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The Topological (as opposed to the analytical) Approach to Fluid and Plasma Flow Problems

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1. INTRODUCTION

This talk is by way of introduction to the whole meeting, and I shall therefore focus on one or two facts and conjectures, which will be explored in more detail in some of the subsequent lectures. My first assertion is that topological, rather than analytical, techniques and language provide the natural framework for many aspects of fluid mechanical research that are now attracting intensive study. The reason for this is very clear: in a fluid flow, in which the velocity field is a continuous function of position and time, the particle paths may be defined by a function $\mathbf{x}(\mathbf{a}, t)$, representing the position at time t of the particle whose initial position is $\mathbf{a}$. For finite t, the function $\mathbf{x}(\mathbf{a}, t)$ is continuous and invertible, and the inverse function $\mathbf{a}(\mathbf{x}, t)$ is also continuous. In these circumstances, any structure (whether described by a scalar, vector or tensor field) which is convected with the flow will have topological properties that are invariant in time (although its geometrical properties are by no means invariant, and indeed in general become exceedingly complex, in time). The simplest such structure is a patch of dye, passively convected with the flow. If the patch is topologically spherical at $t = 0$, then it remains so for all finite t; if it is topologically toroidal, then it remains toroidal, and so on.

The qualification 'at finite t' is important here, because, as t tends to infinity, the mapping $\mathbf{a} \to \mathbf{x}(\mathbf{a}, t)$ induced by a continuous velocity field may become discontinuous. For example, consider a system consisting of two viscous fluids, of densities ρ_1 and ρ_2, the lighter fluid lying above the heavier fluid (e.g. oil on water). Suppose that a drop of oil is introduced in the water, so that it rises under buoyancy forces towards the interface. The velocity field in the entire system is continuous for all t, but the gap between the drop and the overlying layer of oil tends to zero as t tends to infinity so that the induced mapping becomes discontinuous in the limit.

The topological invariance referred to above is represented by invariance of a certain family of integrals involving the convected field. For a scalar field $\theta(\mathbf{x},t)$ satisfying the Lagrangian conservation equation $D\theta/Dt = 0$, these integrals are rather trivial:

$$\int F(\theta)dV = \text{cst}. \tag{1}$$

where $F(\theta)$is any function of θ. The topology of the θ field is characterised by the family of surfaces $\theta = \text{cst}$,and the choice $F(\theta) = \delta(\theta - \theta_c)$ focusses attention on one such surface. The total family of invariants (1) guarantees that the topology of the field θ is conserved.

2. HELICITY INVARIANTS

The situation is less trivial and much more interesting, when we consider convected vector fields. These can be either the gradient of a scalar field ($\mathbf{G} = \nabla\theta$) or the curl of a vector field ($\mathbf{B} = \text{curl}\mathbf{A}$) or a combination of these. The prototype field of the latter type is the magnetic field in a perfectly conducting fluid which satisfies the frozen-field equation

$$\frac{\partial \mathbf{B}}{\partial t} = \text{curl}(\mathbf{u} \wedge \mathbf{B}) \tag{2}$$

or the equivalent Lagrangian equation

$$\frac{D}{Dt}\left(B_i(\mathbf{a},t)\frac{\partial a_j}{\partial x_i}\right) = 0 \tag{3}$$

where $B_i(\mathbf{a},t)$ represents the magnetic field at the current position of the fluid particle which was initially at position $\mathbf{a}$. The invariant integral that characterises the topology of $\mathbf{B}$ was discovered by Woltjer 1958, and is now known as the magnetic helicity $\mathcal{H}_M$ (sometimes denoted K in the plasma physics literature). We define this as follows: for simplicity, suppose that the fluid is contained in a simply connected domain, and let S be any closed surface within this domain, moving with the fluid, on which $\mathbf{B}\cdot\mathbf{n} = 0$ (a condition that clearly persists if it holds at some initial instant). Let V be the volume inside S. Then the magnetic helicity of the field within V is defined by

$$\mathcal{H}_M(V) = \int_V \mathbf{A}\cdot\mathbf{B}dV, \tag{4}$$

where $\mathbf{A}$ is an arbitrary magnetic potential for $\mathbf{B}$. Note that, under the assumed conditions, this integral is gauge invariant. $\mathcal{H}_M(V)$ is also invariant in time, and this is entirely a consequence of equation (2).

