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Turbulence and Nonlinear Dynamics in MHD Flows
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ON THE EXISTENCE, STRUCTURE AND STABILITY OF MHD EQUILIBRIUM STATES

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Steady solutions $\{\mathbf{u}(\mathbf{x}),\ \mathbf{B}(\mathbf{x})\}$ of the MHD equations for an ideal incompressible fluid are classified according to their topological structure. It is shown that solutions of 'sub-Alfvenic' and 'super-Alfvenic' type exist, for which $\mathbf{B}(\mathbf{x})$ has an arbitrarily prescribed topology, with considerable freedom in the possible fields $\mathbf{u}(\mathbf{x})$; it is not however possible to prescribe these topologies independently. Some general considerations are presented concerning the stability of these steady states; it is argued that states that are sub-Alfvenic and 'trans-Alfvenic' are in general stable and it is conjectured that states that are sufficiently super-Alfvenic are unstable. The implications for MHD turbulence are briefly discussed.

1. INTRODUCTION

MHD flows are governed by equations for the velocity field $\mathbf{u}(\mathbf{x},t)$ and the magnetic field $(\mu_0\rho)^{\frac{1}{2}}\mathbf{B}(\mathbf{x},t)$ which may be written in the form

$$\left.\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} &= -\nabla P + \mathbf{u}\wedge\boldsymbol{\omega} + \mathbf{j}\wedge\mathbf{B} + \nu\nabla^2\mathbf{u}\\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla_\wedge(\mathbf{u}\wedge\mathbf{B}) + \eta\nabla^2\mathbf{B}\end{aligned}\right\} \quad (1.1)$$

where $\mathbf{j} = \nabla_\wedge\mathbf{B}$, $\boldsymbol{\omega} = \nabla_\wedge\mathbf{u}$, and $\nabla.\mathbf{u} = \nabla.\mathbf{B} = 0$. We are particularly concerned with the character of the solutions when ν and η are small, i.e. when the Reynolds number and magnetic Reynolds number are large. The effects of diffusion of momentum or magnetic field may then be expected to be confined to singular regions, e.g. vortex sheets or current sheets embedded in the flow and moving with the fluid. It makes sense to consider first the situation when $\nu = \eta = 0$, and then to restore diffusion effects only in such singular regions as and when these arise in the treatment of the ideal fluid.

The evolution of the system (1.1) may be represented by a trajectory in the function space of pairs of solenoidal fields $\{\mathbf{u}(\mathbf{x}),\ \mathbf{B}(\mathbf{x})\}$ on which we may reasonably impose the additional condition of finite energy

$$\int_{\mathcal{D}} (\mathbf{u}^2 + \mathbf{B}^2) dV < \infty \tag{1.2}$$

where $\mathcal{D}$ is the fluid domain. This condition permits certain types of singularity (e.g. vortex sheets and current sheets) but excludes others (e.g. concentrated line vortices or line currents); the condition appears natural for the freely evolving system (1.1), since if it is satisfied at some initial instant $t = 0$, then it is certainly satisfied for all $t > 0$.

We shall suppose that $\mathcal{D}$ is a bounded domain in $\mathbf{R}^3$, although with suitable modifications, the arguments that follow may be adapted either to two-dimensional bounded domains, or domains that are unbounded in one or more directions. We adopt the boundary conditions

$$\mathbf{u}.\mathbf{n} = \mathbf{B}.\mathbf{n} = 0 \text{ on } \partial\mathcal{D} , \tag{1.3}$$

which are compatible with the system (1.1) in the ideal limit $\nu = \eta = 0$.

In dealing with any nonlinear dynamical system, it is natural first to locate the fixed points of the system, then to analyse the structure and stability of the corresponding flows. For the system (1.1), the fixed points in the ideal limit are the steady solutions satisfying

$$0 = -\nabla P + \mathbf{u} \wedge \boldsymbol{\omega} + \mathbf{j} \wedge \mathbf{B} , \tag{1.4}$$

$$0 = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) . \tag{1.5}$$

If $\mathbf{u}(\mathbf{x}) = \pm\mathbf{B}(\mathbf{x})$ and $P =$ cst., these equations are satisfied identically for arbitrary $\mathbf{B}(\mathbf{x})$; these are the well-known 'Elsasser' solutions, which, in an infinite domain with a uniform magnetic field $\mathbf{B}_0$ at infinity and in a frame of reference moving with velocity $\pm\mathbf{B}_0$ represent nonlinear Alfven waves propagating with velocity $\mp\mathbf{B}_0$ relative to the fluid at infinity. We shall describe these solutions, which exhibit exact equipartition of energy between the magnetic field and the velocity field as *Alfvenic*.

