of Hide & Palmer (1982).

7.5 FLUX EXPULSION AND TOPOLOGICAL PUMPING

A further effect which mitigates against efficient dynamo action when η is small is the effect of the expulsion of magnetic flux from any region of closed streamlines. Just as for the homopolar disc dynamo, if magnetic flux cannot penetrate such a region, then any inductive effect in that region will be quite impotent.

Flux expulsion occurs because the velocity field winds up the magnetic field, generally into a tight double spiral, in the region of closed streamlines. Diffusion then acts to eliminate the field from this region. The process is well illustrated by the model problem sketched in figure 7.3: (see Moffatt & Kamkar 1983). Here the initial field $(0, b_o \cos k_o x, 0)$ is sheared by the velocity field $\underline{u} = (\alpha y, 0, 0)$. The problem is easily solved in terms of the vector potential $(0, 0, A)$ of $\underline{B}$ which satisfies the convection-diffusion equation

$$\frac{\partial A}{\partial t} + \underline{u}\,.\,\nabla A = \eta\,\nabla^2 A\ , \tag{7.11}$$

with initial condition

$$A(x,y,0) = -k_o^{-1}\, b_o \sin k_o x\ . \tag{7.12}$$

The solution here is

$$A(x,y,t) = -k_o^{-1} B_o \ \operatorname{Im}\left[a(t)e^{i\underline{k}(t).\underline{x}}\right]\ , \tag{7.13}$$

where

$$\underline{k}(t) = (k_o\, ,\, -\alpha t\, k_o\, ,\, 0) \tag{7.14}$$

and

$$a(t) = \exp\left|\, -\eta k_o^2 (t + \tfrac{1}{3}\alpha^3 t^3)\,\right|\ . \tag{7.15}$$

It is the t^3-term in the latter expression which encapsulates the flux-expulsion effect. The time-scale of this field-elimination process is evidently

$$t_{fe} = \alpha^{-1} R_m^{1/3}$$

where $R_m = \alpha / \eta k_o^2\ (\gg 1)$ is the magnetic Reynolds number associated with the shear. This estimate is consistent with that inferred in the pioneering study of Weiss (1966).

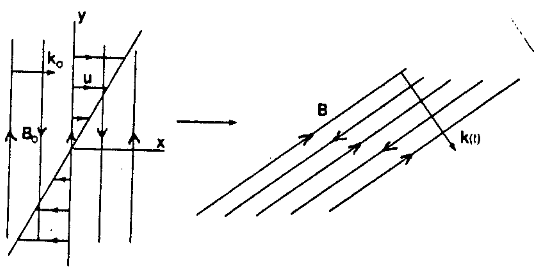

Figure 7.3

If the shear is localised (figure 7.4) then flux expulsion acts only in the region of shear, and reconnection of lines of force is inevitable, as indicated in the figure.

This however is not the whole story. Rhines & Young (1983) have recently studied (7.11) in the context of scalar diffusion, and have observed that a residual field may survive in a region of closed streamlines over the ordinary diffusive time-scale $t_d = \alpha^{-1}R_m$. It is easy to see how this may occur in the magnetic context considered here. If the $\underline{B}$-lines coincide with the $\underline{u}$-lines in the region of closed $\underline{u}$-lines, then there is no 'winding-up' effect (figure 7.5).

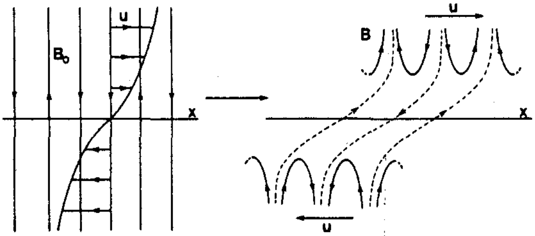

Figure 7.4

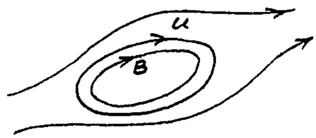

Figure 7.5

A field of this kind will diffuse so that it will not remain exactly aligned with $\underline{u}$; but as shown by Rhines & Young, the strong shearing effect of the $\underline{u}$-field is always such as to maintain a $\underline{B}$-field that is (to leading order) aligned with $\underline{u}$, and this field does indeed survive in the region of closed $\underline{u}$-lines on the long time-scale t_d.

