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Geophysical and Astrophysical Turbulence

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1. Introduction

Turbulence in large-scale geophysical and astrophysical contexts is dominated on the one hand by Coriolis forces associated with global rotation, and on the other by Lorentz forces associated with the currents that flow in the electrically conducting medium of astrophysical bodies and the magnetic fields to which these currents give rise.

Coriolis forces in the upper atmosphere of the Earth constrain the turbulence to be approximately two-dimensional in horizontal planes. Direct numerical simulation (e.g. McWilliams 1984) has proved effective in revealing the characteristic structures that emerge under evolution that is constrained to remain two-dimensional. There are two apparently conflicting theories of two-dimensional turbulence, the 'vortex patch' theory of Saffman (1971) which yields a k^{-4} energy spectrum, and the enstrophy cascade theory of Kraichnan (1967) and Batchelor (1969) which yields a k^{-3} spectrum. We shall argue that these theories are not irreconcilable, and that a k^{-4} spectrum will gradually evolve towards a k^{-3} spectrum through interaction of vortex patches leading to tightly wound-up spiral structures.

In section 3, we briefly review the theory developed in Moffatt (1985, 1986 a, b) which regards fully three-dimensional turbulence as an agglomeration of 'random vortex sheets and coherent helical structures'. The starting point is the Euler equation, regarded as a dynamical system evolving in an infinite-dimensional function space. The fixed points of this system are the steady solutions of the Euler equations (or 'Euler flows'), and by consideration of the problem of magnetic relaxation towards analogous magnetostatic equilibria, it may be shown that there exist Euler flows of arbitrarily complex topology. These flows are characterised by toroidal blobs within which the streamlines are ergodic and the flow has maximal helicity, separated by families of stream-vortex surfaces on which vortex sheets may be located. We tentatively identify the toroidal blobs as the 'coherent structures' of the turbulent flow that evolves from the Euler flow through Kelvin-Helmholtz instability of the vortex sheets. These coherent structures are persistent simply because the vorticity remains nearly parallel to the velocity so that nonlinear energy transfer to smaller scales remains weak. Viscous dissipation is concentrated in the vortex sheets; and the inertial range, with its associated Kolmogorov spectrum is to be found in the spiral structures arising from the Kelvin-Helmholtz instability (cf Lundgren 1982, Moffatt 1984).

In section 4, we consider briefly the process by which current-sheet discontinuities may appear during the magnetic relaxation problem mentioned above. This can occur even in two-dimensional situations, in which the topological constraints are more easily apprehended. The formation of current sheets has been noted in a number of direct numerical simulations of 2-dimensional MHD turbulence (e.g. Frisch et al. 1983) and is known to be associated with the presence of X-type (or hyperbolic) neutral points of the magnetic field, which of course occur randomly when the field is random.

In section 5, we consider what still appears to be the central problem of dynamo theory, at least in astrophysical contexts, namely determination of the parameters α (generation coefficient) and β turbulent diffusivity) in mean-field electrodynamics (see, for example, Moffatt 1978, Krause & Radler 1980) for turbulence with helicity in the limit of large magnetic Reynolds number R_m. If, as frequently maintained in the astrophysical literature,

$$\alpha \sim u_o \quad , \quad \beta \sim u_o \ell \quad \text{as} \quad R_m \to \infty \quad ,$$

where u_o is the rms turbulent velocity, and ℓ is a characteristic turbulent length-scale, then the mean field dynamo equations imply exponential growth of large-scale magnetic field with a growth rate p independent of R_m in the limit $R_m \to \infty$. The dynamo is then a 'fast' dynamo, in the sense of Vainshtein & Zel'dovich (1972). It has been argued by Moffatt & Proctor (1985) that, for the case of a steady velocity field, a fast dynamo has the property that the field becomes non-differentiable nearly everywhere as $R_m \to \infty$, and it seems probable that this property will carry over to unsteady (and turbulent) velocity fields also. The solar dynamo (for which $R_m \sim 10^4$) must inevitably be of this type. It is therefore of vital importance to clarify the behaviour of turbulent dynamos in the large R_m limit. The renormalisation-group procedure described by Moffatt (1983, section 11) still appears to be the best approach to adopt, although difficult to justify with any degree of rigour; this procedure is reviewed in section 5.

