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Generalised vortex rings with and without swirl

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Abstract. Steady solutions of the Euler equations for flow of an inviscid incompressible fluid may be obtained by considering the process of magnetic relaxation to analogous magnetostatic equilibria in a viscous perfectly conducting fluid. In particular, solutions which represent rotational disturbances propagating without change of structure in an unbounded fluid may be obtained by this method. When conditions are axisymmetric, these disturbances are vortex rings of general structure, which may include a swirl component of velocity. This situation is analysed in some detail, and it is shown that the vortex is characterised by two functions: $V(\psi)$, the volume within toroidal surfaces $\psi = \text{cst.}$ and $W(\psi)$, the toroidal volume flux inside the torus $\psi = \text{cst.}$ For each choice of $\{V(\psi), W(\psi)\}$, satisfying appropriate limit conditions, there exists at least one vortex ring of steady structure.

1. Introduction

In a series of papers (Moffatt 1985, 1986 a, b), hereinafter referred to as M85, M86 a, b) a method has been developed for the construction of solutions of the steady Euler equations of arbitrarily prescribed topology. This method is based on the exact analogy between the Euler equations written in the form

$$\boldsymbol{u} \times \omega = \nabla h, \quad \omega = \nabla \times \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{1.1}$$

and the equations of magnetostatic equilibrium in a perfectly conducting fluid

$$\boldsymbol{j} \times \boldsymbol{B} = \nabla p, \quad \boldsymbol{j} = \nabla \times \boldsymbol{B}, \quad \nabla \cdot \boldsymbol{B} = 0, \tag{1.2}$$

and on the technique of magnetic relaxation towards solutions of these equations. This technique has been long recognized in the plasma physics literature (Kruskal and Kulsrud 1958, Taylor 1974, 1986), but it is only recently that the far-reaching implications concerning existence and structure of steady solutions of the Euler equations (or 'Euler flows') have been fully appreciated.

We first review (section 2) the essential features of the approach as applied to fully three-dimensional situations. In section 3, we consider the special case of axisymmetric flows without swirl, including vortex rings of all known form. The importance of the *signature* $V(\psi)$ of these vortex rings is emphasised, and some new properties of $V(\psi)$ are obtained. In section 4, we consider the new features of the problem presented by axisymmetric flows *with* swirl, and we show that the appropriate signature now consists of a pair of functions $\{V(\psi), W(\psi)\}$ where $W(\psi)$ (the *swirl function*) is the azimuthal volume flux around the interior of the torus $\psi = \text{cst.}$

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2. The magnetic relaxation technique

We seek to identify solutions of (1.2) for which the field $\boldsymbol{B}$ has 'prescribed topology', i.e. for which all the knots and links in $\boldsymbol{B}$ are topologically equivalent to the knots and links of some arbitrarily prescribed field $\boldsymbol{B}_0(\boldsymbol{x})$. This field $\boldsymbol{B}_0(\boldsymbol{x})$ does not generally satisfy the magnetostatic conditions (1.2); however if we adopt $\boldsymbol{B}_0(\boldsymbol{x})$ as an *initial condition* in a *viscous, perfectly conducting* fluid, then the field will relax, conserving its topology, but giving up its energy, via the fluid motion that it generates and consequent viscous dissipation.

The equations describing this relaxation process are

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} + \nu \nabla^2 \boldsymbol{v}, \tag{2.1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}), \tag{2.2}$$

and the initial conditions are

$$\boldsymbol{B}(\boldsymbol{x}, 0) = \boldsymbol{B}_0(\boldsymbol{x}), \quad \boldsymbol{v}(\boldsymbol{x}, 0) = 0. \tag{2.3}$$

Both $\boldsymbol{B}$ and $\boldsymbol{v}$ satisfy

$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{v} = 0, \tag{2.4}$$

(although we note that the incompressibility condition is by no means essential – Moffatt 1987b). The boundary conditions depend on whether the domain D of fluid is bounded, or unbounded in one or more directions. For a bounded domain, the appropriate conditions are

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{v} = 0 \quad \text{on } \partial \mathrm{D}. \tag{2.5}$$