It is an open question whether flux expulsion occurs, or not, in more complex three-dimensional situations. One situation of particular current interest is that in which the $\underline{u}$-lines are ergodic (space-filling) in some region V of R^3. Can a magnetic field survive indefinitely in such a region (when $\eta \neq 0$) or is it expelled by a quasi-two-dimensional mechanism on the $R_m^{1/3}$-timescale? No general answer to this question is as yet known.

An interesting three-dimensional variant of the flux expulsion process is the 'topological pumping' mechanism, identified by Drobyshevski & Yuferev (1974). In the topologically asymmetric motion associated with hexagonal cells in a Benard layer, horizontal $\underline{B}$-lines can be convected downwards, but cannot be convected upwards, since the regions of upward moving fluid are disconnected. There is therefore a net pumping effect downwards which becomes more effective as R_m increases from small values.

Recent computations for larger R_m (100-200) by Galloway and Proctor (1984) and by Arter (1984) have shown that here also the effects are much more subtle than originally realised. Not only is flux apparently pumped downwards, but by some mysterious mechanism that is not altogether clear, reversed flux is generated near the top of the layer. (Could this phenomenon have some bearing on the as yet unexplained appearance of reverse field in the Reversed-Field Pinch (Bodin & Newton 1980) ??).

7.6 MEAN-FIELD ELECTRODYNAMICS

There can be no dispute that the major advances in dynamo theory over the past 20 years have been associated with the development of mean-field electrodynamics, in a turbulent context, whose origins may be traced to the work of Parker (1955), Braginskii (1964) and Steenbeck, Krause & Radler (1966). This theory is fully described by Moffatt (1978) and by Krause & Radler (1980), and it will be sufficient here to discuss certain key points of the theory, and to comment on some weak points which call for further investigation.

The theory is based on a decomposition of the total velocity field $\underline{u}_{tot}$ and total magnetic field $\underline{B}_{tot}$ into mean and fluctuating parts

$$\underline{u}_{tot} = \underline{U} + \underline{u} , \qquad \underline{B}_{tot} = \underline{B} + \underline{b} . \tag{7.16}$$

The mean of the induction equation is then

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{U} \times \underline{B}) + \nabla \times \underline{E} + \eta \nabla^2 \underline{B} , \tag{7.17}$$

where $\underline{E} = \langle \underline{u} \times \underline{b} \rangle$ is the electromotive force associated with the turbulence. Consideration of the equation for the fluctuating field $\underline{b}$ establishes (on quite general grounds) a linear relationship between $\underline{E}$ and $\underline{B}$; and provided there is a scale separation (the scale of the fluctuating fields being small compared with the scale of the mean fields) this linear relationship takes the form

$$E_i = \alpha_{ij} B_j + \beta_{ijk} \frac{\partial B_j}{\partial x_k} + \ldots , \tag{7.18}$$

where α_{ij} and β_{ijk} are pseudo-tensors, determined (in principle) by the statistics of the turbulence, and the parameter η. When the scale of $\underline{B}$ is sufficiently large, the series (7.18) may be expected to converge rapidly; and in practice only the first two terms are retained. It is however quite common in dynamo models to find that the β-term in (7.18) is comparable in importance with the α-term, and one may detect here the seeds of a certain inconsistency: if the first two terms are comparable, then what about the third term, to say nothing of the n^{th} term?

The first problem in mean-field electrodynamics (analogous to transport problems in statistical physics) is then to obtain explicit expressions for α_{ij}, β_{ijk} in terms of η and of statistical properties of $\underline{u}$. The astrophysically interesting situation is that in which $\eta \to 0$ (or, more strictly, in which the turbulent magnetic Reynolds number is large); unfortunately this is the limit in which theoretical analysis is peculiarly difficult! If typical magnitudes of α_{ij}, β_{ijk} are denoted by α and β, and if these are independent of η in the limit $\eta \to 0$, then on dimensional grounds one would expect that

$$\alpha = O(u_o) , \qquad \beta = O(u_o \ell_o) , \tag{7.19}$$

where $u_o = \langle \underline{u}^2 \rangle^{1/2}$ and ℓ_o is a characteristic scale of the turbulence; and indeed the estimates (7.19) are commonly used (with suitable numerical coefficients) in the astrophysical literature. But we have already noted the subtleties of the limit $\eta \to 0$ in the laminar context; and there is no reason to suppose that the behaviour will be any less subtle in the turbulent context. If astrophysical dynamo models have to depend only on the dimensional justification of (7.19), this is a shaky foundation for an enormous superstructure!

