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Topological Approach to Problems of Vortex Dynamics and Turbulence

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Abstract

Steady solutions of the Euler equations (i.e., Euler flows) are important in the context of turbulence because they represent the fixed points of the governing dynamical system evolving at high Reynolds number in the function space of solenoidal vector fields of finite energy density. These Euler flows are exactly analogous to magnetostatic equilibria that may be located in the function space by relaxation techniques. By considering relaxation in a perfectly conducting, but viscous, medium and by exploiting the above analogy, the existence of Euler flows of arbitrary streamline topology may be demonstrated. These flows have a characteristic structure, which may be interpreted in the language of turbulence: coherent structures in which the helicity is large are separated by stream-vortex surfaces on which vortex sheets may be located. Kelvin-Helmholtz instability of the vortex sheets provides an inertial-range spectrum and intermittent dissipative structures.

I. Introduction

There are four features of turbulent flow that are more or less universal and that any theory of turbulence should seek to explain:

A. Coherent Structures

These are identifiable structures of length scale and time scale large compared with the scales characterizing the energy-containing

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eddies of the turbulence (for a recent review, see Ref. 1); it is necessary to understand why these structures form and how they persist in the presence of the random strong perturbations associated with the background turbulent velocity field $\mathbf{u}(\mathbf{x}, t)$.

B. Intermittency of Dissipation

The local rate of dissipation of energy per unit mass, $\epsilon = \nu(\partial u_i / \partial x_j)^2$, is intermittent in the following sense: if ϵ_0 is arbitrarily small compared with the mean $\bar{\epsilon}$, then the fraction V_0/V of any volume V of fluid within which $\epsilon > \epsilon_0$ tends to zero as the Reynolds number Re tends to infinity. This means that as $Re \rightarrow \infty$, then $\epsilon(\mathbf{x}, t)$ becomes increasingly spiky, being concentrated almost entirely in this vanishingly small proportion V_0 of the volume V .

C. Inertial Range Spectrum

The universal equilibrium theory of Kolmogorov predicts, on dimensional grounds, an inertial range energy spectrum

$$E(k) = C\bar{\epsilon}^{2/3} k^{-5/3} \quad (1)$$

for $k_0 \ll k \ll Re^{3/4} k_0$, where k_0 is a wave-number characteristic of the energy-containing eddies. This spectrum is weakly modified by the intermittency effect mentioned above (e.g., Frisch et al.,²) but the modification still yields a power-law spectrum $E(k) \sim k^{-\lambda}$ where $\lambda < 2$. The question immediately arises: What characteristic structures in \mathbf{x} space can possibly yield a fractional power law of this kind? The "strained spiral vortex model" of Lundgren⁽³⁾ comes some way towards providing an answer to this central question. Ideally, we would like to see such a model, which applies to inertial range dynamics, embedded within a broader theory that encompasses energy-containing scales and coherent structures also.

D. Enstrophy Production

The enstrophy $\langle \omega^2 \rangle$, where $\omega(\mathbf{x}, t)$ is the vorticity field, is given in homogeneous turbulence by the exact relation

$$\langle \omega^2 \rangle = \bar{\epsilon} / \nu \sim Re \langle \mathbf{u}^2 \rangle \ell_0^{-2} \quad (2)$$

so that, for a given level $\bar{\epsilon}$ of energy dissipation (or equivalently, a given level of energy density $\frac{1}{2} \langle \mathbf{u}^2 \rangle$), $\langle \omega^2 \rangle \rightarrow \infty$ as $\nu \rightarrow 0$. The

equation for enstrophy⁴ is

$$\frac{d}{dt} \langle \omega^2 \rangle = \langle \omega_i \omega_j \frac{\partial u_i}{\partial x_j} \rangle - \nu \langle (\nabla \times \omega)^2 \rangle \quad (3)$$

