(58)

www.moffatt.tc

Rev. BrMec. Rio de Janeiro, V.IX, nº 2 – 1987

ON THE EXISTENCE OF EULER FLOWS THAT ARE TOPOLOGICALLY ACCESSIBLE FROM A GIVEN FLOW

H. Keith Moffatt*
Institute for Nonlinear Science and
Institute of Geophysics and Planetary Physics
University of California
San Diego – USA

ABSTRACT

Through consideration of the process of magnetic relaxation in a perfectly conducting, viscous, barotropic fluid, it is shown that, for each pressure-density relationship $p = k\rho^{\gamma}$, there exists a magnetostatic equilibrium field $\mathbf{b}^E(\mathbf{x})$ *that is topologically accessible from any given field* $\mathbf{b}^0(\mathbf{x})$, *and that the associated magnetic energy is a decreasing function of the compressibility. Exact analogy with the Euler equations of incompressible inviscid flow then leads to the conclusion that, given any kinematically possible flow* $\mathbf{U}(\mathbf{x})$, *there exists (at least) a one-parameter family of distinct Euler flows* $\mathbf{u}^E(\mathbf{x},\lambda)$ *topologically accessible from* $\mathbf{U}(\mathbf{x})$.

1. INTRODUCTION

In a previous paper [1], we have introduced the concept of 'topological accessibility', which is a natural extension of the well-established concept of topological equivalence, and which plays an important role in the theory of the existence of Euler flows, i.e. steady solutions of the classical Euler equations of incompressible flow of an inviscid fluid. In [1], attention was focussed on a restricted form of topological accessibility involving volume-preserving mappings of the fluid domain. In the present paper, we extend the analysis to cover mappings that are not volume-preserving, and thereby show that, corresponding to any given streamline topology, there is (at least) a one-parameter family of distinct Euler flows (which, as in [1], may contain tangential discontinuities). The method involves appeal to the exact analogy between the steady Euler

On leave of absence from DAMTP, University of Cambridge, UK. This paper follows the theme of an invited lecture presented at the Brazilian National Congress in Thermoscience, Rio de Janeiro, December 1986

equations and the equations of magnetostatic equilibrium of a perfectly conducting, compressible fluid, and consideration of the process of relaxation towards such equilibria. The treatment given here is self-contained, but more compact that in [1], to which reference should be made for discussion of physical background and motivation.

2. TOPOLOGICAL ACCESSIBILITY

Let D be a finite connected domain in $\mathbb{R}^3$, and let $\mathbf{v}(\mathbf{x},t)$ be a smooth (C^1) velocity field defined for $\mathbf{x} \varepsilon D$ and $0 \le t < \infty$, and satisfying

$$\mathbf{v} = 0 \text{ on } \partial D \quad (\text{all } t). \tag{2.1}$$

We define the dissipation integrals

$$D_s(t) = \int_D (\nabla \times \mathbf{v})^2 d^3\mathbf{x} \tag{2.2}$$

and

$$D_b(t) = \int_D (\nabla \cdot \mathbf{v})^2 d^3\mathbf{x}, \tag{2.3}$$

and we shall say that $\mathbf{v}$ is a *relaxation velocity field* if

$$\mathbf{v}(\mathbf{x},t) \to 0 \text{ as } t \to \infty, \text{ uniformly in } D, \tag{2.4}$$

and

$$\int_0^\infty D_s(t)dt < \infty, \quad \int_0^\infty D_b(t)dt < \infty. \tag{2.5}$$

Let $\rho(\mathbf{x},t)$ be the associated density field satisfying the mass-conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.6}$$

with initial condition $\rho(\mathbf{x},0) = \rho_0$ (cst.), and let $\mathbf{X}(\mathbf{x},t)$ be the particle displacement field, determined by the dynamical system

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{v}(\mathbf{X},t), \quad \mathbf{X}(\mathbf{x},0) = \mathbf{x}. \tag{2.7}$$