It is important to note first that the *helicity density* $\mathbf{A}\cdot\mathbf{B}$ is *not* invariant in time, although its value following a fluid particle can be forced to be time invariant by choosing a particular gauge for $\mathbf{B}$ (as recognised by Elsasser 1946 in work foreshadowing that of Woltjer); and secondly, that to every magnetic surface, there corresponds an integral invariant of the form (4), and that if there is an infinite family of such surfaces, then we have a corresponding infinite family of invariants. If however the magnetic field is 'space-filling' in any subdomain $\hat{D}$, then there is only one helicity integral for that subdomain.

Note also that the family of invariants (4) exists even if the field $\mathbf{B}$ is not dynamically passive. The velocity field $\mathbf{u}$ in equation (2) may be allowed to depend in any complex nonlinear manner on $\mathbf{B}$, but this does not affect the proof of the invariance of the integrals (4). There are a number of important physical circumstances in which this type of nonlinearity arises. For example, flows can be driven by a 'magnetic buoyancy instability' in which the velocity field $\mathbf{u}$ is quadratically related to $\mathbf{B}$, so that the right-hand side of equation (2) becomes cubic in $\mathbf{B}$. Despite the nonlinearity of the problem, the family of invariants (4) still exists.

The topological significance of these invariants was recognised by Moffatt (1969), through consideration of the trivial case in which the magnetic field is confined to two closed flux tubes of vanishingly small cross-section. Taking V to be the whole domain, $\mathcal{H}_M$ can be explicitly evaluated, with the result

$$\mathcal{H}_M = \pm n \Phi_1 \Phi_2, \tag{5}$$

where Φ_1 and Φ_2 are the two fluxes, and n is the (Gauss) winding number of the two tubes relative to one another. This result of course establishes a very clear relationship between the frozen field equation (2) and the fundamental topological concept of linkage. The relationship has been cemented by Arnol'd (1974) who showed that, in an asymptotic sense, helicity still represents linkage even when the field lines are not closed, but wander chaotically in the fluid domain; Arnol'd's identification of helicity with an asymptotic form of the Hopf invariant provides a powerful bridge between fluid mechanics and topology.

3. HELICITY ASSOCIATED WITH THE VORTICITY FIELD

It was this topological interpretation of helicity that led to the immediate realisation that there must exist an analogous helicity invariant, namely

$$\mathcal{H} = \int \mathbf{u} \cdot \boldsymbol{\omega} dV \tag{6}$$

corresponding to the frozen field Euler evolution of the vorticity field $\boldsymbol{\omega}$. This is the counterpart of Kelvin's (1869) circulation theorem, and it is noteworthy that Kelvin himself recognised that knotted vortex lines would have invariant topology, under evolution governed by the Euler equations. Kelvin developed his 'vortex theory of atoms' on this basis, in collaboration with P.G. Tait, who was thereby motivated (Tait 1898) to classify and catalogue all knots of increasing order of complexity. The vortex theory of atoms turned out to be misconceived, but the catalogue of knots remains as the cornerstone of an established branch of topology, in which there has been a recent renewed upsurge of interest.

In the fluid mechanical context, helicity, like energy, is conserved only for ideal fluid flow, and helicity generally changes under the influence of viscous effects. Change of helicity is associated with diffusion and reconnection of vortex lines, thus changing the topology of the vorticity field. Note that, whereas viscosity always leads to dissipation of energy, it can be responsible for the production, as well as the destruction, of helicity. The interaction of two vortex rings provides the prototype problem, in which the evolution of the total helicity of the flow provides an indicator of viscous interaction and reconnection.

The fact that energy is an 'inviscid invariant' is of course fundamental to the Kolmogorov theory of turbulence, involving a cascade of energy from large scales to small scales. The existence of a second robust inviscid invariant, namely the mean helicity, has naturally raised the question of the influence that non-zero mean helicity may have on this type of cascade process. The influence of helicity in turbulence has excited some controversy in the recent literature. Whatever the outcome of this controversy, we are faced with a problem: if helicity *does* affect the cascade process, then we have to understand exactly how it does so; if helicity *does not* affect the cascade process, then equally we have to understand how the fluid behaves in such a way as to respect one inviscid invariant (namely energy) and ignore another (namely helicity).