and the production term $\langle \omega_i \omega_j \partial u_i / \partial x_j \rangle$ is associated with stretching of vortex lines as these are distorted by the flow. This stretching of vortex lines is presumably associated with the tendency of line elements, on average, to increase in length,⁵ but this type of argument is, in itself, too simplistic (it does not apply, for example, to two-dimensional turbulence, for which line elements parallel to the vorticity field do not increase in length); and what is needed is a dynamical model in which the increase of enstrophy is (by some explicitly three-dimensional mechanism) an essential and natural ingredient. Lundgren's model, referred to above, does incorporate vortex stretching, and it is interesting that this requirement appears to be intimately connected with the extraction of a Kolmogorov spectrum.

In this paper, we review an approach to the problem of turbulence⁶⁻⁸ that provides a possibility of comprehending all four aspects of the problem, as described above, within a single coherent framework. We visualize a turbulent velocity field $u(x, t)$ as evolving in the function space of all solenoidal vector fields of finite energy density. As t increases, the turbulent state is represented by a point that follows a trajectory in this function space. Of particular interest are the fixed points of the associated dynamical system, and (in the limit $Re \rightarrow \infty$) these are the steady solutions of the Euler equations, or 'Euler flows' $u^E(x)$, which may have embedded tangential discontinuities (i.e., vortex sheets) or other singularities, provided these satisfy the constraint of finite energy density (e.g., concentrated line vortices are excluded).

The existence of vortex sheets in nearly all Euler flows of any structural complexity indicates that these flows will generally be unstable to local instabilities of the Kelvin-Helmholtz type (e.g., Ref. 9, Chapter 7) i.e., the fixed points in the function space are generally unstable; we should hardly expect otherwise because unsteadiness is an essential and inescapable feature of the turbulence problem. That fixed points are unstable does not, however, make them any less interesting from the point of view of understanding the global structure of trajectories in the function space. For many low-order dynamical systems having only unstable fixed points, location of these fixed points is nonetheless an essential preliminary to mapping out the strange attractor (or other limit set) to which trajectories asymptote. To take

an even simpler example, a compound pendulum moving *with enough energy* for its center of gravity to describe complete circles in a vertical plane will spend more time in the neighbourhood of its position of unstable equilibrium (vertically upward) than it spends near its position of stable equilibrium (vertically downward); and a time average will reveal the characteristics of the unstable (rather than the stable) equilibrium! It would therefore be foolish to ignore the unstable fixed points, particularly when the energy level of the system is high.

II. The Helicity Invariant

It is important for what follows that the Euler equations of inviscid flow

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p \quad (4)$$

with $\nabla \cdot \mathbf{u} = 0$ and $\rho = \text{const}$ (density) admit an integral invariant for every volume V bounded by a surface S moving with the fluid on which $\omega \cdot \mathbf{n} = 0$, namely,

$$\mathcal{H} = \int_V \mathbf{u} \cdot \boldsymbol{\omega} dV \quad (5)$$

This invariant, the helicity of the flow within V , is associated with the degree of linkage, or knottedness, of the vortex lines in V ¹⁰⁻¹² and therefore has topological significance: it provides a bridge between fluid mechanics and isotopy theory in topology.

It is equally important that a similar invariant exists for any solenoidal vector field that is frozen in the fluid; the prototype example here is the magnetic field $\mathbf{B}(\mathbf{x}, t)$ in a perfectly conducting fluid, for which the invariant analogous to \mathcal{H} is the *magnetic helicity*¹³

$$\mathcal{H}_M = \int_V \mathbf{A} \cdot \mathbf{B} dV \quad (6)$$

where $\mathbf{B} = \nabla \times \mathbf{A}$, and $\mathbf{B} \cdot \mathbf{n} = 0$ on S .