We shall be concerned in this paper with more general solutions of (1.4) and (1.5) which do not exhibit this equipartition of energy. Introducing the norm

$$||\mathbf{u}|| = \int_{\mathcal{D}} \mathbf{u}^2 dV , \tag{1.6}$$

we shall describe solutions as *sub-Alfvenic* or *super-Alfvenic* according as

$$||\mathbf{u}|| < \text{ or } > ||\mathbf{B}|| \tag{1.7}$$

Note immediately that, to each sub-Alfvenic solution $\{\mathbf{u}_1(\mathbf{x}),\ \mathbf{B}_1(\mathbf{x})\}$, there corresponds a super-Alfvenic solution $\{\mathbf{u}_2(\mathbf{x}),\ \mathbf{B}_2(\mathbf{x})\}$, where

$$\mathbf{u}_2(\mathbf{x}) = \mathbf{B}_1(\mathbf{x}) \ , \quad \mathbf{B}_2(\mathbf{x}) = \mathbf{u}_1(\mathbf{x}) \tag{1.8}$$

(and $P_2 = P_0 - P_1$, where P_0 is an arbitary constant reference pressure).

Anticipating the existence of such solutions, we shall first (in section 2) consider their structure. In section 3, we shall develop an appropriate relaxation technique to demonstrate that a very wide family of steady solutions does indeed exist. In section 4, by extension of the techniques of Bernstein et al[1] and Arnold[2], we consider the stability characteristics of these solutions, within the ideal fluid approximation; and finally, in section 5, we comment on the possible relevance of the results in the context of MHD turbulence.

2. STRUCTURE OF STEADY SOLUTIONS OF THE IDEAL MHD EQUATIONS

Consider first equation (1.5), or equivalently

$$\mathbf{u}_\wedge \mathbf{B} = \nabla\phi \tag{2.1}$$

where $\phi(\mathbf{x})$ is the potential of the electric field. It follows immediately from (2.1) that

$$\mathbf{u}\cdot\nabla\phi = 0 \ , \quad \mathbf{B}\cdot\nabla\phi = 0 \ , \tag{2.2}$$

so that, except possibly in subdomains of $\mathcal{D}$ where $\nabla\phi \equiv 0$, the $\mathbf{u}$-lines and $\mathbf{B}$-lines lie on surfaces $\phi =$ cst. The topology of these surfaces is determined by the topology of the sets of points at which $\nabla\phi = 0$; these points may be isolated, or they may cover curves or surfaces, or they may fill three-dimensional subdomains. Let us consider these possibilities in turn:

(i) ISOLATED STATIONARY POINTS

If $\nabla\phi = 0$ at an isolated point $\mathbf{x} = \mathbf{x}_0$ say, then the surfaces $\phi =$ cst. near $\mathbf{x} = \mathbf{x}_0$ are either closed around $\mathbf{x} = \mathbf{x}_0$ or hyperbolic; either way, since both $\mathbf{u}$-lines and $\mathbf{B}$-lines must lie on these surfaces, it is clear that we must also have $\mathbf{u} = \mathbf{B} = 0$ at $\mathbf{x} = \mathbf{x}_0$; hence $\nabla\phi$ must have a double zero at $\mathbf{x} = \mathbf{x}_0$, a situation that seems intrinsically unlikely. [An example is given by

$$\mathbf{u} = (\alpha x, -\alpha y, 0), \quad \mathbf{B} = (0, \beta y,\ -\beta z), \quad \phi = \alpha\beta xyz \ .]$$

(ii) CLOSED CURVE STATIONARY POINTS

If $\nabla\phi = 0$ at all points of a closed curve C, then the surfaces $\phi =$ cst. have toroidal topology in the neighbourhood of C, and obviously C is itself both a **u**-line and a **B**-line. This topology is familiar in the context of plasma fusion devices, and may perhaps be regarded as typical (although note that in general the curve C may be knotted).