For directions in which D is unbounded, appropriate conditions (but not the only possibilities) are

$$\boldsymbol{B} \to \boldsymbol{B}_0(\text{cst.}), \quad \boldsymbol{v} \to 0 \text{ as } |\boldsymbol{x}| \to \infty. \tag{2.6}$$

Eq. (2.2) is the 'frozen-field' equation of magnetohydrodynamics, which for all finite t conserves field topology, i.e. $\boldsymbol{B}(\boldsymbol{x}, t)$ has the same topology as $\boldsymbol{B}_0(\boldsymbol{x})$ for $0 < t < \infty$. However, as $t \to \infty$, due to a 'squeeze-film' mechanism described in M85, tangential discontinuities in $\boldsymbol{B}$ can develop, particularly in the neighborhood of fluid particles at which saddle points of the field are located. This saddle-point behavior is apparent even in two-dimensional relaxation (Moffatt 1987a).

The other essential ingredient of the relaxation process is the viscosity in (2.1) which dissipates energy (although any other dissipative effect acting only on the velocity field would do equally well). For a bounded domain, the energy equation associated with (2.1) and (2.2) is

$$\frac{\mathrm{d}}{\mathrm{d}t}(M(t) + K(t)) = -\Phi(t), \tag{2.7}$$

where $M(t)$ is the magnetic energy, $K(t)$ the kinetic energy and $\Phi(t)$ the rate of viscous dissipation. For an unbounded domain, the only modification required is that $M(t)$ is the magnetic energy associated with the perturbation field $\boldsymbol{B}(\boldsymbol{x}, t) - \boldsymbol{B}_0$ (not necessarily small), and care is needed in consideration of the energy flux as $|\boldsymbol{x}| \to \infty$ (M86b).

Eq. (2.7) implies that the energy $M(t) + K(t)$, which is initially finite, must decrease monotonically for so long as $\Phi(t) \neq 0$, and must therefore tend to a limit. Hence $\Phi(t) \to 0$ as $t \to \infty$, and assuming that ν is sufficiently large to ensure that singularities in the velocity field $\boldsymbol{v}$ do not form, we must have

$$\boldsymbol{v} \to 0 \quad \text{and} \quad \boldsymbol{B}(\boldsymbol{x}, t) \to \boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x}) \text{ as } t \to \infty. \tag{2.8}$$

The limit field $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ is the required field that is 'topologically accessible' from $\boldsymbol{B}_0(\boldsymbol{x})$. If $\boldsymbol{B}_0(\boldsymbol{x})$ is topologically nontrivial, then clearly $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ (or $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x}) - \boldsymbol{B}_0$ in the case of an unbounded domain) is non-zero, since $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ retains the topology of the initial field.

Evidently, with $\boldsymbol{v} = 0$ in (2.1) and (2.2), we have relaxed to a solution of the magnetostatic equations (1.2). Only at this stage do we appeal to the analogy $\boldsymbol{B} \to \boldsymbol{u}$ between (1.2) and (1.1); by virtue of this analogy, we have in effect found a steady solution $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$ of the Euler equations, which in the same sense is topologically accessible from an arbitrary kinematically possible field $\boldsymbol{U}(\boldsymbol{x})(\equiv \boldsymbol{B}_0(\boldsymbol{x}))$.

Note that there is no sense in which $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$ is *dynamically* accessible from $\boldsymbol{U}(\boldsymbol{x})$; indeed if $\boldsymbol{u} = \boldsymbol{U}(\boldsymbol{x})$ is adopted as an initial condition for the unsteady Euler equations, then the flow will never approach $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$, since the kinetic energy associated with $\boldsymbol{u}^{\mathrm{E}}$ is less than the kinetic energy associated with $\boldsymbol{U}$. This however is immaterial; the power of the technique described above is in identifying classes of steady Euler flows; it gives no information about the behaviour of the fluid system in a neighborhood of such a steady state. (The question of stability of the Euler flows can be approached by Arnold's 1966 method, as described in M86a.)