The Jacobian of the mapping $\mathbf{x} \to \mathbf{X}(\mathbf{x},t)$ is

$$J = \frac{\partial(X_1,X_2,X_3)}{\partial(x_1,x_2,x_3)} = \frac{\rho_0}{\rho(\mathbf{X},t)} \ (> 0). \tag{2.8}$$

Components of the deformation tensor $\partial X_i/\partial x_j$ cannot increase more rapidly than exponentially, and so are finite for all finite t; hence, for $t < \infty$, the mapping $\mathbf{x} \to \mathbf{X}(\mathbf{x}, t)$ is a homeomorphism. However, as $t \to \infty$, the limit mapping induced by a relaxation velocity field

$$\mathbf{x} \to \mathbf{X}^E(\mathbf{x}) = \lim_{t \to \infty} \mathbf{X}(\mathbf{x}, t) \tag{2.9}$$

may exhibit discontinuities, despite the partial control implied by (2.5); this is because material surfaces that are initially apart may, asymptotically, be squeezed together even when the conditions (2.4) and (2.5) are satisfied.

Now let $\mathbf{b}^0(\mathbf{x})$ be a smooth field satisfying

$$\nabla \cdot \mathbf{b}^0 = 0 \text{ in } D, \mathbf{n} \cdot \mathbf{b}^0 = 0 \text{ on } \partial D \tag{2.10}$$

but otherwise arbitrary; in particular, the topology of $\mathbf{b}^0$ is arbitrary, the knots and linkages in the lines of force of $\mathbf{b}^0$ ('$\mathbf{b}^0$-lines') being arbitrarily complex. Let $\mathbf{b}(\mathbf{x}, t)$ be the vector field that evolves from $\mathbf{b}^0$ under 'frozen-field' distortion by $\mathbf{v}(\mathbf{x}, t)$, i.e. $\mathbf{b}(\mathbf{x}, t)$ is determined by

$$\frac{\partial \mathbf{b}}{\partial t} = \text{curl}\,(\mathbf{v} \times \mathbf{b}), \quad \mathbf{b}(\mathbf{x}, 0) = \mathbf{b}_0(\mathbf{x}). \tag{2.11}$$

Note that, by virtue of (2.1), (2.10) and (2.11), the conditions

$$\nabla \cdot \mathbf{b} = 0 \text{ in } D, \mathbf{n} \cdot \mathbf{b} = 0 \text{ on } \partial D \tag{2.12}$$

are automatically satisfied for all $t > 0$.

The (Lagrangian) solution of (2.11) is given by

$$b_i(\mathbf{X}, t) = \frac{\rho(\mathbf{X},t)}{\rho_0} b_j^0(\mathbf{x}) \frac{\partial X_i}{\partial x_j} = J^{-1} b_j^0(\mathbf{x}) \frac{\partial X_i}{\partial x_j} \tag{2.13}$$

(see, for example, [2], § 3.1). For each finite $t > 0$, this relationship establishes a homeomorphism between the fields $\mathbf{b}^0(\mathbf{x})$ and $\mathbf{b}(\mathbf{x}, t)$ which are therefore topologically equivalent: $\mathbf{b}^0$-lines map faithfully to $\mathbf{b}$-lines, and the fluxes of $\mathbf{b}^0$ and $\mathbf{b}$ along corresponding flux-tubes are equal. However, as $t \to \infty$, as noted above, the limit mapping $\mathbf{x} \to \mathbf{X}^E(\mathbf{x})$ may exhibit discontinuities, and so therefore may the limit field $\mathbf{b}^E(\mathbf{x})$ defined by

$$\mathbf{b}^E(\mathbf{x}) = \lim_{t \to \infty} \mathbf{b}(\mathbf{x}, t). \tag{2.14}$$

In general therefore, the relationship between $\mathbf{b}^0$ and $\mathbf{b}^E$ is *not* a homeomorphism, and the fields are not topologically equivalent. We say nevertheless that $\mathbf{b}^E(\mathbf{x})$ is *topologically accessible* from $\mathbf{b}^0(\mathbf{x})$, being the result of deformation of $\mathbf{b}^0(\mathbf{x})$ by a relaxation velocity field. The property of topological accessibility is weaker than that of topological equivalence, but it is just the property that is needed in the Euler flow context.