There is however some evidence from numerical simulation experiments that (7.19) may, despite the apparent naivety, be essentially correct. Formally exact expressions for α_{ij} and β_{ijk} were obtained by Langrangian analysis by Moffatt (1975) and these were used in a numerical simulation by Kraichnan (1976) who showed that, except possibly in the artificial case of 'frozen' turbulence, α and β do settle down to values of order u_o and $u_o \ell_o$ respectively. Current work of Drummond, Duane & Horgan (1984), which incorporates weak diffusion via a Brownian 'jiggle' superposed on the turbulence, finds results so far consistent with Kraichnan's study, and this is at least reassuring. The calculations are however at the limit of available computer power, and one must question whether true asymptotic ($t \to \infty$) conditions are attained in these computations.

The case of isotropic turbulence (statistically invariant under rotations of the frame of reference) deserves particular comment. In this case, α_{ij} and β_{ijk} are isotropic, i.e.

$$\alpha_{ij} = \alpha\,\delta_{ij}\ , \qquad \beta_{ijk} = \beta\,\epsilon_{ijk} \tag{7.20}$$

where, now, α is a pseudo-scalar and β is a scalar. This difference is highly significant: α can be non-zero only in turbulence that 'lacks reflexional symmetry'; β, on the other hand, is generally non-zero, whether the turbulence lacks reflexional symmetry or not.

The simplest measure of the lack of reflexional symmetry in a field of turbulence is the mean helicity

$$H = \langle \underline{u}\,.\,\mathrm{curl}\,\underline{u}\rangle\ . \tag{7.21}$$

At <u>low</u> turbulent magnetic Reynolds number, there is a direct relationship between α and H: α is a weighted integral of the spectrum of H (Moffatt 1978, §7.8).

It is known that, when $\underline{U} = 0$ and $\alpha \neq 0$, equation (7.17) admits dynamo solutions provided $|\alpha| R/(\eta + \beta)$ exceeds a critical value dependent only on the shape of the fluid domain, where R is a typical scale of this domain. Hence, a <u>sufficient</u> condition for dynamo action in such a domain is that $|\alpha|$ be non-zero and R be sufficiently large; the former condition is generally satisfied if the turbulence in the domain lacks reflexional symmetry. This is the sufficient condition referred to in §7.4 above.

7.7 SOME PROPERTIES OF THE PSEUDO-TENSORS α_{ij} and β_{ijk}

If the turbulence is not isotropic (and it seldom is!) then there are certain other effects concealed in α_{ij} and β_{ijk} in addition to the simple α-effect and the eddy diffusivity (β-) effect that are present in isotropic conditions. Firstly, α_{ij} need not be symmetric; if we decompose it into symmetric and antisymmetric parts, i.e.

$$\alpha_{ij} = \alpha_{ij}^{(s)} + \epsilon_{ijk}\gamma_k\ , \tag{7.22}$$

then it is evident that $\underline{\gamma}$ is a polar vector which need not va reflexionally symmetric turbulence. The symmetric part $\alpha_{ij}^{(s)}$ do ever vanish unless the turbulence lacks reflexional symmetry.