(iii) CLOSED SURFACE STATIONARY POINTS

It may happen that $\nabla\phi = 0$ at all points of a closed surface S (but $\nabla\phi \neq 0$ on surfaces close to S); then $\mathbf{u} \wedge \mathbf{B} \equiv 0$ on S, so that **u**-lines and **B**-lines coincide on S. Again the most likely possibility is that S has toroidal topology, the **B**-lines being helical, either closed curves or possibly ergodic, on S.

(iv) $\nabla\phi \equiv 0$ IN A SUBDOMAIN $\mathcal{D}^{(1)}$

Finally, it may happen that $\nabla\phi \equiv 0$ throughout some subdomain $\mathcal{D}^{(1)}$ of $\mathcal{D}$; then $\mathbf{u}\wedge\mathbf{B} \equiv 0$ in $\mathcal{D}^{(1)}$ so that

$$\mathbf{u} = \alpha(\mathbf{x})\mathbf{B} \quad \text{with} \quad \mathbf{B}\cdot\nabla\alpha = 0 \quad \text{in} \quad \mathcal{D}^{(1)} \tag{2.3}$$

Hence now the **B**/**u**-lines are constrained to lie on surfaces $\alpha =$ cst, and similar considerations now apply to the possible topologies of these surfaces. In particular, it may happen that $\nabla\alpha \equiv 0$ throughout some subdomain $\mathcal{D}^{(2)}$, of $\mathcal{D}^{(1)}$; then

$$\mathbf{u} = \alpha\mathbf{B} \quad (\alpha = \text{cst.}) \quad \text{in} \quad \mathcal{D}^{(2)} \tag{2.4}$$

There may of course be separate components of $\mathcal{D}^{(2)}$, with differeent constant values of α in each component region.

So far, the discussion has centred entirely on (2.1). In the subdomain $\mathcal{D}^{(2)}$ we can however proceed further by appeal to the momentum balance equation (1.4), which in this region reduces to

$$0 = -\nabla h + \mathbf{j}\wedge\mathbf{B} \tag{2.5}$$

where $h = P(1-\alpha^2)^{-1}$ (using (2.4) and $\boldsymbol{\omega} = \alpha\mathbf{j}$) . Hence

$$\mathbf{B}\cdot\nabla h = 0 \quad \text{and} \quad \mathbf{j}\cdot\nabla h = 0 \tag{2.6}$$

and we have not yet escaped from the constraint that **B**-lines lie on a family of surfaces, namely $h =$ cst.. It may happen that $\nabla h \equiv 0$ in a subdomain $\mathcal{D}^{(3)}$ of $\mathcal{D}^{(2)}$; then

$$\mathbf{j} = \beta(\mathbf{x})\mathbf{B} \quad \text{with} \quad \mathbf{B}\cdot\nabla\beta = 0 \quad \text{in} \quad \mathcal{D}^{(3)}$$

and **B**-lines now lie on surfaces $\beta = \text{cst}$. However, if $\nabla\beta = O$ in a subdomain $\mathcal{D}^{(4)}$ of $\mathcal{D}^{(3)}$ then

$$\mathbf{j} = \beta\mathbf{B}\ , \quad \mathbf{u} = \alpha\mathbf{B} \ \text{ in } \ \mathcal{D}^{(4)}$$

with α and β constant, and only at this level do we finally escape from the topological constraint that **B**-lines lie on surfaces. The **B**-lines may be chaotic (space-filling) in $\mathcal{D}^{(4)}$ or in some subdomain $\mathcal{D}^{(5)}$ of $\mathcal{D}^{(4)}$.

3. EXISTENCE OF STEADY SOLUTIONS OF THE IDEAL MHD EQUATIONS

Suppose we first attempt to find fields $\{\mathbf{u}_O(\mathbf{x}),\ \mathbf{B}_O(\mathbf{x}),\ \phi_O(\mathbf{x})\}$ satisfying (2.1), and for which the topology of $\mathbf{B}_O(\mathbf{x})$ (as characterised by all its knots and links) is prescribed. We may do this by trivial modification of the relaxation technique of Moffatt[3]: let $\hat{\mathbf{B}}_O(\mathbf{x})$ be an arbitrary initial field with prescribed non-trivial topology, and let this field relax through topologically accessible states to a magnetostatic equilibrium of minimum magnetic energy, i.e.

$$\hat{\mathbf{B}}_O(\mathbf{x}) \to \mathbf{B}_O(\mathbf{x}) \tag{3.1}$$

where

$$\mathbf{j}_O \wedge \mathbf{B}_O = \nabla h_O \quad , \quad \mathbf{j}_O = \nabla \wedge \mathbf{B}_O\ . \tag{3.2}$$

In this relaxed state, the $\mathbf{B}_O$-lines and $\mathbf{j}_O$-lines lie on surfaces $h_O = \text{cst.}$, except possibly in regions where $\nabla h_O \equiv 0$.