3. MAGNETIC RELAXATION IN A BAROTROPIC FLUID

Let us now interpret $\mathbf{b}^0(\mathbf{x})$ as the magnetic field at time $t=0$ in a perfectly conducting, viscous, compressible fluid, with barotropic pressure-density relationship $p=p(\rho)$. To be specific, we shall assume the perfect gas relationship

$$p = k\rho^{\gamma}, \tag{3.1}$$

where $\gamma(>1)$ and k are constants. The compressibility of the fluid in the uniform density state $\rho=\rho_0$ is

$$\lambda = (\rho\frac{dp}{d\rho})_0^{-1} = (\gamma k\rho^{\gamma})_0^{-1}, \tag{3.2}$$

so that the incompressible limit corresponds to $\lambda \to 0$ (or $k \to \infty$), and the perfectly compressible (or pressureless) limit corresponds to $\lambda \to \infty$ (or $k \to 0$).

In general, the Lorentz force $(\nabla\times\mathbf{b}^0)\times\mathbf{b}^0$ is rotational, so that the fluid must move for $t>0$, even if initially at rest. Let $\mathbf{v}(\mathbf{x},t)$ be the velocity field that develops, and let $\rho(\mathbf{x},t)$, $p(\mathbf{x},t)$, and $\mathbf{b}(\mathbf{x},t)$ be the associated density, pressure and magnetic fields. These fields are governed by the MHD equations, namely (2.6), (2.11) and (3.1) together with the Navier-Stokes equation

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + (\nabla\times\mathbf{b})\times\mathbf{b} - \mu\nabla\times(\nabla\times\mathbf{v}) + \zeta\nabla(\nabla\cdot\mathbf{v}). \tag{3.3}$$

Here, $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla$, and μ and ζ represent shear and bulk viscosity coefficients respectively. We shall suppose that μ and ζ are sufficiently large to guarantee smoothness of $\mathbf{v}(\mathbf{x},t)$ for all $t>0$.

From these equations, we may easily construct an energy equation. Defining

$$M(t) = \tfrac{1}{2}\int \mathbf{b}^2 d^3\mathbf{x} \quad \text{(magnetic energy)} \tag{3.4}$$

$$K(t) = \tfrac{1}{2}\int \mathbf{v}^2\rho\, d^3\mathbf{x} \quad \text{(kinetic energy)} \tag{3.5}$$

$$\Pi(t) = \int Q(\rho)\rho\, d^3\mathbf{x} \quad \text{(elastic energy)} \tag{3.6}$$

where

$$Q(\rho) = \int_{\rho_0}^{\rho} \rho^{-2}p(\rho)d\rho = \frac{k}{\gamma-1}(\rho^{\gamma-1}-\rho_0^{\gamma-1}), \tag{3.7}$$

and

$$\Phi(t) = \mu D_s(t) + \zeta D_b(t) \geq 0 \qquad \text{(energy dissipation rate)} \tag{3.8}$$

we find

$$\frac{d}{dt}(M(t) + K(t) + \Pi(t)) = -\Phi(t). \tag{3.9}$$

$\Pi(t)$ represents the energy stored in the fluid by virtue of the compression or expansion of fluid elements, and is positive (when $\gamma > 1$) (since work must be done on the fluid to generate density fluctuations from an initial state of uniform density). At time $t = 0$, we have

$$M(0) = M_0 \neq 0,\ K(0) = 0,\ \Pi(0) = 0, \tag{3.10}$$

and for all $t \geq 0$, we have

$$0 \leq M(t) \leq M_0,\ 0 \leq K(t) \leq M_0,\ 0 \leq \Pi(t) \leq M_0. \tag{3.11}$$