4. HELICITY AND THE SPONTANEOUS GROWTH OF MAGNETIC FIELDS

Helicity has long been known to be of fundamental significance in the context of dynamo theory, that is the theory of the spontaneous growth of magnetic fields in electrically conducting fluids in motion. When the flow is turbulent, magnetic field fluctuations are generated, and these interact with the turbulence to provide a mean electromotive force $\mathcal{E}$, which can in principle be expanded as a series in the mean field $\mathbf{B}$ and its derivatives:

$$\mathcal{E} = \alpha\mathbf{B} - \beta\nabla\wedge\mathbf{B} + \dots \quad (7)$$

The coefficients α, β, etc. in this equation are determined exclusively by the statistical properties of the turbulence, and by the physical properties of the fluid, particularly its magnetic diffusivity η. The leading coefficient α is of particular importance, since it is this term which gives rise to dynamo instability. This 'α-effect' was anticipated in the early work of Parker (1955), and again in the nearly axisymmetric dynamo model of Braginskii (1964), but it reached maturity with the work of Steenbeck, Krause & Rädler (1966), who recognised explicitly the relationship between α and the underlying helicity of the flow. Evidently, α is a pseudo-scalar quantity (i.e. one which changes sign under change from a right-handed to a left-handed frame of reference) and, insofar as helicity is the simplest measure of lack of reflexional symmetry in a turbulent flow, such a relationship is to be expected. However, helicity is not the *only* measure of lack of reflexional symmetry, and, only in the large η (i.e. low magnetic Reynolds number) limit is the relationship between α and helicity straightforward; in this limit, α may be expressed as a weighted integral of the helicity spectrum function:

$$\alpha = -\tfrac{1}{3\eta}\int k^{-2}H(k)dk. \tag{8}$$

The mean field equation then takes the form

$$\frac{\partial \mathbf{B}}{\partial t} = \alpha\nabla\wedge\mathbf{B} + \eta\nabla^2\mathbf{B}, \tag{9}$$

and unstable modes of Beltrami form ($\mathbf{B} = K\nabla\wedge\mathbf{B}$) exist provided

$$|\alpha K| > \eta K^2. \tag{10}$$

There are great difficulties in the mean field theory in the alternative *low* diffusivity limit, which have not yet been fully resolved. This limit is very important in astrophysical contexts, and it is customary to suppose that both α and β are determined in order of magnitude by the velocity and length scales (u_0, l_0) of the turbulence (and independent of the magnetic diffusivity) in this limit, i.e.

$$\alpha \sim u_0 \quad , \quad \beta \sim u_0 l_0. \tag{11}$$

The maximum growth rate for the mean field occurs on a length scale of order β/α (according to equation (9)), and we are now faced with the difficulty that, if the estimates (11) are correct, then the mean field instability progresses most efficiently on the scale l_0 of the turbulence itself. This cuts at the heart of mean field theory, which relies on a separation of scale between fluctuating and mean quantities. (The only escape would be if a small numerical coefficient ϵ were to appear in the expression for α, i.e. $\alpha = \epsilon u_0$.)

Current efforts are being increasingly directed at the 'fast' dynamo problem, in which it is assumed from the outset that the dominant scale of the magnetic field is not much greater than that of the velocity field. In fact, it may be very much *smaller*: a generic feature of 'fast' dynamos (for which, by definition, the growth rate is independent of magnetic diffusivity η as this tends to zero) is that the scale of the magnetic field must nearly everywhere be of order $\eta^{\frac{1}{2}}$, and this means of course that the field is nondifferentiable nearly everywhere as η tends to zero. The beginnings of such structures have emerged from the numerical simulations of the ABC dynamo by Galloway & Frisch (1986), which emphasise the central role played by the saddle points of the flow, where maximal stretching of the magnetic field takes place.

These magnetic structures presumably have their counterpart in the vorticity field structures of turbulent flow. Vortex stretching is like magnetic field stretching, but of course the velocity field is itself determined (in a nonlocal way) by the vorticity field. The question of whether the vorticity field can become singular, under Euler evolution, has attracted intense interest, and quite rightly so, because it is crucial to the understanding of the turbulent process. If a singularity *does* form, then its structure is of seminal importance for the process of energy dissipation; if it does *not* form, as now appears to be the case (see Pumir & Siggia, this volume) then there is some mechanism at work which tends to suppress nonlinear transfer of energy to very high wave numbers.

5. MAGNETIC RELAXATION: AN UNCONVENTIONAL ROUTE TO THE DETERMINATION OF EULER FLOWS

The concept of a fluid that is perfectly conducting, but nevertheless viscous, is a mathematical abstraction, rather than a physical reality, but has nevertheless proved remarkably fruitful at a fundamental level. In such a fluid, magnetic lines of force are frozen in the fluid, so that their topology is conserved (apart from the possibility that discontinuities may form by the 'squeeze film' process described in the introduction) but at the same time, energy is dissipated by viscosity. In these circumstances, a magnetic field will relax to a magnetostatic equilibrium state compatible with its initial topology. The formation of discontinuities (i.e. current sheets) is an inescapable part of this process, and this will occur even when the initial field and all its derivatives are continuous.