Finally, in Section 6, we consider certain features of what may be described as 'chromospheric turbulence', i.e. turbulence in the solar atmosphere outside the photosphere. Here, the density is low, and the dynamics is controlled by the magnetic field, whose field lines are 'rooted' on the solar surface. The footpoints move randomly with the sub-photospheric turbulent motion, and the field in the chromosphere is then instantaneously determined. Particular interest attaches to the question of formation of discontinuities (current sheets) in this region; such sheets are the site of strong Ohmic heating, and may provide an explanation for solar flares and similar intense bursts of activity. One mechanism by which discontinuities can appear is presented in section 6.

2. Spiral Structures in Two-dimensional Turbulence

It is well-known (see, for example, Aref 1983) that when 4 or more point vortices move under their mutual influence, the corresponding dynamical system is in general non-integrable, and each point vortex moves on a chaotic trajectory. It may therefore be readily appreciated that fluid particles in a two-dimensional turbulent field $\underline{u} = (u(x,y,t),\ v(x,y,t),\ o)$ will follow an equally chaotic path, and that any small 'patch' of fluid marked by dye will rapidly spread under the random distorting effect.

There are two apparently conflicting theories of two-dimensional turbulence; the first is the 'vortex patch' theory of Saffman (1971) which conceives of turbulence as the mutual interaction of a large number of patches in each of which the vorticity takes a constant value, a situation that persists since the vorticity equation reduces to $D\omega/Dt = 0$. The vorticity on any straight-line transversal then has a finite number of discontinuities per unit length, and so the vorticity spectrum $k^2E(k)$ falls off as k^{-2}. Viscous effects will of course provide an exponential cut-off for wave-numbers $k_v = \ell_v^{-1}$ characterising the real width of the 'discontinuities', but we suppose that viscosity is so weak that this effect may be ignored for all times of interest.

The second theory is the 'enstrophy cascade' theory of Kraichnan (1967) and Batchelor (1969). This starts from the enstrophy (or mean-square vorticity) equation,

$$\frac{d}{dt} \langle\omega^2\rangle = \eta - \nu\langle(\nabla\omega)^2\rangle , \tag{2.1}$$

where η is the rate of supply of enstrophy at low wave-numbers of order k_o (and equally the rate of dissipation of enstrophy when a statistically steady state is attained). It is then argued, in parallel with the Kolmogorov theory of 3-dimensional turbulence, that enstrophy cascades through the spectrum at a rate η towards the viscous sink at high k. On dimensional grounds, the energy spectrum in the inertial range $k_o \ll k \ll k_v$ $(\sim(\eta^{1/3}/\nu)^{1/2})$ is given by

$$E(k) = C_2 \eta^{2/3} k^{-3}, \qquad (2.2)$$

where C_2 is a universal constant (the suffix 2 referring to the two-dimensional situation). Kraichnan (1967) has indicated that refinements of the above argument indicate the need for logarithmic corrections to this result, but this can be ignored for the purpose of the present discussion

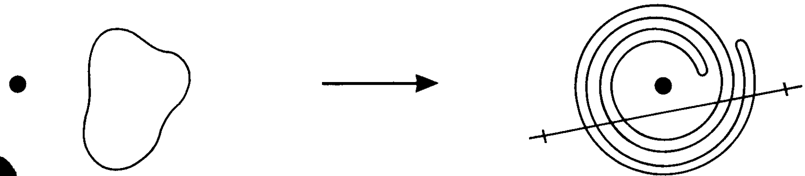

Fig. 1. Interaction of a small strong patch of vorticity and a weak large patch: the large patch is wound up into a tight spiral structure which makes many intersections with any straight line transversal.

We then have two theories, one giving a k^{-4} spectrum, the other a k^{-3} spectrum, in apparent conflict. How are we to reconcile these results?

Let us consider more closely the interaction of vortex patches. Suppose in particular that a small strong patch strays into the neighbourhood of a large weak patch (Fig. 1). The large patch is then distorted like a passive scalar in the r^{-1} velocity field of the small patch (which may be treated like a point vortex). The large patch is thus wound up into a spiral of increasingly fine-structure, its area being conserved. A transversal passing near the centre of the patch now 'sees' a large number N of discontinuities of vorticity per unit length, and $N \to \infty$ as $t \to \infty$. In calculating the spectrum, we are thus dealing with the Fourier transform of a function having an accumulation point of discontinuities, a situation similar to that considered in the context of 3-dimensional turbulence by Moffatt(1984). The resulting spectrum falls off as $k^{-\lambda}$ where $\lambda < 4$ over a range of wave-numbers (the 'spiral range') characterising this type of spiral structure which may be supposed to be randomly distributed. Detailed calculations (A.D. Gilbert 1986, in preparation) have shown that $\lambda = {}^{11}/_{3}$, account being taken of all possible positions of the transversal. The spectrum is then as indicated in Fig. 2.