In the 'first-order smoothing approximation' in which ter dratic in fluctuating quantities are neglected in the fluctuatic tion, it turns out that α_{ij} is symmetric, i.e. $\underline{\gamma} = 0$. At t order, however, 'second-order smoothing', $\underline{\gamma}$ can be expresse weighted integral of triple spectra (i.e. Fourier transforms of velocity correlations), and is in general non-zero. A more in situation is perhaps that in which the turbulence is inhomogene this case a contribution to $\underline{\gamma}$ is obtained at the first-order sm level, in the direction of decreasing turbulence intensity:

$$\underline{\gamma} = \frac{-k}{\eta}\nabla(\ell_0^2 \langle \underline{u}^2 \rangle) ,$$

where again ℓ_0 is the scale of the turbulence, and k is a dimens constant of order unity; the factor η^{-1} is a product of the firs smoothing approximation. Note that for inhomogeneous turbulen vector $\underline{\gamma}$ given by (7.23) will be a function of position, $\underline{\gamma}$ When substituted in the mean-field equation, via (7.22) and (7. gives a contribution

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{\gamma} \times \underline{B}) + \ldots$$

i.e. $\underline{\gamma}$ acts like an effective velocity, transporting the mea relative to the fluid. It is important however to note that $\underline{\gamma}$ general non-solenoidal, i.e. $\nabla . \underline{\gamma} \neq 0$, so that the qualitative of $\underline{\gamma}$ is quite different from that of the actual fluid mean velo which is assumed to satisfy $\nabla . \underline{U} = 0$. In fact, the $\underline{\gamma}$-effect fied here is none other than the flux-expulsion effect (incorp topological pumping also), reappearing within the mean-field fram

Turning now to β_{ijk}, a first-order smoothing analysis gi contributions (Moffatt & Proctor 1982). The first is a w integral of the symmetric part of the spectrum tensor of the turb and admits interpretation as an anisotropic eddy diffusivity second part is a weighted integral over the helicity spectrum f $H(\underline{k},\omega)$, viz

$$\beta_{ijk}^{(2)} = \iint \left[\frac{\omega}{\omega^2 + \eta^2 k^4}\right] \left[\frac{4\eta^2 k_i k_j k_k}{\omega^2 + \eta^2 k^4} + \frac{k_i \delta_{jk} - k_j \delta_{ki}}{2k^2}\right] H \, d\underline{k} \, d\omega .$$

This full expression is given here just to indicate the measure of tensorial complexity that arises even at the lowest order of approximation. In the special case of axisymmetric turbulence, it can be shown that the expression (7.25) contains the Radler effect (Radler 1969):

$$\beta^{(2)}_{ijk} \frac{\partial B_j}{\partial x_k} = R(\underline{e} \quad \underline{J})_i + \ldots . \qquad (7.26)$$

where $\underline{J}$ is the mean current, $\underline{e}$ is a unit vector along the axis of symmetry, and R is the Radler coefficient (a pseudo-scalar). As shown by Moffatt & Proctor (1983), if the turbulence is statistically symmetric about a plane perpendicular to the axis of symmetry, then (at first-order smoothing level), $\alpha_{ij} = 0$ but $R \neq 0$; in this situation the Radler effect may be important for field generation.