The magnetostatic equations are exactly analogous to the steady Euler equations (with magnetic field now analogous to velocity, not vorticity). This means that to every magnetostatic equilibrium, there corresponds a steady Euler flow. Mag-

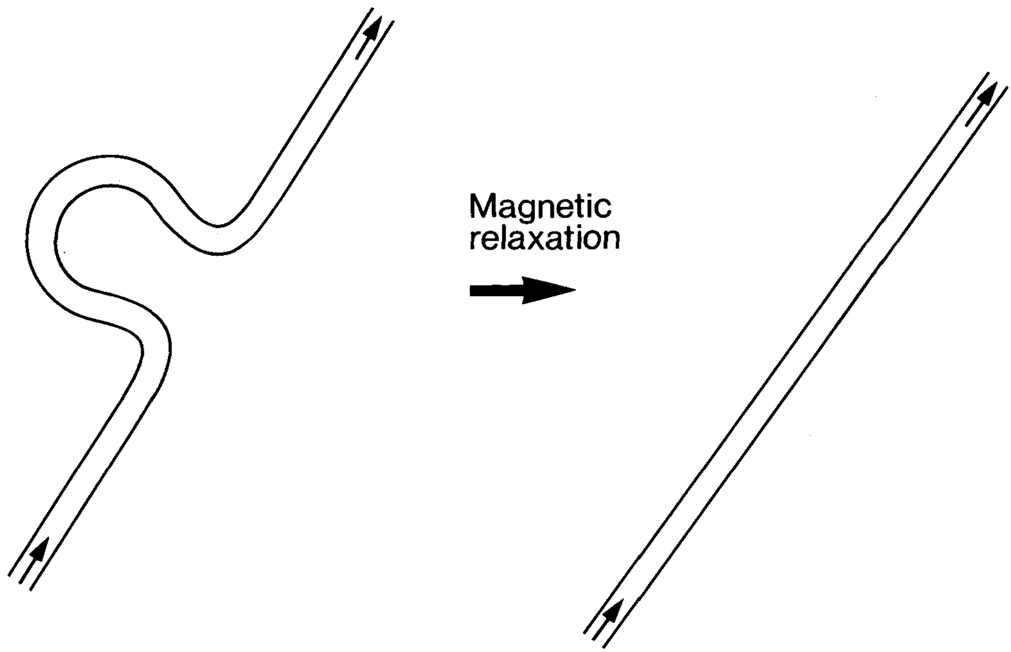

FIGURE 1. Relaxation of a magnetic flux tube in a perfectly conducting, viscous fluid.

netic relaxation therefore provides a route to the determination of Euler flows whose streamline topology may be prescribed in advance. This approach is quintessentially topological, rather than analytical in character, and it has proved powerful in establishing the existence of flows with particular topological properties (Moffatt 1985, 1986). Helicity plays a central role also in the magnetic relaxation problem, since through a combination of Schwarz and Poincaré inequalities (Arnol'd 1974) the magnetic energy of a field configuration is bounded below by a positive constant when the helicity of the configuration is non-zero. In the more esoteric situation when the helicity is zero, but there is nevertheless a higher order linkage present, the magnetic energy is still bounded below (Freedman 1988) but an estimate of this lower bound has yet to be found.

Alternative 'artificial' relaxation processes may be devised which conserve vorticity, rather than streamline, topology (see Vallis et al, and Carnevale & Vallis, this volume). These techniques appear to have tremendous potential, constrained at present only by the limits of computational power.

6. CONNECTIONS AND RECONNECTIONS

Consider a magnetic flux tube with an Ω-shaped kink in it. If the fluid is perfectly conducting, but viscous, the Lorentz force associated with the kink will cause this to relax to rectilinear form (figure 1). If the viscous effect is weak, there will of course be Alfven oscillations involved (as for a weakly damped pendulum); if viscosity is strong then these oscillations do not occur.

Suppose now that we start with two magnetic flux tubes of equal strength in the configuration of figure 2: each tube has an Ω-shaped kink, and these are 'connected'; the axes of the two tubes far from these kinks are non-parallel and do not intersect. Now the tubes obviously interfere with each other during the relaxation process, which for simplicity we assume to be dominated by viscosity. A field discontinuity tends to form, and if η is small, but non-zero, field diffusion and reconnection of field lines must occur.

This reconnection can occur in two ways (figures 2(c), (d)) or more probably as a mixture of these, each tube being in effect forced to bifurcate. Note that, whereas in case (c) the relaxation asymptotes to a state with two non-intersecting rectilinear flux tubes, in case (d), relaxation will continue indefinitely (there being initially an infinite reservoir of magnetic energy).