3. Axisymmetric flows without swirl

The particular case of axisymmetric flow without swirl has been considered in M86b. In this case, with cylindrical polar coordinates (r, ϕ, z), the velocity field may be expressed in terms of a Stokes stream function $\psi(r, z)$:

$$u = \left(\frac{1}{r}\frac{\partial\psi}{\partial z}, 0 - \frac{1}{r}\frac{\partial\psi}{\partial r}\right), \tag{3.1}$$

and the vorticity takes the form

$$\omega = \left(0, \frac{1}{r}\mathrm{D}^2\psi, 0\right), \tag{3.2}$$

where

$$\mathrm{D}^2\psi = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial z^2}. \tag{3.3}$$

It is well-known that the condition for steady flow is

$$\omega_\phi = \frac{1}{r}\mathrm{D}^2\psi = rF(\psi), \tag{3.4}$$

for some function $F(\psi)$. The known analytic solutions of this equation are few, and are for the most part limited to situations in which $F(\psi)$ is either constant or a linear function of ψ, and even then, analysis of vortex ring structures is a matter of considerable complexity (see for example Fraenkel 1970, 1972; Norbury 1973; Friedman and Turkington 1981). When $F(\psi)$ is a general nonlinear function, no general statements can be made concerning the existence of solutions, far less their structure. The technique of magnetic relaxation is however well-adapted for the more general treatment that is required.

We suppose that $\psi_0(r, z)$ is the streamfunction of an arbitrary, kinematically possible, flow with the streamline topology indicated in fig. 1, and with symmetry about the plane $z = 0$. This is the streamline topology of Hill's spherical vortex, and likewise (Batchelor 1967, p. 525) of any vortex ring whose core radius is not too small compared with the ring radius. We suppose that $\psi_0(r, z)$ is a C^1 function satisfying

$$\psi_0 \sim -\tfrac{1}{2}U_0 r^2 \text{ as } |\boldsymbol{x}| \to \infty \tag{3.5}$$

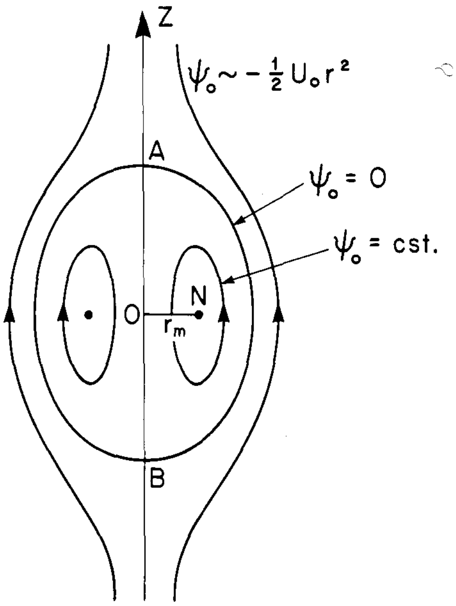

Fig. 1. Streamline topology for the reference flow with streamfunction $\psi_0(r, z)$. The surface $\psi = 0$ bounds the region of closed streamlines, and ψ_0 takes its maximum value ψ_m at the elliptic stagnation point N. The signature $V(\psi_0)$ is the volume inside the torus $\psi_0 = \text{cst.}$ $(0 < \psi_0 < \psi_m)$.

and that $\psi_0 = 0$ on the surface bounding the region of closed streamlines, of volume V_0. Within this region, the surfaces $\psi_0 = \text{cst.}$ form a family of nested tori, with innermost (degenerate) member a circle $r = r_m$, $z = 0$ where ψ_0 takes its maximum value, ψ_m say.

The *signature* of the flow $V(\psi)$ is defined for $0 \leq \psi_0 \leq \psi_m$ as the volume of fluid contained within the torus $\psi_0(r, z) = \psi$. This satisfies

$$V(\psi_m) = 0, \; V(0) = V_0, \tag{3.6}$$

and $V(\psi)$ is evidently continuous and monotonic decreasing in $0 < \psi < \psi_m$.