Now let

$$\mathbf{u}_O = F(h_O)\mathbf{j}_O + G(h_O)\mathbf{B}_O \tag{3.3}$$

in regions where $\nabla h_O \neq 0$ (with suitable modifications in regions where $\nabla h_O \equiv 0$). This choice of $\mathbf{u}_O$ evidently satisfies $\nabla \cdot \mathbf{u}_O = 0$; moreover it gives

$$\mathbf{u}_O \wedge \mathbf{B}_O = \nabla\phi_O \tag{3.4}$$

where $\phi_O = f(h_O)$ and $f'(h_O) = F(h_O)$. In fact, (3.3) probably provides the most general solenoidal field $\mathbf{u}_O$ satisfying (3.4).

The fields $\{\mathbf{u}_O,\ \mathbf{B}_O\}$ thus constructed do not of course in general satisfy the dynamical equilibrium equation (1.4); we may however adopt them as the initial condition for a second relaxation problem which *will* lead to such an equilibrium. This second problem, involving fields $\{\mathbf{u}(\mathbf{x},t),\ \mathbf{B}(\mathbf{x},t),\ \phi(\mathbf{x},t)\}$ must be formulated so that the condition

$$\mathbf{u} \wedge \mathbf{B} = \nabla\phi \tag{3.5}$$

is satisfied for all $t \geq 0$. We also require that the topology of $\mathbf{B}(\mathbf{x},t)$ be conserved, so that we can be sure of arriving at an equilibrium whose topology can be prescribed in advance.

We achieve this by introducing an *auxiliary* solenoidal velocity field $\mathbf{v}(\mathbf{x},t)$, which is assumed to convect *both* the $\mathbf{B}$ field *and* the $\mathbf{u}$-field according to the frozen field equations

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{v} \wedge \mathbf{B}) \ , \quad \frac{\partial \mathbf{u}}{\partial t} = \nabla \wedge (\mathbf{v} \wedge \mathbf{u}) \tag{3.6}$$

and to evolve according to the relaxation equation

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla P + \mathbf{u} \wedge \boldsymbol{\omega} + \mathbf{j} \wedge \mathbf{B} + \lambda \nabla^2 \mathbf{v} \tag{3.7}$$

where λ is a 'viscosity parameter', and the boundary condition $\mathbf{v} = 0$ on $\partial\mathcal{D}$. These equations may be easily combined to give the 'pseudo energy equation'

$$\frac{d}{dt}\int_{\mathcal{D}} (\mathbf{B}^2 - \mathbf{u}^2 + \mathbf{v}^2)dV = -2\lambda \int_{\mathcal{D}} (\nabla \wedge \mathbf{v})^2 dV \tag{3.8}$$

where the prefix 'pseudo' is used because of the minus sign in front of $\mathbf{u}^2$. This equation is useful only if

$$\int (\mathbf{B}^2 - \mathbf{u}^2)dV \geq C \quad \text{for all} \quad t \geq 0\ , \quad C = \text{cst.} \tag{3.9}$$

If this condition is satisfied, then the integral on the left of (3.8) is monotonic decreasing and bounded below, and therefore tends to a limit, which is non-trivial if the initial fields $\mathbf{u}_0$ and $\mathbf{B}_0$ are distinct, and the topology of $\mathbf{B}_0$ is non-trivial. Obviously under the condition (3.9), the asymptotic equilibrium state is sub-Alfvenic; but by the argument of section 1, corresponding super-Alfvenic states are then simultaneously determined.