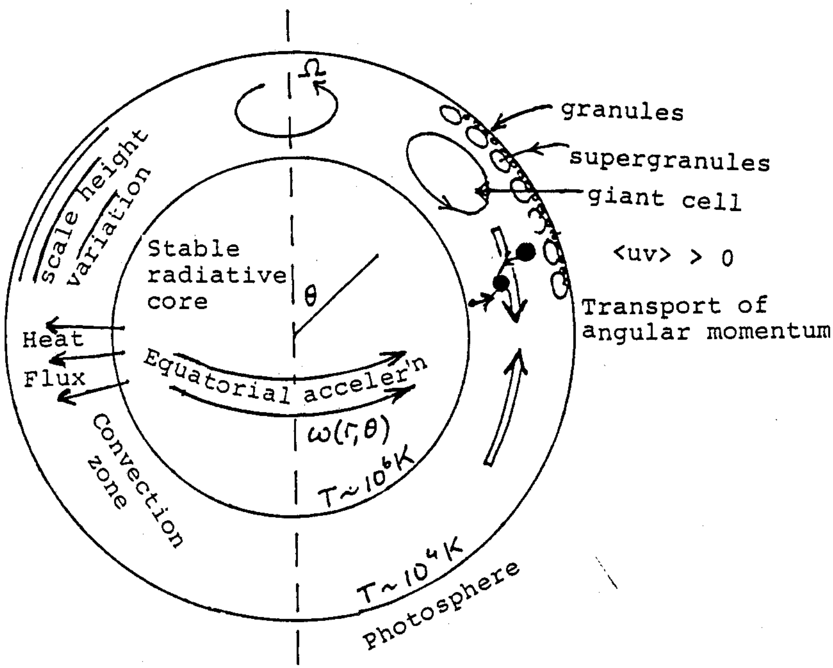

Figure 7.6

7.8 THE SOLAR DYNAMO

Let us now consider some aspects of the solar dynamo problem. The solar scenario for dynamo action is indicated in figure (7.6). The rotation of the Sun has an important double influence on the convective cells in the convection zone: first, Coriolis forces cause a deflection of rising blobs of fluid; this causes the generation of a Reynolds stress distribution, which in turn is believed to be responsible for the differential rotation $\omega(r,\theta)$ of the Sun. Secondly, as blobs rise, they expand and therefore tend to rotate more slowly (conserving their intrinsic angular momentum); this establishes a correlation between vertical velocity and vertical vorticity, i.e. a helicity distribution, which in turn leads to an α-effect. Thus, the two ingredients of an $\alpha\omega$-dynamo, the α-effect and differential rotation, are both a consequence of Coriolis forces; from a dynamical point of view, we are not free to specify $\alpha(r,\theta)$ and $\omega(r,\theta)$ independently - they should both be derived in a self-consistent manner from the governing dynamical equations. This desirable aim has not as yet been attained.

Let us however look at the two processes in a little more detail. The equation of motion, whatever else it may contain, contains a Coriolis force,

$$\frac{\partial \underline{u}}{\partial t} = -2\underline{\Omega}\times\underline{u} + \ldots \tag{7.27}$$

where, in local Cartesian coordinates (south, east, and vertically up) at colatitude θ,

$$\underline{\Omega} = (-\Omega\sin\theta\,,\,0\,,\,\Omega\cos\theta)\,. \tag{7.28}$$

With $\underline{u} = (u,v,w)$, and with u and v initially zero, we find an initial tendency (from (7.27))

$$v = -2w\sin\theta\,.\,\Omega t + O(t^3)\,, \tag{7.29}$$

$$u = -2w\cos\theta\sin\theta\,(\Omega t)^2 + O(t^4)\,, \tag{7.30}$$

so that the Reynolds stress is

$$\langle uv\rangle = 4(\Omega t)^3\langle w^2\rangle\cos\theta\sin^2\theta + O(t^5)\,. \tag{7.31}$$

This suggests that a reasonable approximation in a statistically steady state should be

$$\langle uv\rangle = 4(\Omega t_c)^3\langle w^2\rangle\cos\theta\sin^2\theta\,, \tag{7.32}$$

where t_c is a coherence time for the rising blobs ($t_c = 3 \times 10^5$ s, $\Omega t_c = 0.2$ for supergranular scales). This generates differential rotation ω whose θ-dependence is given by

$$\nu_T \frac{\partial \omega}{\partial \theta} = \langle uv \rangle \tag{7.33}$$

where ν_T is an eddy viscosity ($80\ km^2/s$) associated with granular and sub-granular scales. Integrating (7.33) gives

$$\omega(r,\theta) = \frac{4\langle w^2 \rangle}{3\nu_T} (\Omega t_c)^3 (\sin^3\theta - \frac{4}{3\pi}) , \tag{7.34}$$

where the constant of integration is chosen so that $\langle\omega\rangle = 0$, i.e. $\omega(r,\theta)$ represents the fluctuation about the mean. The expression (7.34) indicates equatorial acceleration, as observed in the Sun, and indeed the difference in ω between equator and poles,

$$\omega(r,\frac{\pi}{2}) - \omega(r,0) = \frac{4}{3\nu_T} \langle\omega^2\rangle (\Omega t_c)^3 \quad 6.6 \times 10^{-7}\ s^{-1} , \tag{7.35}$$

which compares very favourably with the observed value ($7.9 \times 10^{-7}\ s^{-1}$).