The importance of $V(\psi)$ arises from the magnetic relaxation problem, in which $\psi_0(r, z)$ is interpreted as the flux function of an initial magnetic field $\boldsymbol{B}_0(\boldsymbol{x})$. For this configuration, eq. (2.2) reduces to

$$\frac{\mathrm{D}\psi}{\mathrm{D}t} = 0, \tag{3.7}$$

where $\psi(\mathrm{r}, z, t)$ is the evolving flux function of $\boldsymbol{B}(\boldsymbol{x}, t)$; hence the surfaces $\psi = \text{cst.}$ (which are magnetic surfaces) moves with the fluid, and so the volume $V(\psi)$ within tori $\psi = \text{cst.}$ remains constant (the flow being incompressible). Hence the signature $V(\psi)$ is an invariant under the relaxation process, and it is proper to think of it as a topological invariant since its invariance is associated with the conserved ordering of the nested tori, i.e. if $\psi = \psi_1$, is nested inside $\psi = \psi_2$ at some time t_1, then it remains so nested for all $t > t_1$.

An important feature of axisymmetric relaxation is that in this case, tangential discontinuities of $\boldsymbol{B}$ *cannot* form at $t \to \infty$. This is because the formation of discontinuities would require that two initially distinct surfaces $\psi = \text{cst.}$ should come together over a finite area as $t \to \infty$; this would involve infinite field stretching in the intervening layer, and an unbounded *increase* of magnetic energy which clearly cannot occur. Note that this argument applies only for the topology of fig. 1; if there were any hyperbolic neutral points *off* the axis of symmetry, then there would be nothing to prevent the formation of tangential field discontinuities in their neighbourhoods.

Since the field $\boldsymbol{B}_0$ with flux function $\psi_0(r, z)$ relaxes to a continuous field $\boldsymbol{B}^{\mathrm{E}}$ with the same signature function $V(\psi)$, we may immediately deduce the existence of an Euler flow

$\boldsymbol{u}^{\mathrm{E}}(\equiv \boldsymbol{B}^{\mathrm{E}})$ with signature function $V(\psi)$. The vorticity of this flow clearly satisfies (3.4) for some $F(\psi)$, and evidently $F(\psi)$ is determined (in principle) by $V(\psi)$ and U_0. The magnetic relaxation procedure indicates a method by which $F(\psi)$ may be (computationally) determined when $V(\psi)$ and U_0 are given.

3.1. The signature of Hill's spherical vortex

It will be helpful, for reference purposes, to calculate the signature $V_{\mathrm{H}}(\psi)$ for Hill's spherical vortex, for which the stream function is

$$\left.\begin{aligned} \psi_{\mathrm{H}}(r, z) &= 4\psi_{\mathrm{m}}(r/a)^2\left(1 - \frac{r^2+z^2}{a^2}\right) \quad (r^2+z^2<a^2) \\ &= -\tfrac{1}{2}U_0 r^2\left(1 - \frac{a^3}{(r^2+z^2)^{3/2}}\right) \quad (r^2+z^2>a^2) \end{aligned}\right\} \tag{3.8}$$

with $U_0 = 16\psi_{\mathrm{m}}/3a^2$ and $V_0 = \frac{4}{3}\pi a^3$. For this case, assuming $\psi_{\mathrm{m}} > 0$, we have

$$\left.\begin{aligned} F_{\mathrm{H}}(\psi) &= -40\psi_{\mathrm{m}}/a^4 \quad (\psi_{\mathrm{m}} > \psi > 0) \\ &\ 0 \qquad\qquad\quad (\psi < 0) \end{aligned}\right\}. \tag{3.9}$$

The calculation of $V_{\mathrm{H}}(\psi)$ is straightforward. We find

$$V_{\mathrm{H}}(\psi) = \frac{3V_0}{4\sqrt{2}}\left(1 - \frac{\psi}{\psi_{\mathrm{m}}}\int_{-1}^{1}\left[\frac{1-x^2}{1+x(1-\psi/\psi_{\mathrm{m}})^{1/2}}\right]^{1/2} \mathrm{d}x, \tag{3.10}$$

an elliptic integral, with asymptotic behaviour

$$\begin{aligned} V_{\mathrm{H}}(\psi) &\sim \frac{3\pi V_0}{8\sqrt{2}}\left[\left(1 - \frac{\psi}{\psi_{\mathrm{m}}}\right) + \frac{3}{16}\left(1 - \frac{\psi}{\psi_{\mathrm{m}}}\right)^2 + \cdots\right] \quad \text{as } \psi \to \psi_{\mathrm{m}}, \\ V_{\mathrm{H}}(\psi) &\sim V_0\left[1 - \left(\frac{3}{16}\ln\frac{4\psi_{\mathrm{m}}}{\psi} + 1\right)\frac{\psi}{\psi_{\mathrm{m}}} + \cdots\right] \quad \text{as } \psi \to 0. \end{aligned} \tag{3.11}$$