Consider now the situation when we have *vortex* tubes, rather than magnetic flux tubes, in the configurations of either figure 1 or 2. Suppose for example that the kinks have the form of Hasimoto solitons (Hasimoto 1972). The soliton of figure 1 can propagate along the vortex tube. The solitons of figure 2 may try to propagate but will very soon interfere with each other in a manner that can only be resolved by viscous diffusion and reconnection. Again reconnection can proceed to the configuration (c) or (d), or a mixture of these. The configuration (c) is relatively stable, while the configuration (d) will continue to evolve rapidly due to the twisted configuration of the tubes.

Examples of this kind may serve as prototypes of the reconnection processes associated with Joule dissipation in the MHD context and with viscous dissipation in high Reynolds number turbulence [see Melander & Hussain; Kerr et al; and Kida et al; this volume; also Meiron et al 1989].

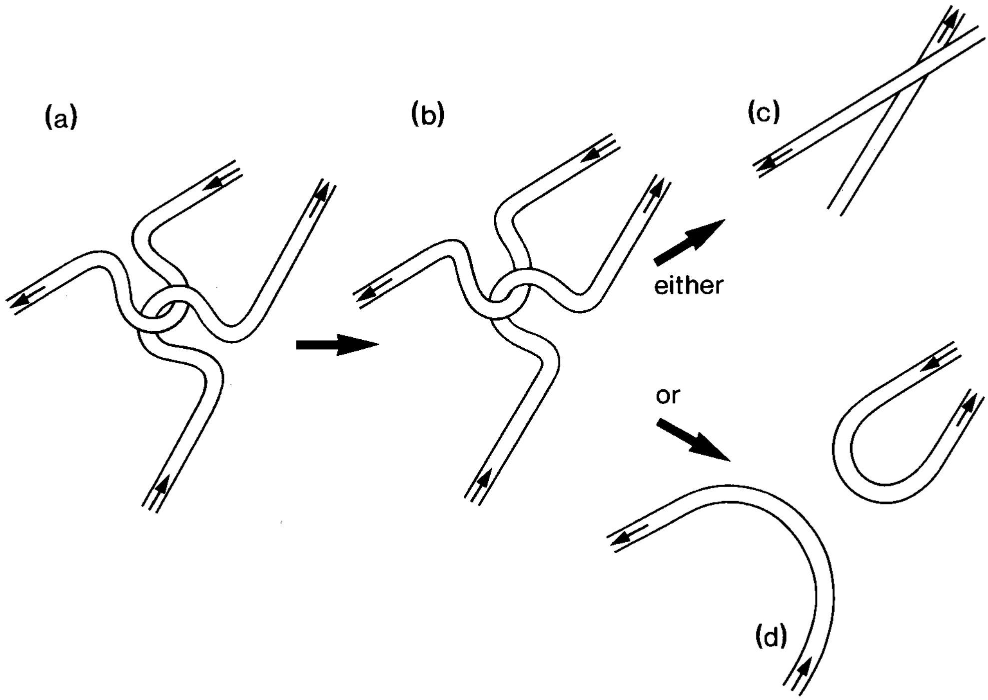

FIGURE 2. Relaxation of two connected flux tubes. A discontinuity forms (*b*), and reconnection occurs to the configuration (*c*) or the configuration (*d*), or to a mixture of these. The figure may also be interpreted in terms of vortex tubes supporting Hasimoto solitons.

7. CONCLUDING REMARKS

In this introductory survey, I have endeavoured to cover a range of topics in which topological, rather than analytical, considerations play a critical part. Helicity, being the natural, and simplest, measure of topological complexity of a convected vector field, plays a prominent role. In turbulent flow, it is the mean helicity that is an inviscid invariant. When this mean helicity is zero, then, as shown by Levich & Tsinober (1983), there still exists an integral invariant characterising helicity fluctuations. Current experimental and numerical investigations of the role of helicity fluctuations in turbulent flow are greatly to be welcomed, and it is to be hoped that these will shed new light on the fundamental mechanisms of turbulence.

The method of magnetic relaxation, and analogous relaxation techniques which conserve either streamline or vorticity topology, seems to hold great promise as a means of determining Euler flows of highly complex form. Tangential discontinuities (i.e. vortex sheets, or vortex gradient sheets, depending on the type of problem considered) may appear naturally in the relaxed fields, even if the initial fields are smooth.

For relaxation problems of this kind an important unsolved problem presents itself: what is the appropriate 'minimal' function space that contains all asymptotically relaxed fields starting from smooth initial conditions? An answer to this question would characterise the 'typical' degree of irregularity of *steady* solutions of the Euler equations, and would represent one step towards analysis of structures that develop under *unsteady* conditions and that may be expected to be generic for the problem of turbulence.

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