We now argue as in [1]: the total energy $E(t) = M(t) + K(t) + \Pi(t)$ is non-negative and monotonic decreasing, according to (3.9), and therefore tends to a limit. Hence

$$\Phi(t) \downarrow 0 \text{ as } t \to \infty, \tag{3.12}$$

and hence also, since $\mathbf{v}(\mathbf{x}, t)$ is smooth,

$$K(t) \downarrow 0 \text{ as } t \to \infty. \tag{3.13}$$

Hence

$$\mathbf{v}(\mathbf{x}, t) \to 0 \text{ as } t \to \infty, \text{ uniformly in } D. \tag{3.14}$$

Moreover

$$\int_0^\infty \Phi(t)dt = E(0) - E(\infty) \leq M_0 \tag{3.15}$$

so that the integrals (2.5) both converge. Hence $\mathbf{v}(\mathbf{x}, t)$ is a relaxation velocity field as defined in section 2.

The constraints (3.11) further imply that both ρ and $|\mathbf{b}|$ remain uniformly bounded in D. To see this, consider the contribution to Π and M from a small mass element $\delta m = \rho\, \delta^3\mathbf{x}$:

$$\delta\Pi \sim Q(\rho)\delta m, \quad \delta M \sim \tfrac{1}{2}|\mathbf{b}|^2 \delta m / \rho \tag{3.16}$$

If $\rho \to 0$ (δm being fixed) then clearly $|\mathbf{b}| \to 0$ also, since δM remains finite. Also ρ cannot *increase* without limit, since then $\delta\Pi \to \infty$ also, in conflict with (3.11c) (a positive infinity cannot be compensated by a negative infinity, since $Q(\rho)$ is bounded below). Hence, $|\mathbf{b}|^2 - 2\rho\,\delta M/\delta m$ remains finite also.

Since $\mathbf{v} \to 0$ and ρ and $|\mathbf{b}|$ remain bounded, equations (2.6) and (2.11) become asymptotically

$$\frac{\partial\rho}{\partial t} = 0, \quad \frac{\partial \mathbf{b}}{\partial t} = 0 \tag{3.17}$$

i.e.

$$\rho \to \rho^E(\mathbf{x}), \quad \mathbf{b} \to \mathbf{b}^E(\mathbf{x}), \text{ say, as } t \to \infty. \tag{3.18}$$

Moreover (3.3) becomes

$$(\nabla \times \mathbf{b}^E) \times \mathbf{b}^E = \nabla p^E, \tag{3.19}$$

where $p^E = p(\rho^E)$; i.e. the asymptotic state is magnetostatic.

If the topology of the initial field $\mathbf{b}^0$ is trivial (in the sense that each $\mathbf{b}^0$-line is a closed curve which may be shrunk continuously in D to a point without cutting any other $\mathbf{b}^0$-line) then the asymptotic field may be identically zero. This cannot happen however if the topology of $\mathbf{b}^0$ is non-trivial, since all linkages are conserved by (2.11). One measure of topological complexity is the magnetic helicity

$$H = \int_D \mathbf{a} \cdot \mathbf{b}\, d^3\mathbf{x}, \tag{3.20}$$

where $\mathbf{a}$ is any vector potential of $\mathbf{b}$ (i.e. $\mathbf{b} = \text{curl}\,\mathbf{a}$). H is conserved under the frozen-field evolution described by (2.11) even when the flow is compressible [4,3]. Moreover when $H \neq 0$, the magnetic energy is bounded [5,1] by an inequality of the form

$$M \geq q_0 |H|, \tag{3.21}$$

where q_0 depends only on the geometry of D. As $t \to \infty$ therefore, during the relaxation process, the magnetic energy $M(t)$ must tend to a steady value M^E compatible with (3.21).