Under what conditions then can we expect the inequality (3.9) to persist, assuming it to be satisfied at time $t = 0$? First consider any subdomain $\mathcal{D}_0$ of $\mathcal{D}$ in which $\mathbf{v}_0$ is parallel to $\mathbf{B}_0$ (and so $\nabla\phi_0 \equiv 0$). Under the convective action of the velocity field $\mathbf{v}(\mathbf{x},t)$, this domain is distorted into a time-dependent domain $\mathcal{D}_0 \to \mathcal{D}_1(t)$, say, and $\mathbf{u}$ remains parallel to $\mathbf{B}$ in $\mathcal{D}_1(t)$. In fact, since both $\mathbf{u}$ and $\mathbf{B}$ are subject to the same stretching action,

$$\int_{\mathcal{D}_1(t)} (\mathbf{B}^2 - \mathbf{u}^2)dV = \int_{\mathcal{D}_1} s(\mathbf{x},t)\ (\mathbf{B}_0^2 - \mathbf{u}_0^2)dV \tag{3.10}$$

where $s(\mathbf{x},t)$ is a strictly positive factor representing the square of the net stretch acting on either field up to time t. Hence, if $|\mathbf{u}_0| < |\mathbf{B}_0|$ everywhere in $\mathcal{D}_0$, then

$$\int_{\mathcal{D}_1(t)} (\mathbf{B}^2 - \mathbf{u}^2)dV > 0 \quad (\text{all } t \geq 0) \tag{3.11}$$

A small modification of this argument is needed to cover the region $\hat{\mathcal{D}}_1(t)$ in which $\nabla\phi \neq 0$. This region is 'foliated' by surfaces $\phi = \text{cst.}$, which for simplicity we shall suppose to be of toroidal topology. Let $\Phi_P(\phi)$ be the poloidal flux of $\mathbf{B}$ through the hole of the torus $\phi = \text{cst.}$ and let $\Phi_T(\phi)$ be the toroidal flux around its interior. Similarly, let $Q_P(\phi)$, $Q_T(\phi)$ be the volume fluxes determined by the velocity field $\mathbf{u}(\mathbf{x},t)$. Under the frozen-field evolution described by (3.6), these four fluxes are all conserved, i.e.

$$\Phi_P(\phi) = \Phi_P(\phi_0) \quad \text{etc.} \tag{3.12}$$

where $\phi(\mathbf{x},0) = \phi_0(\mathbf{x})$.

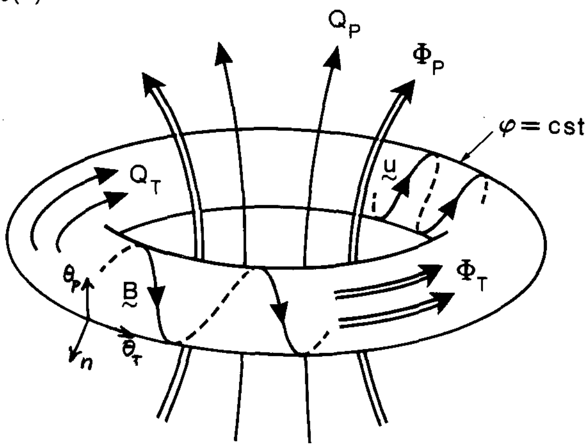

Figure 1. Poloidal fluxes Φ_P, Q_P and **toroidal** fluxes Φ_T, Q_T associated with toroidal surface $\phi = \text{cst.}$

Now if we consider adjacent surfaces labelled by $(\phi, \phi + \delta\phi)$, there are corresponding flux increments $\delta\Phi_P, \delta\Phi_T, \delta Q_P, \delta Q_T$ in the toroidal shell between these surfaces; and if we set up an *orthogonal* coordinate system (θ_P, θ_T, n) (fig. 1), the corresponding components of $\mathbf{u}$ and $\mathbf{B}$ are initially related in order of magnitude by

$$\frac{u_P}{B_P} \sim \frac{\delta Q_P}{\delta \Phi_P} \;, \quad \frac{u_T}{B_T} \sim \frac{\delta Q_T}{\delta \Phi_T} \,. \tag{3.13}$$

The initial conditions (3.2), (3.3) are such that these order of magnitude relations are likely to persist for $t > 0$; on this assumption,

$$\int_{\mathcal{D}_1(t)} (\mathbf{B}^2 - \mathbf{u}^2)dV \sim \int_{\hat{\mathcal{D}}_1(t)} \left[B_P^2(1 - (\frac{dQ_P}{d\Phi_P})^2) + B_T^2(1 - (\frac{dQ_T}{d\Phi_T})^2)\right] dV \tag{3.14}$$

and this remains positive for all $t > 0$ provided

$$|\frac{dQ_P}{d\Phi_P}| < 1 \quad \text{and} \quad |\frac{dQ_T}{d\Phi_T}| < 1 \tag{3.15}$$

These conditions in turn are guaranteed provided the initial fields $\mathbf{u}_0$, $\mathbf{B}_0$ satisfy

$$|\mathbf{u}_{P0}| \leq |\mathbf{B}_{P0}| \quad , \quad |\mathbf{u}_{T0}| \leq |\mathbf{B}_{T0}| \tag{3.16}$$

for all $\mathbf{x}$.