The helicity satisfies the Schwarz inequality

$$|\mathcal{H}| \leq \left\{ \int \mathbf{u}^2 dV \int \boldsymbol{\omega}^2 dV \right\}^{\frac{1}{2}} \quad (7)$$

Moreover, if the flow is confined to a fixed bounded domain D , then we have a Poincaré inequality,

$$\int_D \omega^2 dV \geq q_0^2 \int_D u^2 dV \quad (8)$$

where q_0 depends only on the geometry of D . Hence,

$$\int_D \omega^2 dV \geq q_0 |\mathcal{K}| \quad (9)$$

and, by the same token, in the magnetic context,

$$\int_D \mathbf{B}^2 dV \geq q_0 |\mathcal{K}_M| \quad (10)$$

provided $\mathbf{B} \cdot \mathbf{n} = 0$ on ∂D . The significance of this lower bound on the energy associated with a frozen-in field was noted by Arnold¹²: it has an important bearing on the question of existence of Euler flows (i.e., fixed points of the Euler system), as will now be described.

III. Fixed Points of the Euler Equations Regarded as a Dynamical System

The Euler equations (4) may be written in the equivalent form

$$\partial \mathbf{u} / \partial t = \mathbf{u} \times \boldsymbol{\omega} - \nabla h \quad (11)$$

where $h = p/\rho + \frac{1}{2}u^2$. Thus, Euler flows $\mathbf{u}(\mathbf{x})$ satisfy

$$\mathbf{u} \times \boldsymbol{\omega} = \nabla h \quad (12)$$

There is a very well-known exact analogy between this equation and the equation of magnetostatic equilibrium in a perfectly conducting fluid, namely,

$$\mathbf{j} \times \mathbf{B} = \nabla p \quad (13)$$

The great significance of this analogy is that it provides a means of constructing a wide class of solutions of Eq. (12) of arbitrarily complex topological structure. The details are given in Moffatt,⁶ and only the briefest outline will be given here. Let $\mathbf{B}_0(\mathbf{x})$ be a solenoidal field of arbitrarily complex structure and generally not satisfying the condition of magnetostatic equilibrium.¹³ Suppose that this field is

embedded in a *viscous* but *perfectly conducting* fluid (a mathematical idealization, which is nonetheless appropriate to the purpose) and that the fluid is initially at rest. Obviously the Lorentz force $\mathbf{j} \times \mathbf{B}$ will cause the fluid to move, with velocity $\mathbf{v}(\mathbf{x}, t)$, say, for $t > 0$, and energy will be dissipated by viscosity for as long as this motion continues. The total energy therefore decreases monotonically, but is bounded below by virtue of the inequality (10) associated with nontrivial (conserved) topology of the frozen-in field. Hence, ultimately, as $t \rightarrow \infty$, $\mathbf{v}(\mathbf{x}, t)$ must tend to zero everywhere, and

$$\mathbf{B}(\mathbf{x}, t) \rightarrow \mathbf{B}^E(\mathbf{x}) \quad (14)$$

where $\mathbf{B}^E(\mathbf{x})$ is a magnetostatic equilibrium field that is *topologically accessible* from the initial field $\mathbf{B}_0(\mathbf{x})$. The analogy between Eqs. (12) and (13), and specifically between \mathbf{B} in the magnetic problem and \mathbf{u} in the Euler flow problem, permits us to conclude that if $\mathbf{U}(\mathbf{x})$ is a kinematically possible flow of arbitrarily complex structure, then there exists at least one Euler flow $\mathbf{u}^E(\mathbf{x})$ that is topologically accessible from $\mathbf{U}(\mathbf{x})$, in the sense that the streamlines of \mathbf{u}^E may be obtained from the streamlines of \mathbf{U} by a frozen-field distortion associated with a subsidiary velocity field $\mathbf{v}(\mathbf{x}, t)$ ($0 < t < \infty$) of finite total dissipation.