This of course only brings us part way towards reconciling the k^{-4} and k^{-3} predictions. Observe now however that the same effect may

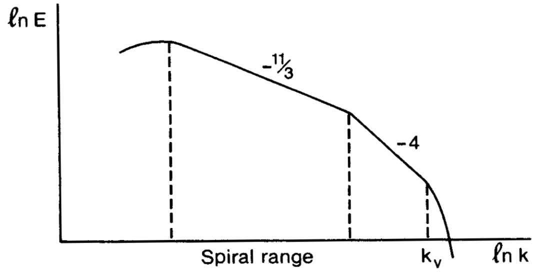

Fig. 2. Form of energy spectrum resulting from random distribution of spiral vortex structures; a spiral range with spectral exponent -11/3 appears (A.D. Gilbert 1986).

be repeated, as indicated in Fig 3: if another strong vortex patch should wander into the neighbourhood of an already formed spiral structure, then it will induce a secondary spiral deformation, thus increasing further the spatial complexity of the vorticity field and leading to a decrease in the relevant index λ in the spiral range. Repeated spiral distortion of this kind will lead to a vorticity field structure that is self-similar on all scales, and it is this property that characterises a k^{-3}-spectrum. We thus conjecture that a k^{-3}-spectrum will emerge from repeated spiral distortions of the kind indicated above. A possible reconciliation between the k^{-4} and k^{-3} theories may thus be recognised.

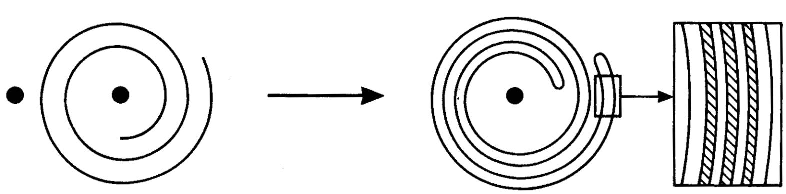

Fig. 3. Interaction of a second strong vortex with a pre-existing spiral structure: a secondary spiral is formed in which the beginnings of a fractal structure may be discerned.

We note that substantial evidence for the 'stacking' of vorticity discontinuities is provided by the numerical simulations of Brachet et al (1986), presented in the poster session at this meeting. The formation of spirals is one mechanism by which such stacking may be understood.

3. Fixed Points of the Euler Equations

Turning now to three-dimensional turbulence, it is natural to consider the general structure of solutions of the Euler equations ($\nu = 0$), recognizing that vortex sheets may be present in such solutions, and that viscosity will eventually resolve the structure of such sheets (cf. Townsend 1951). The Euler equations may be written in the form

$$\frac{\partial \underline{u}}{\partial t} = \underline{u} \times \underline{\omega} - \nabla h, \qquad (3.1)$$

where $\underline{\omega}$ = curl $\underline{u}$ and $h = {}^{p}/_{\rho} + \frac{1}{2}\underline{u}^2$, and we regard this as an infinite-dimensional dynamical system, analogous to a finite dimensional system of the form

$$\frac{d\underline{u}}{dt} = \underline{F}(\underline{u})$$

where $\underline{u} = (u_1(t), u_2(t), \ldots u_n(t))$. From such a viewpoint, it is natural to locate first the fixed points of the system (where $\underline{F}(\underline{u}) = 0$), and then to consider the stability of these fixed points. Even if they are all unstable, and therefore in a sense unphysical, their location provides valuable information about the evolution of the system in its phase-space, as reference to such well-known examples as the Lorenz system will make clear.

The fixed points $\underline{u}^E(\underline{x})$ of the Euler equations, described as Euler flows, are solutions of the steady equations

$$\underline{u} \times \underline{\omega} = \nabla h. \qquad (3.2)$$

An approach, suggested first by Arnold (1974), has been developed by Moffatt (1985, 1986 a, b), and is remarkably powerful in revealing an incredible richness in the structure of Euler flows. We first note the well-known exact analogy with the problem of magnetostatics of a perfectly conducting fluid

$$\underline{j} \times \underline{B} = \nabla p \qquad (3.3)$$

with $\underline{j}$ = curl $\underline{B}$, $\nabla \cdot \underline{B} = 0$. Here, $\underline{B}$ represents magnetic field, and $\underline{j}$ electric current. The analogy is between the variables

$$\underline{u} \longleftrightarrow \underline{B}\,, \quad \underline{\omega} \longleftrightarrow \underline{j}, \quad h \longleftrightarrow -p. \qquad (3.4)$$

Location of solutions of (3.3) lends itself to a method of 'magnetic relaxation' that is not available (without introducing very artificial physics) for the problem (3.2). Nevertheless, to every solution of (3.3) found in this way, there corresponds an Euler flow, via the exact analogy (3.4). The analogy does not extend to questions of stability of the steady states thus determined.