Under this condition then of *uniform boundedness* of $\mathbf{u}_0$, as regards both its poloidal and toroidal components, the inequality (3.9) is satisfied, and we may conclude from (3.8) that

$$\mathbf{v} \to 0 \;\; \text{everywhere} , \tag{3.17}$$

and hence that

$$\mathbf{u}(\mathbf{x}, t) \to \mathbf{u}_1(\mathbf{x}) \; , \quad \mathbf{B}(\mathbf{x}.t) \to \mathbf{B}_1(\mathbf{x}) \tag{3.18}$$

where $\mathbf{u}_1(\mathbf{x})$, $\mathbf{B}_1(\mathbf{x})$ satisfy the steady MHD equation (1.4) and (3.5), and are topologically accessible from $\mathbf{u}_0(\mathbf{x})$ and $\mathbf{B}_0(\mathbf{x})$. This solution is sub-Alfvenic by construction; a corresponding super-Alfvenic solution is then given by (1.8).

The above argument hinges on the estimates (3.13) which are difficult to justify with any degree of rigour for the unconventional type of problem considered. Computational experiment will be required to test whether relaxation (3.18) does indeed occur, or whether a runaway situation $|\mathbf{u}| \to \infty$, $|\mathbf{v}| \to \infty$ may develop.

4. STABILITY OF MHD EQUILIBRIA

We have shown that (i) given any field $\hat{\mathbf{B}}_0(\mathbf{x})$ of arbitrarily complex topology, we can construct a family of fields ($\mathbf{u}_0(\mathbf{x}$, $\mathbf{B}_0(\mathbf{x})$, $\phi_0(\mathbf{x})$) satisfying

$$\mathbf{u}_0 \wedge \mathbf{B}_0 = \nabla\phi_0$$

and having the property that $\mathbf{B}_0(\mathbf{x})$ is topologically accessible from $\hat{\mathbf{B}}_0(\mathbf{x})$; and (ii) that provided certain natural conditions are satisfied, there exists a steady solution $\{\mathbf{u}_1(\mathbf{x}),\ \mathbf{B}_1(\mathbf{x})\}$ of the MHD equations, topologically accessible from $\{\mathbf{u}_0,\ \mathbf{B}_0\}$. This implies the existence of an enormous class of steady solutions of these equations, of both sub-Alfvenic and super-Alfvenic type. We now consider briefly the stability of these solutions. Let $\{\mathbf{u}(\mathbf{x}),\ \mathbf{B}(\mathbf{x})\}$ be one such steady solution, and let $\boldsymbol{\xi}(\mathbf{x})$ be a small virtual displacement of the medium satisfying $\nabla\cdot\boldsymbol{\xi}=0$, and $\boldsymbol{\xi}\cdot\mathbf{n}=0$ on $\partial\mathcal{D}$. This displacement may be thought of as the result of a large velocity $\mathbf{V}(\mathbf{x})$ acting over a small time interval τ, with

$$\boldsymbol{\xi}(\mathbf{x})=\tau\mathbf{V}(\mathbf{x})\ . \tag{4.1}$$

The velocity $\mathbf{V}(\mathbf{x})$ convects both the $\mathbf{B}$-lines and the $\boldsymbol{\omega}$-lines during this small time-interval and the resulting change of energy (magnetic plus kinetic) is given by[4]

$$\delta E=\delta^{(1)}E+\delta^{(2)}E+O(\boldsymbol{\xi}^3) \tag{4.2}$$

$$\text{where }\delta^1 E=\int_{\mathcal{D}}(\mathbf{u}\cdot\delta^1\mathbf{u}+\mathbf{B}\cdot\delta^1\mathbf{B})dV \tag{4.3}$$

$$\delta^2 E=\frac{1}{2}\int_{\mathcal{D}}[2\mathbf{u}\cdot\delta^2\mathbf{u}+(\delta^1\mathbf{u})^2+2\mathbf{B}\cdot\delta^2\mathbf{B}+(\delta^1\mathbf{B})^2]dV \tag{4.4}$$