The process of magnetic relaxation (14) may, and in general does, involve the appearance of tangential discontinuities where magnetic surfaces are squeezed together by opposing Lorentz forces. The resulting magnetostatic equilibria therefore involve current sheets embedded within them but are nevertheless stable within the 'perfectly conducting' framework because the magnetic energy will obviously be minimal with respect to frozen-field perturbations about the state $\mathbf{B}^E(\mathbf{x})$.

The analogous Euler flows $\mathbf{u}^E(\mathbf{x})$ have vortex sheets embedded within them, which are presumably subject to the Kelvin-Helmholtz instability. There is no conflict here with the analogy between Eqs. (12) and (13) because this analogy applies strictly to the equilibrium states but not to questions of stability about these states.

IV. Structure and Stability of Euler Flows

The argument in sec. III establishes the existence of a wide class of fully three-dimensional solutions of the steady Euler equation (12) that are all topologically distinct if the reference fields $\mathbf{U}(\mathbf{x})$ from which they are derived are topologically distinct. Each flow, nev-

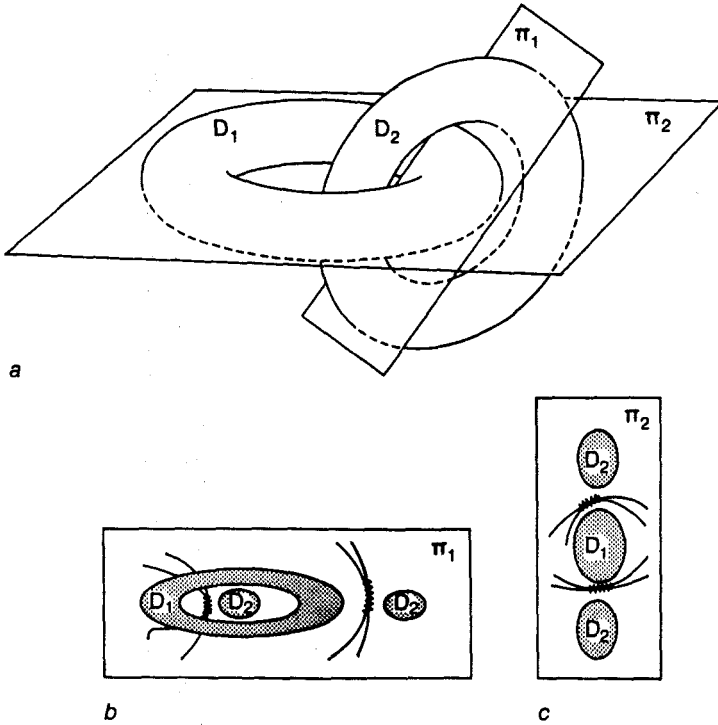


Fig. 1 a) Typical structure of Euler flow; within the toroids D_1 and D_2 , the streamlines are ergodic and the flow has maximal helicity; b) and c) Poincaré sections Π_1 , Π_2 of the flow indicating the ergodic blobs (or 'coherent structures') separated by surfaces $h = const$ on which vortex sheets may be located.

ertheless, has certain structural properties that can be very simply described. First, note from Eq. (12) that since $\mathbf{u} \cdot \nabla h = 0$, the streamlines generally lie on surfaces $h = const$. The only situation that permits streamlines to escape this constraint occurs when, in some subdomain, D_1 say, ∇h is zero and

$$\omega = \alpha_1 \mathbf{u} \tag{15}$$

where α_1 is constant. This is a Beltrami flow (in D_1) of maximal helicity in the sense that the upper bound on $|\mathcal{H}|$ permitted by the Schwarz inequality (7) is attained (note that $\omega \cdot \mathbf{n} = 0$ on ∂D_1 , so that the helicity in D_1 is well defined). Within D_1 or at least within parts of D_1 , streamlines may be space filling (i.e., ergodic in the three-

dimensional subspace) in the manner identified for a particular space-periodic Beltrami flow by Hénon⁽¹⁴⁾ and Dombre et al.¹⁵