The magnetic relaxation argument proceeds as follows. We suppose that $\underline{B}_o(x)$ is an arbitrary initial field (at $t = o$) in a domain D (which may be infinite) and that the fluid is perfectly conducting (so that $\underline{B}$-lines are frozen in the fluid) but viscous (so that energy is dissipated when the fluid moves). Such assumptions are physically artificial, but no matter: they are merely an artifice, a means towards an end. In general the Lorentz force $\underline{j} \times \underline{B}$ is rotational at time $t = o$, so thefluid will move. The total energy (magnetic plus kinetic) then decreases, but subject to the vital constraint that the topology of the $\underline{B}$-field is conserved. As $t \to \infty$, the viscous dissipation must tend to zero, and so the velocity field must everywhere tend to zero. In the final stages of this process, discontinuities of $\underline{B}$ may form through a 'squeeze-film' mechanism in which flux tubes embrace each other in increasingly close contact. The final field $\underline{B}^E(\underline{x})$ is not for this reason strictly topologically equivalent (or homeomorphic) to the initial field $\underline{B}_0(\underline{x})$, but it is nevertheless 'topologically accessible' from it, being obtained by the convective action of a smooth velocity field $\underline{u}(\underline{x},t)$ $(0 < t < \infty)$ of finite total dissipation.

This argument implies that to every field $\underline{B}_0(\underline{x})$, there corresponds at least one magnetostatic equilibrium $\underline{B}^E(\underline{x})$ topologically accessible from it, the lines of force, magnetic surfaces, and ergodic regions of $\underline{B}_0$ mapping faithfully to those of $\underline{B}^E(\underline{x})$. By the analogy referred to above, we may then make a similar statement about Euler flows: to every solenoidal velocity field $\underline{U}(\underline{x})$, there corresponds at least one Euler flow $\underline{u}^E(\underline{x})$, whose streamlines may be obtained by distortion of the streamlines of $\underline{U}(\underline{x})$ by an (imagined) velocity field $\underline{v}(\underline{x},t)$ $(0<t<\infty)$ of finite total 'dissipation integral', i.e.

$$\int_0^\infty \int_D \left(\frac{\partial v_i}{\partial x_j}\right)^2 dV\, dt < \infty . \tag{3.5}$$

We emphasise that the Euler flow $\underline{u}^E(\underline{x})$ will in general contain vortex sheets (which may well be stacked on top of each other, or exhibit other curious features) even although the 'reference' field $\underline{U}(\underline{x})$ is everywhere differentiable. With this qualification, it may be confidently asserted that there is an uncountable infinity of topologically distinct Euler flows in any domain D.

Consider now the 'generic' structure of such a flow, satisfying (3.2). Obviously

$$\underline{u}.\nabla h = 0, \quad \underline{\omega}.\nabla h = 0 , \tag{3.6}$$

so that both streamlines and vortex lines lie in surfaces $h = cst$. It may however happen that in some subdomain D' of D, h is identically constant, so that $\underline{u} \times \underline{\omega} \equiv 0$ in D', i.e.

$$\underline{\omega} = \alpha(\underline{x})\, \underline{u} \quad \text{in } D' \tag{3.7}$$

for some scalar field $\alpha(\underline{x})$, satisfying

$$\underline{u}.\nabla\alpha = 0 \tag{3.8}$$

since both $\underline{u}$ and $\underline{\omega}$ are solenoidal. Hence now the streamlines (coinciding with the vortex lines) lie on surfaces $\alpha = cst$. Yet again, it may happen that, in some subdomain D_1 of D', α is identically constant, so that

$$\underline{\omega} = \alpha_1\, \underline{u} \quad \text{in } D_1\ , \ \alpha_1 = cst. \tag{3.9}$$

Only in this circumstance can the streamlines be ergodic in D_1 (or in some subdomain of D_1). Conversely, if we know that the streamlines of $\underline{u}$ are ergodic in a subdomain D_1, then the Beltrami property (3.9) must be satisfied. This is important, because the arbitrary reference field $\underline{U}(\underline{x})$ will in general have ergodic regions which will map into corresponding regions D_1, D_2, - - -, in which the associated Euler flow is likewise ergodic. The picture is indicated schematically in Fig. 4: blobs of maximal helicity ($\underline{\omega} = \alpha_n \underline{u}$) in which the streamlines are ergodic are separated by families of stream-vortex surfaces, on which, through confluence of stream-surfaces, vortex sheets may be located.