Notice that $V_{\mathrm{H}}(\psi)$ is not differentiable at $\psi = 0$; this is because of the behavior of ψ near the stagnation points on the axis of symmetry, and it is evidently a property of the signature of any flow with the topology of fig. 1.

Note further that the function $V_{\mathrm{H}}(\psi)$ is by its construction continuous and monotonic decreasing in $0 < \psi < \psi_{\mathrm{m}}$; the inverse function $\psi_{\mathrm{H}}(V)$ is therefore equally well-defined.

3.2. Construction of a stream function $\psi_0(r, z)$ with a given signature $V(\psi)$

In order to commence the magnetic relaxation procedure with a given signature function $V(\psi)$, we need to be able to construct an appropriate initial flux function $\psi_0(r, z)$. First, let $\psi = \Psi(V)$ be the function inverse to $V(\psi)$, i.e. $\Psi(V(\psi)) = \psi$. Secondly, let $\psi_0(r, z) = K(\psi_{\mathrm{H}}(r, z))$, where $K(\psi)$ is a function to be determined. Then the surfaces $\psi_{\mathrm{H}} = \text{cst.}$ coincide with the surfaces $\psi_0 = \text{cst.}$ so that

$$\begin{aligned} V_{\mathrm{H}}(\psi) &= \text{volume inside } \psi_{\mathrm{H}} = \psi \\ &= \text{volume inside } \psi_0 = K(\psi) \\ &= V(K(\psi)). \end{aligned} \tag{3.12}$$

Hence the required signature function is given by the choice

$$K(\psi) = \Psi(V_{\mathrm{H}}(\psi)). \tag{3.13}$$

4. Vortex rings with swirl

Suppose now that a vortex ring with stream function $\psi(r, z)$ also has a swirl component of velocity $u_\phi(r, z)$. Then it is well-known that, under steady conditions, the ϕ-components of velocity and vorticity must satisfy

$$u_\phi = \frac{1}{r} G(\psi), \qquad \frac{\omega_\phi}{r} = F(\psi) - \frac{G(\psi)G'(\psi)}{r^2}. \tag{4.1}$$

A particular family of solutions involving spherical Bessel functions is known for the case when $G(\psi) = \alpha\psi$, $F(\psi) = \beta\psi$ where α and β are constants. This family has been described by Moffatt (1969): the streamlines are helical on the tori $\psi = \text{cst.}$, and are generally ergodic on these surfaces (fig. 2). Those streamlines that are closed are in fact torus knots, the knottedness and linkage of the streamlines being associated with the non-zero helicity of the flow.

We now enquire whether there exist more general solutions of the Euler equations for which $F(\psi)$ and $G(\psi)$ are not simple linear functions. Again the method of magnetic relaxation to analogous magnetostatic equilibria is well-adapted to address this problem. We simply suppose that the initial magnetic field includes a ϕ-component:

$$\boldsymbol{B}_0(\boldsymbol{x}) = \left(\frac{1}{r}\frac{\partial \psi_0}{\partial z},\ B_{0\phi}(r, z),\ -\frac{1}{r}\frac{\partial \psi_0}{\partial r} \right), \tag{4.2}$$

where $\psi_0(r, z)$ satisfies (3.5) as before. The Lorentz force now has a ϕ-component, so that the velocity field generated by (2.1) has the form

$$\boldsymbol{v} = \left(v_r(r, z, t),\ v_\phi(r, z, t),\ v_z(r, z, t) \right), \tag{4.3}$$

and the magnetic field evolves according to (2.2). The ϕ-component of this equation may be written in the form

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{B_\phi}{r} \right) = \left(\boldsymbol{B}_\mathrm{p} \cdot \nabla \right)\left(\frac{v_\phi}{r} \right), \tag{4.4}$$

an equation that describes the generation of toroidal field from poloidal field $\boldsymbol{B}_\mathrm{p}$ by differential rotation ,a process well-known in the context of dynamo theory (Moffatt 1978, section 3.11).