and where

$$\delta^1\mathbf{B}=\nabla\wedge(\boldsymbol{\xi}\wedge\mathbf{B})\ ,\quad \delta^2\mathbf{B}=\frac{1}{2}\nabla\wedge(\boldsymbol{\xi}\wedge\delta^1\mathbf{B}) \tag{4.5}$$

$$\delta^1\mathbf{u}=(\boldsymbol{\xi}\wedge\boldsymbol{\omega})_s\ ,\quad \delta^2\mathbf{u}=\frac{1}{2}[\boldsymbol{\xi}\wedge[\nabla\wedge(\boldsymbol{\xi}\wedge\boldsymbol{\omega}]]_s \tag{4.6}$$

and the suffix s denotes 'solenoidal part of'. It may easily be verified that

$$\delta^1 E=0 \tag{4.7}$$

in consequence of the equilibrium conditions. Moreover, $\delta^2 E$ may be manipulated to the form $\delta^2 E=\delta^2 M+\delta^2 K$ where

$$\delta^2 M=\int((\delta^1\mathbf{B})^2+(\boldsymbol{\xi}\wedge\mathbf{j})\cdot\delta^1\mathbf{B})dV\ , \tag{4.8}$$

$$\delta^2 K=\int[(\boldsymbol{\xi}\wedge\boldsymbol{\omega})_0^2+(\boldsymbol{\xi}\wedge\boldsymbol{\omega})\cdot\delta^1\mathbf{u}]dV\ . \tag{4.9}$$

Now when $||\mathbf{u}||<<||\mathbf{B}||$, the field $\mathbf{B}$ is only weakly perturbed from the magnetostatic field $\mathbf{B}_o$, which, by its construction, is stable (in an ideal fluid); it is reasonable to infer therefore that, for sufficiently sub-Alfvenic flows (i.e. $||\mathbf{u}||/||\mathbf{B}||$ sufficiently small), $\delta^2 E$ will remain positive for all (non-trivial) displacements $\boldsymbol{\xi}$, and these flows will therefore be stable. We conjecture that the corresponding super-Alfvenic flows, constructed by the recipe (1.8), will be unstable, simply because there will in general be virtual displacements in the neighbourhood of inflexion points of the velocity field $\mathbf{u}$ which can then extract energy from the system.

For pure Alfvenic states (i.e. $\mathbf{u} = \pm\mathbf{B}$),

$$\delta^2 E = \int (\delta^1 \mathbf{B} + (\boldsymbol{\xi} \wedge \mathrm{j})_s)^2 dV > 0 \tag{4.10}$$

for all non-trivial $\boldsymbol{\xi}$, and so these states are stable. There is moreover a band of "trans-Alfvenic" states in the neighbourhood of these pure Alfvenic states (e.g. states with $\mathbf{u} = (1 \pm \epsilon)\mathbf{B}$ with $\epsilon << 1$) for which $\delta^2 E$ should remain positive for all $\boldsymbol{\xi}$), and which should also therefore be stable.

5. DISCUSSION

If the fixed points (i.e. steady states) of the MHD dynamical system do indeed play a role in the general time-dependent evolution of this system, then the above discussion suggests that there should be a very different qualitative behaviour according as the initial state, as characterised by the ratio $||\mathbf{u}||/||\mathbf{B}||$, is strongly sub-Alfvenic or super-Alfvenic. In the former case, many stable fixed points are topologically accessible to the system, which may therefore be expected to oscillate in the neighbourhood of such fixed points, with time-dependent nonlinearity playing a relatively weak role. In the latter case, it seems likely that the only fixed points near the initial state will be unstable, and that the system will exhibit large oscillations with strong time-dependent nonlinearity, and that it may evolve towards a sub-Alfvenic state; this just means that magnetic energy will increase at the expense of kinetic energy; i.e. from these general considerations, dynamo action[5] is likely to occur, with field saturation only when the magnetic energy reaches the same order of magnitude as the kinetic energy, when stable 'field-saturated' states may be available to the system. These conclusions hold only when molecular diffusion effects are weak, and may require further that the magnetic Prandtl number ν/η be large, as is frequently the case in fusion or astrophysical plasma.

ACKNOWLEDGEMENTS I am grateful to Konrad Bajer and Andrew Gilbert for their critical reading of the manuscript and their constructive comments which led to the inclusion of the final paragraph of section 3.

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