It would appear therefore that, for each topologically nontrivial field $\mathbf{b}^0(\mathbf{x})$, and for each pressure-density relationship $p = k\rho^\gamma$ (with $\gamma > 1$), the field must relax to a field $\mathbf{b}^E(\mathbf{x})$ that is topologically accessible from $\mathbf{b}^0(\mathbf{x})$, and that satisfies the equation of magnetostatic equilibrium (3.19).

Consider now how the field $\mathbf{b}^E(\mathbf{x})$ changes if the compressibility of the fluid, represented by the parameter λ (eqn. (3.2)), is gradually changed. Suppose that, when the fluid is incompressible ($\lambda=0$) with uniform density ρ_0, the relaxed state is described by a field $\mathbf{b}_1^E(\mathbf{x})$, with corresponding pressure field $p_1^E(\mathbf{x})$, and magnetic energy M_1^E. Suppose that we now introduce a small compressibility $\delta\lambda$ into the system, the new initial conditions being

$$\mathbf{b}(\mathbf{x},0)=\mathbf{b}_1^E(\mathbf{x}),\ \rho(\mathbf{x},0)=\rho_0,\ p(\mathbf{x},0)=p(\rho_0)\ (\text{cst.}) \tag{3.22}$$

This is no longer a magnetostatic equilibrium, and the field $\mathbf{b}$ will proceed to relax, as described above, to a new magnetostatic field $\mathbf{b}_2^E(\mathbf{x})$ say, topologically accessible from $\mathbf{b}_1^E(\mathbf{x})$, with energy M_2^E. During this process, some energy is dissipated by viscosity, and some magnetic energy is converted to elastic energy. Hence

$$M_2^E < M_1^E\,. \tag{3.23}$$

This argument may now be repeated. With each small increase $\delta\lambda$ in compressibility, there will be an adjustment to a new magnetostatic equilibrium, with a small decrease of magnetic energy: $\delta M^E<0$. We thus infer the existence of a family of magnetostatic equilibria $\mathbf{b}^E(\mathbf{x},\lambda)$, each one of which is topologically accessible from the initial field $\mathbf{b}^0(\mathbf{x})$, with magnetic energy $M^E(\lambda)$ satisfying

$$\frac{dM^E}{d\lambda}<0\,. \tag{3.24}$$

This inequality of course implies that distinct values of λ give distinct fields:

$$\mathbf{b}^E(\mathbf{x},\lambda_1)\not\equiv\mathbf{b}^E(\mathbf{x},\lambda_2)\quad \text{if } \lambda_1\neq\lambda_2\,. \tag{3.25}$$

4. THE ANALOGOUS EULER FLOWS

The steady Euler equations of incompressible inviscid flow may be written in the form

$$\mathbf{u}\times(\nabla\times\mathbf{u})=\nabla h \tag{4.1}$$

with $\nabla\cdot\mathbf{u}=0$. The analogy with (3.19) is well-known, the analogous variables being $(\mathbf{u},\mathbf{b})$ and $(h,-p)$. To each solution $\mathbf{b}^E(\mathbf{x})$ of (3.9), there then corresponds an Euler flow $\mathbf{u}^E(\mathbf{x})$, satisfying the same boundary condition that $\mathbf{b}^E$ satisfies, namely

$$\mathbf{u}^E\cdot\mathbf{n}=0\ on\ \partial D\,. \tag{4.2}$$

By the argument of § 3, we have shown that for any field $\mathbf{b}^0(\mathbf{x})$ satisfying $\nabla \cdot \mathbf{b}^0 = 0$ in D, $\mathbf{n} \cdot \mathbf{b}^0 = 0$ on ∂D, there exists at least a one-parameter family of magnetostatic fields $\mathbf{b}^E(\mathbf{x},\lambda)$ that are topologically accessible from $\mathbf{b}^0$. We can now translate this to the language of Euler flows: given any kinematically possible flow $\mathbf{U}(\mathbf{x})$ in D, satisfying $\nabla \cdot \mathbf{U} = 0$ in D, $\mathbf{n} \cdot \mathbf{U} =$ on ∂D, there exists at least a one-parameter family of Euler flows $\mathbf{u}^E(\mathbf{x},\lambda)$ $(0 \le \lambda < \infty)$, with kinetic energy $K^E(\lambda)$ satisfying $dK^E/d\lambda < 0$.