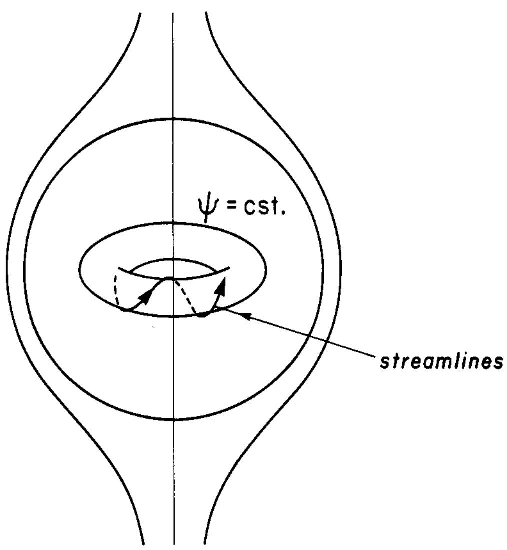

Fig. 2. Typical streamline topology of vortex ring with swirl; the streamlines are helices on the nested tori $\psi = \text{cst.}$ (> 0), and are generally ergodic on these surfaces.

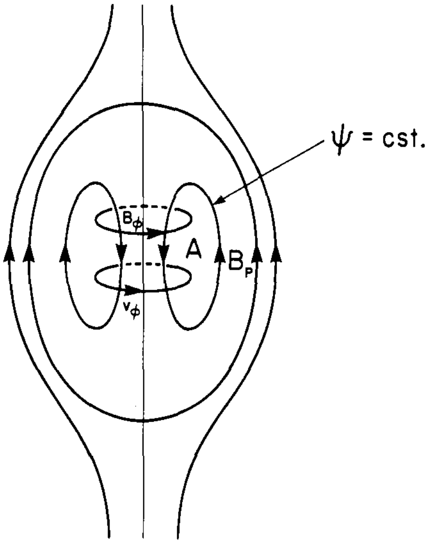

Fig. 3. During the process of magnetic relaxation considered in section 4, the Lorentz force drives a velocity component v_ϕ which generates a field component B_ϕ from $\boldsymbol{B}_\mathrm{p}$, but conserving the flux W of B_ϕ through curves $\psi = \text{cst}$. Hence $W = W(\psi)$ is invariant during the relaxation process.

Eq. (4.4) guarantees that the flux of B_ϕ around the interior of the torus $\psi = \text{cst.}$ is conserved (fig. 3); for, with

$$W(\psi) = \int_A B_\phi \, \mathrm{d}r \, \mathrm{d}z = \frac{1}{2\pi} \int_V \left(\frac{B_\phi}{r} \right) \mathrm{d}V, \tag{4.5}$$

where A is the cross section of the torus in the r, z plane, and V is its volume,

$$\begin{aligned} 2\pi \frac{\mathrm{d}W}{\mathrm{d}\psi} &= \int_V \frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{B_\phi}{r} \right) \mathrm{d}V = \int (\boldsymbol{B}_\mathrm{p} \cdot \nabla) \frac{v_\phi}{r} \mathrm{d}V \\ &= \int_{\psi = \text{cst.}} (\boldsymbol{n} \cdot \boldsymbol{B}_\mathrm{p}) \frac{v_\phi}{r} = 0, \end{aligned} \tag{4.6}$$

since $\boldsymbol{B}_\mathrm{p} \cdot \boldsymbol{n} = 0$ on the torus $\psi = \text{cst}$. Hence $W(\psi)$ is, like $V(\psi)$, an invariant during the magnetic relaxation process.