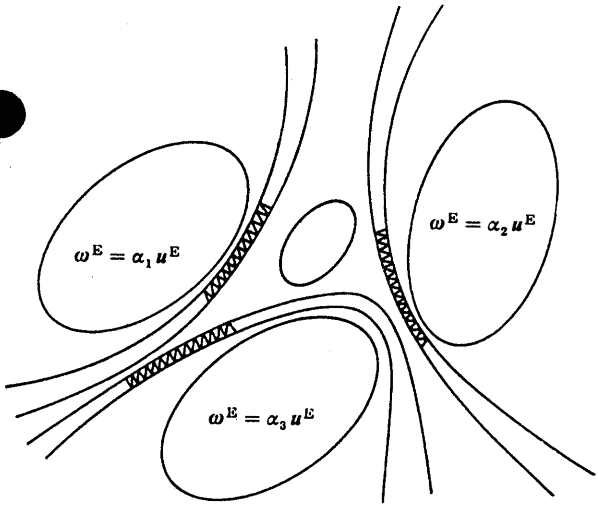

Fig. 4. Generic structure of Euler flow: ergodic blobs are separated by families of stream-vortex surfaces on which vortex sheets may be located.

We have observed above that the analogy between Euler flows and magnetostatic equilibria does not extend to questions of stability. The magnetostatic equilibria, arrived at through a relaxation process, may be expected to be stable within the perfect conductivity framework, the magnetic energy being minimal in the subspace of topologically accessible fields. The analogous Euler flow $\underline{u}^E$ may however be unstable in the different subspace of velocity fields whose vorticity fields $\underline{\omega}$ are topologically accessible from $\underline{\omega}^E$. This dichotomy has been explored in detail (Moffatt 1986a) where it is shown, by way of example, that the ABC-flow studied by Dombré et al (1986) is unstable to large-scale helical perturbations, whereas the analogous magnetostatic equilibrium is stable.

When vortex sheets are present in the Euler flow, these are of course subject to the fundamental Kelvin-Helmholtz instability leading to intrinsic and inescapable wind-up into the familiar double spiral structures. When the wave-crests of the instability are inclined to the vorticity in the sheet as is inevitable for a general toroidal topology (Moffatt 1984), the instability stretches the vorticity perpendicular to the crests, so that enstrophy increases (unlike the two-dimensional situation); this of course is an essential feature of 3-dimensional turbulence.

The elemental picture of random vortex sheets (Townsend 1951) gives a k^{-2} energy spectrum, modified by an exponential cut-off on the inner (Kolmogorov) scale. We conjecture that the double spiral wind-up associated with the Kelvin-Helmholtz instability, possibly occurring on a succession of decreasing length-scales, will convert this spectrum to the slower $k^{-5/3}$ fall-off required by the cascade (similarity) theory of Kolmogorov (see, for example, Batchelor 1953) in parallel with the corresponding 2-dimensional problem described in §2 above. Here, the picture is close to that of Lundgren (1982) who has succeeded in extracting a $k^{-5/3}$-spectrum from a 'strained spiral vortex' model of turbulence.

As regards the ergodic blobs of maximal helicity, we may tentatively associate these with the 'coherent structures' that have been identified in a variety of turbulent flows. To be sure, the condition $\underline{\omega} = \alpha_n \underline{u}$ is no longer exactly satisfied in the blob D_n when perturbed by the influence of Kelvin-Helmholz instabilities in adjacent layers; but if, in some sense, the real turbulent flow follows a trajectory (in function space) around an unstable fixed point (as suggested by the behaviour of simpler finite-dimensional systems) then ergodic blobs in which $\underline{\omega}$ is nearly parallel to $\underline{u}$ are to be expected; within these blobs, nonlinear energy transfer represented by the term $\underline{u} \times \underline{\omega}$ of the Euler equation (3.1) is then weak, and so they will be relatively persistent, and therefore identifiable. Independent evidence has been provided by Tsinober & Levich (1983) that coherent structures in real turbulent flows do indeed generally exhibit non-zero helicity in the appropriate co-moving frame of reference; and (Shtilman et al 1985) that coherent structures of non-zero helicity (positive and negative) do emerge in direct numerical simulations of certain space-periodic flows, of which the Taylor-Green vortex may be regarded as the prototype (see also Levich 1985).