Consider now the mechanism of generation of an α-effect (Steenbeck, Krause & Radler 1966). As a blob rises into a region of decreasing density, the vertical component of $(\underline{\omega} + 2\underline{\Omega})/\rho$ tends to be conserved (where $\underline{\omega}$ is the vorticity). Hence for small t,

$$\omega_3 \approx 2\Omega \cos\theta\, (wt) \frac{d}{dz} (\ln \rho_0(z)) \tag{7.36}$$

where $\rho_0(z)$ is the basic density stratification, and so the helicity is

$$H = \langle \underline{u} \cdot \underline{\omega} \rangle \approx \langle w\omega_3 \rangle = -(\Omega t_c) \langle w^2 \rangle \cos\theta / H_\rho \tag{7.37}$$

where H_ρ is the density scale-height. The associated α-effect (on the simplest theory) is

$$\alpha = -\frac{1}{3} H t_c = \frac{1}{3} \Omega t_c^2 \cos\theta \langle w^2 \rangle / H_\rho . \tag{7.38}$$

Equations (7.34) and (7.38) provide a pair of dynamically consistent expressions for α and ω, which could usefully be employed in numerical investigation of dynamo modes.

7.9 MAGNETIC BUOYANCY AS AN EQUILIBRATION MECHANISM

It is well-known that when R_m is large as in the Sun, that α-effect in conjunction with differential rotation will yield solutions of (7.17) in a spherical geometry having an oscillatory dynamo character, i.e.

$$\underline{B}(\underline{x},t) = \mathrm{Re}\left\{ \hat{\underline{B}}(\underline{x})\, e^{(p_r+ip_i)t} \right\}, \tag{7.39}$$

where $p_r > 0$, $p_i \neq 0$. The field then grows in intensity from one cycle of its periodic behaviour to the next, and ultimately it must react back upon the dynamical system through some equilibration mechanism. There are three possibilities here: (i) a strong field will tend to suppress the turbulent convection, and thus to decrease the α-effect, an effect studied by Moffatt (1972); (ii) likewise, a strong field will react upon the mean velocity field, and in particular will tend to damp the differential rotation; this mechanism was first studied by Malkus and Proctor (1975), and it has recently been identified by Gilman (1984) in his monumental numerical investigation of the solar dynamo, as a mechanism of crucial importance. The third mechanism, not included in the Gilman model, is probably equally important: this is magnetic buoyancy (Parker 1955). When a strong toroidal magnetic field $\underline{B}_T$ is generated deep in

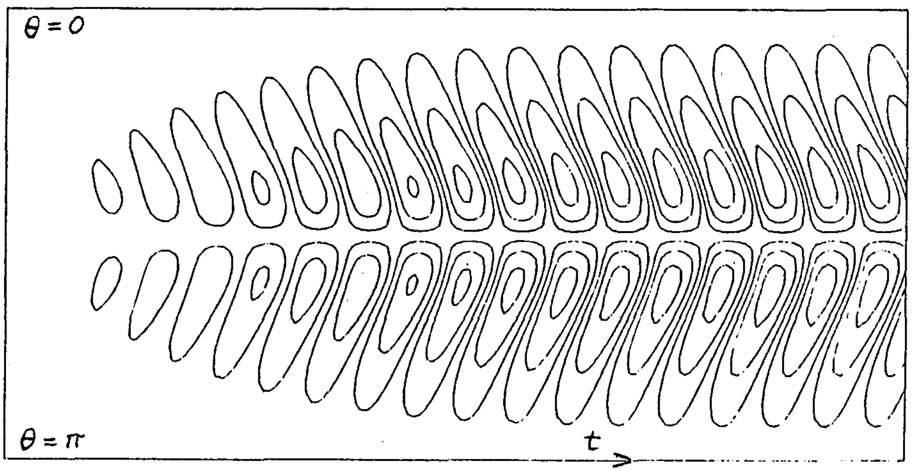

Figure 7.7 (from Nightingale 1985)