It is easy to visualize such regions as of toroidal topology as in Fig. 1a (although they may, in general, be much more complex). Different 'ergodic toroids' of maximal helicity D_1, D_2, \dots may be linked and knotted with arbitrary complexity (determined by the topology of the reference field $\mathbf{U}(\mathbf{x})$ and different Poincaré sections of the flow Π_1, Π_2 then have the structures indicated in Figs. 1b and 1c. The space separating the toroids is filled with a family of surfaces $h = \text{const}$ (or possibly $\alpha = \text{const}$, where $\omega = \alpha(\mathbf{x})\mathbf{u}$, $\mathbf{u} \cdot \nabla\alpha = 0$), and it is on these surfaces that tangential discontinuities of \mathbf{u} may appear. It is important to note that vortex sheets cannot be present in the ergodic regions since $\mathbf{u} = \alpha^{-1}\omega$ would then be singular also, with infinite energy density, and this is excluded. So the vortex sheets, if present, are necessarily separate from the ergodic structures and occur through confluence of surfaces $h = \text{const}$ with discontinuities related by

$$[h] = \frac{1}{2}\rho[\mathbf{u}^2] \quad (16)$$

across the sheet (i.e., $[p] = 0$).

The relevance of such solutions to the problem of turbulence should now be clear: we interpret the toroidal ergodic 'blobs' as the 'coherent structures' of the flow; these are bounded by a stream surface and would therefore be detectable in a visualization process; indeed, if a dye is injected at any point in such a region, it will rapidly spread to fill the region.

The vortex sheets, on the other hand, are the site of strong viscous dissipation (when viscosity is restored to the equations). These vortex sheets, as has already been indicated, are subject to instabilities of the Kelvin-Helmholtz type, so that characteristic 'double-spiral' structures may be expected to develop and to induce unsteadiness in otherwise stable parts of the flow field. Under such perturbations, the exact alignment of ω and \mathbf{u} within a coherent structure will be disturbed, but nonlinear energy transfer associated with the $\mathbf{u} \times \omega$ term in the Navier-Stokes equation may be expected to remain small, which is why these structures may persist over long time scales.

V. Spiral Structures and the Kolmogorov Spectrum

It has been argued previously¹⁶ that the development of double-spiral structures (see Fig. 2) can be associated with a spectral power

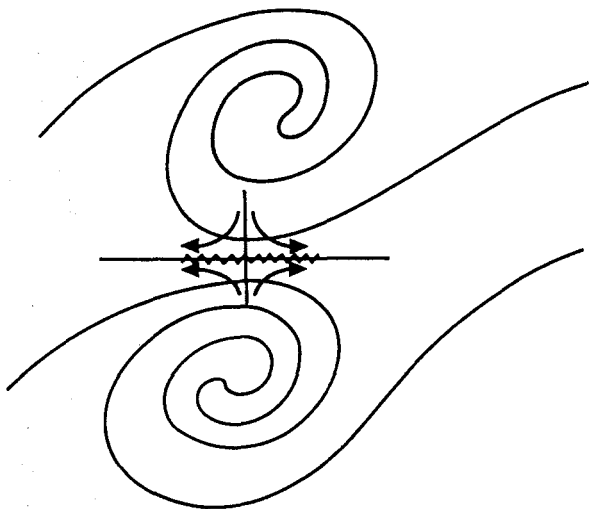


Fig. 2 Growth of double spirals associated with Kelvin-Helmholtz instability; a new vortex sheet can form due to stretching action associated with growth of adjacent spirals.

law $E(k) \sim k^{-\lambda}$ with fractional exponent $1 < \lambda < 2$. If the structure of the spiral is known, with a (near) accumulation point of velocity discontinuities on any straight-line transversal passing near the 'eye' of a spiral, then the value of λ may be calculated. There is here an important difference between two-dimensional Kelvin-Helmholtz instability (for which the wave-crests are parallel to the vorticity vector within the sheet) and three-dimensional modes for which the wave-crests are inclined to the vorticity vector, as must occur, for example, if the vortex sheet is wrapped on a torus, the vorticity being ergodic on the torus. In the latter case, the component of vorticity perpendicular to the wave-crests is stretched by the instability, the associated velocity jump remaining constant across the sheet.