Again, magnetic energy decreases, but now subject to the topological constraint that the signature $\{V(\psi), W(\psi)\}$ is conserved, and an equilibrium characterised by this signature must be attained. During this process, B_ϕ is redistributed on each surface $\psi = \text{cst.}$, until in equilibrium, rB_ϕ is constant on each such surface, i.e. $rB_\phi = G(\psi)$. Moreover, in equilibrium, $(\nabla \times \boldsymbol{B})_\phi$ necessarily satisfies

$$\frac{(\nabla \times \boldsymbol{B})_\phi}{r} = F(\psi) - \frac{G(\psi) G'(\psi)}{r^2} \tag{4.7}$$

analogous to (4.1), for some function $F(\psi)$. Thus in principle, $F(\psi)$ and $G(\psi)$ are determined by $(V(\psi), W(\psi))$ together with the value of the field B_0 at infinity.

In order to implement this procedure numerically, for given $(V(\psi), W(\psi)$ it is necessary to construct an initial field (4.2) having this signature. We have already indicated in section 3 how $\psi_0(r, z)$ may be constructed, in the form $\psi_0 = K(\psi_\mathrm{H})$. Now let

$$B_{0\phi}(r, z) = \frac{\mathrm{d}W}{\mathrm{d}\psi} \frac{\mathrm{d}K}{\mathrm{d}\psi_\mathrm{H}} \left(\frac{\mathrm{d}A_\mathrm{H}}{\mathrm{d}\psi_\mathrm{H}} \right)^{-1}, \tag{4.8}$$

where $A_H(\psi_H)$ is the area inside a closed curve $\psi_H = \text{cst.}$ of Hill's vortex. Then

$$\begin{aligned}\int_A B_{0\phi}\,\mathrm{d}A &= \int_{A_H} B_{0\phi}\frac{\mathrm{d}A_H}{\mathrm{d}\psi_H}\mathrm{d}\psi_H = \int_{A_H}\frac{\mathrm{d}W}{\mathrm{d}\psi}\frac{\mathrm{d}K}{\mathrm{d}\psi_H}\mathrm{d}\psi_H \\ &= \int_{A_H}\frac{\mathrm{d}}{\mathrm{d}\psi_H}W(K(\psi_H))\,\mathrm{d}\psi_H = W(\psi_0)\end{aligned} \tag{4.9}$$

as required. Hence (4.8) gives the required construction for $B_{0\phi}(r, z)$.

5. Conclusion

We have applied the method of magnetic relaxation to demonstrate the existence of a wide family of vortex rings with swirl, which are steady in a frame of reference that moves relative to the fluid at infinity with constant velocity $-U_0$. These vortex rings are characterised by a signature consisting of a pair of functions $\{V(\psi), W(\psi)\}$ defined for $0 < \psi < \psi_m$, where $V(\psi)$ is the volume of fluid within the torus $\psi = \text{cst.}$, and $W(\psi)$ is the azimuthal volume flux within the same torus. The shape of the boundary of the region of closed streamlines (on which $\psi = 0$) is determined by $\{V(\psi), W(\psi)\}$ and U_0, as are also the functions $F(\psi)$ and $G(\psi)$ determined the distributions of azimuthal vorticity and velocity in the vortex.

The streamlines within these vortices are topologically similar to those of the special case when $F(\psi)$ and $G(\psi)$ are linear in ψ, i.e. they are helices wrapped on the family of nested tori $\psi = \text{cst.}$ $(0 < \psi < \psi_m)$, the pitch of the helix varying continuously from zero on the 'vortex axis' $\psi_m = 0$ to infinity on the surface $\psi = 0$ (where the magnetic field in the anologous magneto-static equilibrium must be purely poloidal to match continuously with an irrotational field in the 'external'region where $\psi \le 0$).

Finally, we note that similar considerations apply to two-dimensional configurations for which $V(\psi)$ must simply be replaced by $A(\psi)$, the area inside a closed curve $\psi(x, y) = \text{cst.}$, and $W(\psi)$ is the volume flux in the z-direction through the same curve; of course for this situation, the results analogous to (4.1) are

$$u_z = G(\psi), \quad \omega_z = F(\psi), \tag{5.1}$$

and the z-component of motion is dynamically passive as far as determination of $\psi(x, y)$ is concerned.

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