4. Appearance of Discontinuities in the Magnetic Relaxation Problem

The appearance of discontinuities is a very important aspect of the magnetic relaxation problem, foreshadowed by the discussion of Arnold (1974) who simply pointed out that the problem would in general have no solution 'within the class of smooth vector fields'. Discontinuities form even in two-dimensional situations, by a mechanism that we shall now indicate.

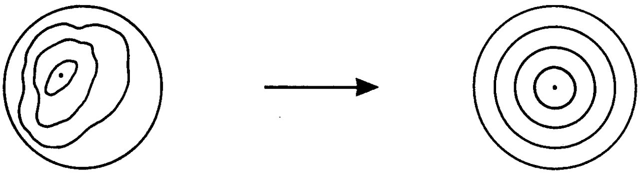

Fig. 5. Relaxation of field with one elliptic neutral point: each line of force becomes a circle.

Consider first the very simple topology indicated in Fig. 5, in which the initial field $\underline{B}_0(\underline{x})$ within a cylinder of circular cross-section has a single elliptic (or 'O-type') neutral point in the interior of the domain. Magnetic relaxation proceeds through contraction of $\underline{B}$-lines, but here subject to the essentially topological constraint that the area within each convected field-line remains constant. Equilibrium is attained when the length of each $\underline{B}$-line is minimised subject to this constraint, i.e. when each $\underline{B}$-line is circular; moreover these circles must be concentric with the cylinder in order to equalise the magnetic pressure along $\underline{B}$-lines. There is thus a unique equilibrium towards which the field relaxes, and this has no discontinuities.

Consider now the more complex topology of figure 6 in which the initial field $\underline{B}_0(\underline{x})$ has two elliptic neutral points and one hyperbolic (or 'X-type') neutral point in the interior of the domain. Again, each field line would like to become circular in order to minimise length subject to constant enclosed area, but this is no longer compatible with the topology. The system must seek an alternative route to equilibrium. The disposition of Lorentz forces in the neighbourhood of the X-type neutral point is such as to eject fluid from the acute angles, and so to decrease these angles. This process continues until (as $t \to \infty$) a field discontinuity (or current sheet) must form. This effect has been confirmed numerically by Bajer (1986, in preparation) who finds an asymptotic configuration with cusped structure as indicated in Fig. 6.

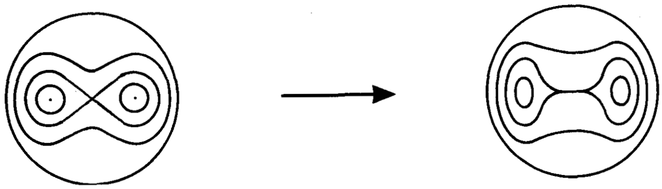

Fig. 6. Relaxation of field with two elliptic neutral points and one hyperbolic point: a current sheet discontinuity forms with a cusped structure at its end-points (K. Bajer 1986).

For a more general topology, and in particular for a random magnetic field, such behaviour is to be expected in the neighbourhood of every X-type neutral point. This indeed is the behaviour that has been revealed in the direct numerical simulations of Frisch et al (1983). These simulations were carried out with non-zero magnetic diffusivity η (in fact with $\eta = \nu$), but the tendency to form near-discontinuities was nevertheless apparent.

Formation of current sheet discontinuities is an important process in the solar atmosphere, as persuasively argued by Parker (1979, chap. 14) who considers the possible structure of force-free fields and of magnetostatic equilibria of prescribed non-trivial topology, with the conclusion that in general these must exhibit discontinuities. This is entirely consistent with the point of view developed in the present paper. We return to the solar atmosphere context in section 6 below.

5. The Fast Turbulent Dynamo

As indicated in the introduction, a turbulent dynamo acting in the convection zone of any late-type star (e.g. the Sun), and in which the magnetic Reynolds number is of order 10^4 or greater, must presumably be of 'fast' type in the sense of Vainshtein & Zel'dovich (1972). The growth rate of such a dynamo (neglecting back-reaction effects) is independent of R_m in the limit $R_m \to \infty$, i.e. it exhibits a 'magnetic Reynolds number similarity' analogous to the Reynolds number similarity that is familiar in conventional turbulence. It has been shown by Moffatt & Proctor (1985) that if the magnetic diffusivity η is zero (i.e. $R_m = \infty$), then dynamo action is impossible with a steady velocity field $\underline{u}(\underline{x})$, even if this is a random function of position. It seems likely that this is true also for unsteady (turbulent) fields $\underline{u}(\underline{x},t)$, although opinions are divided on this matter.