In the process of spiral winding, the component of vorticity *parallel* to the spiral axis becomes more and more concentrated near this axis (the beginnings of this process being seen in the analysis of Moore¹⁷), so that the winding process becomes similar to that due to a concentrated line vortex superposed on the sheet. Here the results of Gilbert¹⁸ on a related two-dimensional problem are illuminating: Gilbert considered the wind-up of a weak vorticity discontinuity by a neighboring concentrated line vortex and found that the energy spec-

trum associated with the developing spiral structure has a $k^{-11/3}$ fall-off (a result that receives some support from numerical simulations of vortex interactions in a mixing layer¹⁹). The corresponding result for the wind-up of a vortex sheet by a superposed line vortex perpendicular to the vorticity in the sheet would be a $k^{-5/3}$ spectrum; but the three-dimensional problem is complicated by the fact that, for the Kelvin-Helmholtz instability, the strength of the vortex sheet (i.e., of the surface vorticity) becomes non-uniform, as indicated above, and the winding is not precisely that due to a concentrated vortex. Lundgren's³ model of a strained spiral vortex suggests that the $k^{-5/3}$ spectrum may not, in fact, be too sensitive to the details of the winding process, but this is an area that clearly merits further study.

The picture that then emerges is that the inertial range spectrum is associated with a particular type of developing singularity, namely, a Kelvin-Helmholtz double spiral, distributed randomly throughout the flow field. Each spiral will grow until it interacts with neighboring spirals, when pairing of spirals may occur, the 'mature' spirals acting as possible foci for new coherent structures.

At the same time, we see a mechanism for the formation of new vortex sheets (see Fig. 2), through stretching of the vorticity component in the plane of the straining motion associated with spiral expansion. The vorticity in the new sheet is perpendicular to the axes of the 'parent' spirals, and the process (related to the process of production of enstrophy) is essentially three-dimensional in character. (There is no comparable mechanism for the production of vortex sheets in two-dimensional turbulence.)

VI. Summary

The aim of this paper has been to provide a global description of turbulence that incorporates in a natural way the four features specified in the introduction, namely, coherent structures, intermittency of dissipation, an inertial range spectrum, and a three-dimensional process of enstrophy production. We achieve this by focusing attention on the fixed points of the Euler equations (i.e., Euler flows of general three-dimensional structure) and the instabilities to which these flows may be subject. The coherent structures emerge as regions in which streamlines may be ergodic and helicity density is high. Dissipation is concentrated in the vortex sheets located in regions separating the coherent structures and is thus spatially intermittent. The vortex sheets are unstable by the Kelvin-Helmholtz mechanism and form

spiral structures that we conjecture are intimately bound up with the appearance of an inertial range spectrum with fractional power law $k^{-\lambda}$. Finally, this same instability is responsible for stretching of vortex lines inclined to the wave-crests and thus for enstrophy production and formation of new vortex sheets.

Development of a detailed model of this kind is important because, if the characteristic structures and instabilities are understood, then it should be possible to derive not only second-order statistics such as the energy spectrum but also higher-order statistics without appeal to closure assumptions, which are hard, if not impossible, to justify.

Topological (as opposed to analytical) considerations enter through the helicity invariant (5), which is maximal in the ergodic blobs of Euler flows. There is some evidence from direct numerical simulation of space-periodic flows such as the Taylor-Green vortex^{20,21} that coherent structures of nonzero helicity where dissipation is low do tend to emerge as the flow develops. Further work along these lines, as well as direct experimental measurement of helicity fluctuations in turbulent flow, is urgently needed to complement the theoretical developments.

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