the solar convection zone, it is subject to a self-induced instability which causes flux tubes to rise and burst through the photosphere. If downward topological pumping is present, then this magnetic buoyancy instability is what must limit the accumulation of toroidal flux near the bottom of the convection zone. Magnetic buoyancy can be incorporated in an $\alpha\omega$-dynamo via the γ-effect described in §7.7 above, and with $\underline{\gamma}$ a vertical effective velocity proportional to $-\frac{d}{dr}(B_T^2)$ (Nightingale 1985). The boundary condition adopted on the photospheric surface $r = R$ must be such as to allow the toroidal field to escape when it gets there - e.g. a boundary condition of the form

$$B_T + R\frac{\partial B_T}{\partial r} = 0 \quad \text{on} \quad r = R \tag{7.40}$$

is one possibility. Figure (7.7) shows contours of $B_T(r,\theta,t)$ at a fixed value of r in the (θ,t) plane (butterfly diagrams), for a particular choice of α, ω and γ. The initial exponential growth is clear, as is the equilibration at constant amplitude induced by the magnetic buoyancy term in the equations. Nightingale's choice of α and ω was based on the previous purely kinematic study of Roberts (1972), and is not dynamically consistent in the sense of §7.8 above - nevertheless it does succeed in establishing that magnetic buoyancy can equilibrate, and it points the way for future studies that should aim in addition at dynamical consistency.

REFERENCES

Arter W.: 1983 Fluid Mech. 132, 25-48.
Backus, G.E.: 1958 Ann. Phys. 4, 372-447.
Bodin, H.A.B. and Newton, A.A.: 1980 Nuclear Fusion 20, 1255.
Bondi, H. and Gold, T.: 1950 Mon. Not. Roy. Astr. Soc. 110, 607-611.
Braginskii, S.i.: 1964 Sov. Phys. JETP 20, 726-735.
Childress, S.: 1969 Lectures on Dynamo Theory Inst. Henri Poincare, Paris.
Cowling, T.G.: 1934 Mon. Not. Roy. Astr. Soc. 94, 39-48.
Drummond, I.T., Duane, S. and Horgan, R.R.: 1984 J. Fluid Mech. 138, 75--91.
Galloway, D.J. and Proctor, m.R.E.: 1983 Geoph. Astr. Fluid Dyn. 34, 109-136.
Hide, R. and Palmer, T.N.: 1982 Geoph. Astr. Fluid Dyn. 19, 301-319.
Knobloch, E.: 1981 Phys. Lett. 82A, 439-440.

Kraichnan, R.h.: 1976 J. Fluid Mech. 77, 753-768.

Krause, F. and Radler, K.-H.: 1980 Mean-field magnetohydrodynamics and dynamo theory. Pergamon.

Malkus, W.V.R. and Proctor, M.R.E.: 1975 J. Fluid Mech. 67, 417-444.

Moffatt, H.K.: 1972 J. Fluid Mech. 53, 385-399.

Moffatt, H.K.: 1974 J. Fluid Mech. 65, 1-10.

Moffatt, H.K.: 1978 Magnetic field generation in electrically conducting fluids. Cambridge University Press.

Moffatt, H.K.: 1979 Geophys. Astr. Fluid Dyn. 14, 147-166.

Moffatt, H.K. and Kamkar, H.: 19183 In Stellar and Planetary Magnetism (ed. A.D. Soward), Gordon & Breach, 91-98.

Moffatt, H.K. and Proctor, M.R.E.: 1983 Geophys. Astr. Fluid Dyn. 21, 265-283.

Moffatt, H.K. and Proctor, M.R.E.: 1984 J. Fluid Mech. 154, 493-507.

Nightingale, S.: 1985 Magnetic flux pumping and magnetic buoyancy in mean-field dynamos Ph.D. Thesis, Cambridge University, in preparation.

Parker, E.N.: 1955 Astrophys. J. 122, 293-314.

Rhines, P.B. and Youngs, w.R.: 1983 J. Fluid Mech. 133, 133-145.

Roberts, P.H.: 1972 Phil. Trans. Roy. Soc. A 272, 663-698.

Steenbeck, M., Krause, F. and Radler, K.-H.: 1966 Z. Naturforsch. 21a, 1285-1296.

Vainshtein, S. and Zel'dovich, Ya.B.: 1978 Sov. Phys. Usp. 15, 159-172.

Weiss, N.O.: 1966 Proc. Roy. Soc. A293, 310-328.