It seems intuitively clear that, as $R_m \to \infty$, the magnetic field fluctuations will have progressively finer scale, and will inevitably be larger (by some positive power of R_m) than the mean field. This has been used by Piddington (1972) as an argument against the applicability of the methods of mean-field electrodynamics developed by Steenbeck, Krause & Rädler (1966) (for a comprehensive account of this and subsequent papers, see Krause & Rädler 1980); the criticism is however not sound: the mean-field method does not require that the fluctuation be small compared with the mean. This is a requirement of the frequently used 'first-order smoothing' approximation (see Moffatt 1978, Chap.7) which is certainly not valid when $R_m >> 1$. What is needed when $R_m >> 1$ is an alternative approach, still within the general framework of the mean-field method. This means essentially that we need a good theory for the determination of the 'generation coefficient α (in the 'α-effect') and the turbulent magnetic diffusivity β, which remains valid no matter how large R_m may be.

One possible approach, which seems to incorporate the right physics, involves the renormalisation group technique, as described by Moffatt (1983). Here we simply review the elements of this procedure. The energy spectrum E(k) and the helicity spectrum H(k) (on which the α-effect depends) are first discretised, and first order smoothing is applied to the smallest scales (or largest wave-numbers), whose associated magnetic Reynolds number $R_m(k)$ is small. This yields a 'mean-field' equation involving an α-effect and a (weak) turbulent diffusivity which supplements the (weak) molecular diffusivity. This process is then repeated: another 'bite' is taken of the energy and helicity spectra, these being progressively devoured from the '$k = \infty$' end. The mean-field equation is modified only through change in the values of α and β (i.e. through renormalisation). After n similar steps, we obtain values α_n, β_n in terms of α_{n-1}, β_{n-1}, $E(k_n)$ and $H(k_n)$. Now, we make the essential assumption, which is hard to justify with any rigour, that cumulative errors are small, and that we may return to continuum (rather than discrete) spectra. This yields differential equations for $\alpha(k)$ and $\beta(k)$, the effective α and β associated with all scales of motion smaller than k^{-1}.

These equations are

$$\left.\begin{aligned} \frac{d\alpha}{dk} &= \frac{1}{3}\,\frac{2\alpha E(k) + (\eta+\beta)H(k)}{k^2(\eta+\beta)^2 - \alpha^2}\,, \\ \frac{d\beta}{dk} &= -\frac{1}{3}\,\frac{2(\eta+\beta)E(k) + \alpha k^{-2}H(k)}{k^2(\eta+\beta)^2 - \alpha^2}\,, \end{aligned}\right\} \qquad (5.1)$$

and these are to be integrated in from $k = \infty$, with 'initial' conditions $\alpha(\infty) = 0$, $\beta(\infty) = 0$.

The interesting thing now is that a little molecular diffusivity η is needed to, as it were, get the system started: if $\eta = 0$, the first step of the smoothing process cannot be justified. At the same time, if $\eta > 0$ (no matter how small), then $|\alpha(k)|$ increases as k decreases to a maximum value $|\alpha(k)|_{max}$ independent of η. In fact, $\eta+\beta$ may be eliminated from (5.1) giving

$$\frac{d}{dk}\alpha^2 = \frac{2\alpha^2(2\alpha^2 E - hH)}{3(kh - \alpha^2)(kh + \alpha^2)} \tag{5.2}$$

where $h(k) = \frac{1}{3}\int_k^\infty k^{-2} H(k)\, dk.$ (5.3)

When k is large (and $|\alpha|$ small), (5.2) becomes

$$\frac{d}{dk}\alpha^2 = \frac{-2\alpha^2 H}{3k^2 h} . \tag{5.4}$$

Suppose, for example, that $H(k) \sim C k^{-m}$ for large k, so that $h(k) \sim \frac{1}{3}C\,(m+1)^{-1} k^{-(m+1)}$, and

$$\frac{d}{dk}\alpha^2 \sim -2k^{-1}(m+1)\alpha^2 . \tag{5.5}$$

Then $\alpha^2 \sim k^{-2(m+1)}$ as $k \to \infty$. (5.6)

Thus, as k decreases from ∞, α^2 will 'latch on' to this solution. The typical behaviour of α^2 is indicated in figure 7.