We should perhaps emphasise that, for general three-dimensional configurations, these flows may exhibit tangential discontinuities (which are an inescapable feature of the magnetic relaxation problem as described in [1]). As λ increases, it seems likely (from physical considerations like those preceding (3.24)) that these discontinuities will become stronger and more densely distributed.

The limiting situation $\lambda \to \infty$ is of particular interest. For the MHD relaxation problem of section 3, this corresponds to the 'pressureless limit' and the equilibrium field in then a force-free field satisfying

$$(\nabla \times \mathbf{b}^E) \times \mathbf{b}^E = 0. \tag{4.3}$$

(Minimisation of magnetic energy subject to the constraints of magnetic helicity invariance is known to yield force-free fields [4,6].) The analogous Euler flow is then a Beltrami flow satisfying

$$(\nabla \times \mathbf{u}^E) \times \mathbf{u}^E = 0. \tag{4.4}$$

It is remarkable that there must exist a Beltrami flow that is topologically accessible from any kinematically possible flow $\mathbf{U}(\mathbf{x})$; but the above reservations concerning the probable singular character of the flow $\mathbf{u}^E(\mathbf{x},\lambda)$ as $\lambda \to \infty$ should be borne in mind.

5. DISCUSSION

It is an extraordinary fact that new insights concerning Euler flows, i.e., the steady flows of a fluid that is (i) inviscid (ii) incompressible and (iii) non-conducting, can be obtained, as described above, through consideration of unsteady relaxation processes in a fluid that is (i) viscous (ii) compressible and (iii) perfectly conducting, and through argument by analogy. The same results could be obtained without using the language of magnetohydrodynamics, but the formulation would appear artificial, since the subsidiary relaxation velocity field $\mathbf{v}(\mathbf{x}, t)$ that has to be introduced could be interpreted only in terms of mappings of the fluid domain (via (2.7)), and would lack simple physical interpretation.

The fact [1] that there is at least one Euler flow topologically accessible from an arbitrary solenoïdal flow $\mathbf{U}(\mathbf{x})$ is already remarkable since this immediately implies the existence of an uncountable infinity of topologically distinct Euler flows for a given domain. We have shown in the present paper that the complete family of Euler flows is wider still: for any solenoïdal $\mathbf{U}(\mathbf{x})$, there is a whole family $\mathbf{u}^E(\mathbf{x},\lambda)$ of Euler flows, each member of the family being topologically accessible from $\mathbf{U}(\mathbf{x})$ via mappings of the domain that are not volume-preserving.

REFERENCES

[1] Moffatt, H.K. - Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 1, Fundamentals. *J. Fluid Mech.*, **159**, 359-378.

[2] Moffatt, H.K. - Magnetic field generation in electrically conducting fluids. Cambridge Univ. Press, 1978

[3] Moffatt, H.K. - The degree of knottedness of tangled vortex lines. *J. Fluid Mech.*, **35**, 117-129.

[4] Woltjer, L. - A theorem on force-free magnetic fields. *Proc. Nat. Acad. Sci.*, **44**, 489-491, 1958.

[5] Arnold, V. - The asymptotic Hopf invariant and its applications. In: Proc. Summer School in Differential Equations. Armenian SSR Acad. Sci. (in Russian), 1974.

[6] Taylor, J.B. - Relaxation of toroidal plasma and generation of reverse magnetic fields. *Phys. Rev. Lett.*, **33**, 1139, 1974.