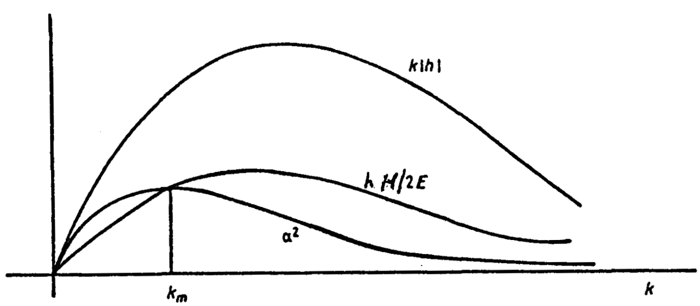

Fig. 7. Form of the function $\alpha^2(k)$ as determined by integration of equation (5.2).

In a real system such as the Sun, it is of course unrealistic to pursue the integration all the way to k = 0. Rather, one should stop around the wave-number k_m where α^2 is maximal and use equations (5.1) to determine $\alpha(k_m)$, $\beta(k_m)$ (this is just a form of sub-grid modelling). Scales larger than k_m^{-1} should ideally be treated by full numerical simulation, in the manner of Gilman & Miller (1981).

6. Chromospheric Turbulence, and the Formation of Discontinuities

Outside the visible surface of the Sun (the photosphere) the density of the solar atmosphere falls off very rapidly, and the dynamics is totally dominated by the magnetic field emanating from the surface.

This field presumably adjusts itself more or less instantaneously to be force-free ($\underline{j} \times \underline{B} = 0$), since otherwise infinite accelerations would occur. Actually, the process by which this adjustment occurs in response to a changing photospheric field is not at all clear; let us assume however, following Low (1977), that it does occur and that the field in the chromosphere is essentially force-free at all times. The field lines emerge from the photosphere, where they are rooted to the sub-photospheric plasma which may be assumed perfectly conducting on the time-scale characteristic of solar magnetic activity. This field has a complex random structure, which is constantly changing as the footpoints of the field-lines move with the plasma.

Adopting, for simplicity, a two-dimensional model of this field, we may expect a random distribution of O-type and X-type neutral points below the photosphere. The field topology in the neighbourhood of any X-type neutral point is as indicated in figure 8. Clearly one field line ABC above the neutral point must touch the photosphere (at B). Consider now two field lines A'C' and $A_1B_1B_2C_2$ just above and below this 'critical' field line. As the footpoints move due to sub-photospheric turbulence, the field instantaneously adjusts itself to remain force-free. This means that the field in the chromosphere is instantaneously determined by the motion of the footpoints, i.e. by the motion of A' and C' for the field line A'C', and by A_1 and B_1 for the field line A_1B_1. Since the motions of B_1 and C' are quite independent, we must therefore expect in general that a tangential discontinuity (or current sheet) will form on AB and likewise on BC. Detailed calculation confirms this conclusion.

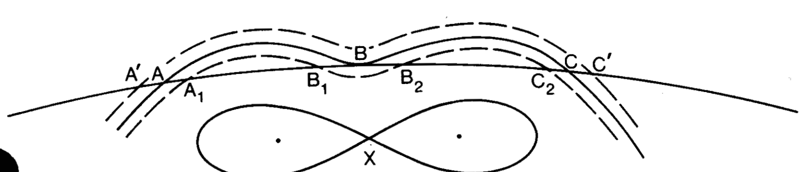

Fig. 8. Field structure above a hyperbolic neutral point just below the photosphere: when the footpoints move, a current sheet appears on the critical line of force ABC.

The importance of X-type neutral points in the chromospheric context was first pointed out by Sweet (1969) and Syrovatsky (1966) — see also Priest (1982, chap.10). The fact that such neutral points below the photosphere may equally have an important influence on chromospheric activity has not however been previously recognized. Current sheet discontinuities, and the means by which they may appear, have an important bearing on the phenomenon of solar flares and similar bursts of magnetic activity on the Sun.

8. Summary

This review of geophysical and astrophysical turbulence has necessarily been very selective, and we have focussed mainly on aspects of both two-dimensional and three-dimensional turbulence that may fairly be described as topological in character. When a magnetic field is present (as it almost inevitably is in a high conductivity plasma) the fact that $\underline{B}$ - lines are frozen in the fluid on time-scales small compared with the (long) diffusion time makes topological (as opposed to purely analytic) arguments even more appealing. This point of view has been urged by Parker (1979) and others; and the fact that helicity, already known to be of vital importance in the dynamo context, has a topological interpretation, lends further force to this standpoint. The method described in §3, involving topological ideas in an essential way, is remarkably powerful in providing new insights both for the problem of turbulence, and for certain classical problems of laminar flow theory (Moffatt 1986b). It seems that we may be on the threshold of major advances in basic understanding, which will have a profound influence on further analytical and numerical work.

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