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Structure and Stability of Solutions of the Euler Equations: A Lagrangian Approach

H. K. Moffatt

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Structure and stability of solutions of the Euler equations: a lagrangian approach

By H. K. Moffatt

Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, U.K.

This paper reviews methods that are essentially lagrangian in character for determination of solutions of the Euler equations having prescribed topological characteristics. These methods depend in the first instance on the existence of lagrangian invariants for convected scalar and vector fields. Among these, the helicity invariant for a convected or 'frozen-in' vector field has particular significance. These invariants, and the associated topological interpretation are discussed in §§1 and 2. In §3 the method of magnetic relaxation to magnetostatic equilibria of prescribed topology is briefly described. This provides a powerful method for determining steady Euler flows through the well-known exact analogy between Euler flows and magnetostatic equilibria. Stability considerations relating to magnetostatic equilibria obtained in this way and to the analogous Euler flows are reviewed in §4. In §5 the related relaxation procedure is discussed; for two-dimensional and axisymmetric situations this technique provides stable solutions of the Euler equations for which the vorticity field has prescribed topology. The concept of flow signature is described in §6: this is the relevant topological characteristic for two-dimensional or axisymmetric situations, which is conserved during frozen-field relaxation processes. In §§7 and 8, the formation of tangential discontinuities as a normal part of the relaxation process when saddle points of the frozen-field are present is discussed. Section 9 considers briefly the application of these ideas to the theory of vortons, i.e. rotational disturbances that propagate without change of structure in an unbounded fluid. The paper concludes with a brief discussion, with comment on the possible development of the results in the context of turbulence.

1. Lagrangian invariants for convected scalar and vector fields

A lagrangian approach to problems of fluid mechanics naturally forces us to focus on 'material domains', i.e. on curves, surfaces or volumes (or more generally any set of 'marked' fluid particles) that move with the fluid. Of particular interest are quantities which, when integrated over a material domain, are constants of the motion. Such integrals may be described as 'lagrangian invariants' of the flow.

We consider a velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ in some fluid domain D and its associated density field $\rho(\boldsymbol{x}, t)(\geqslant 0)$ satisfying the equation of mass conservation

$$\mathrm{D}\rho/\mathrm{D}t \equiv \partial\rho/\partial t + \boldsymbol{u}\cdot\nabla\rho = -\rho\nabla\cdot\boldsymbol{u}. \tag{1.1}$$

Let $\theta(\boldsymbol{x}, t)$ be a scalar field (e.g. dye concentration), which we assume to be passively

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convected by the flow, without any molecular diffusion. The flux of θ is $\boldsymbol{u}\theta$, so that θ satisfies the same conservation equation as ρ, namely

$$\mathrm{D}\theta/\mathrm{D}t \equiv \partial\theta/\partial t + \boldsymbol{u}\cdot\nabla\theta = -\theta\nabla\cdot\boldsymbol{u}. \tag{1.2}$$

It follows from (1.1) and (1.2) that

$$\mathrm{D}/\mathrm{D}t(\theta/\rho) = -(\theta/\rho)\,\nabla\cdot\boldsymbol{u} + (\theta/\rho^2)\rho\nabla\cdot\boldsymbol{u} = 0. \tag{1.3}$$

Now let S be any closed material surface imbedded within D, containing a (material) volume V, and consider, for any function F, the lagrangian integral

$$I_\theta = \int_V F(\theta/\rho)\,\rho\,\mathrm{d}V. \tag{1.4}$$

Obviously,

$$\frac{\mathrm{d}I_\theta}{\mathrm{d}t} = \int_V F'(\theta/\rho)\,\mathrm{D}/\mathrm{D}t(\theta/\rho)\,\rho\,\mathrm{d}V = 0. \tag{1.5}$$

Hence, for arbitrary F, I_θ is a lagrangian invariant of the flow.

The topological structure of the field θ/ρ is associated with the structure of the family of surfaces $\theta/\rho = \text{const.}$, each of which moves with the fluid. The volume integral (1.4) yields a family of surface integrals through the choice

$$F(\theta/\rho) = \delta(\theta/\rho - (\theta/\rho)_\mathrm{c}), \tag{1.6}$$

where $(\theta/\rho)_\mathrm{c}$ labels one of these material surfaces, S_c say. Let $\boldsymbol{n}$ be the unit normal on S_c; then writing

$$\mathrm{d}V = \mathrm{d}n\,\mathrm{d}S = \mathrm{d}(\theta/\rho)\,\mathrm{d}S/(\boldsymbol{n}\cdot\nabla)\,(\theta/\rho), \tag{1.7}$$

the integral (1.4) converts to the surface integral

$$I_\mathrm{c} = \int_{S_\mathrm{c}} \frac{\rho\,\mathrm{d}S}{(\boldsymbol{n}\cdot\nabla)\,(\theta/\rho)}, \tag{1.8}$$

and this also (for each S_c) is a lagrangian invariant.

When the flow is incompressible, and of uniform density, then (with $\rho = 1$) the invariants (1.4) and (1.8) take the simpler form

$$I_\theta = \int_V F(\theta)\,\mathrm{d}V, \quad I_\mathrm{c} = \int_{S_\mathrm{c}} \frac{\mathrm{d}S}{\boldsymbol{n}\cdot\nabla\theta}. \tag{1.9}$$

In this case, the gradient $\boldsymbol{G} = \nabla\theta$ satisfies the equation (Batchelor 1952)

$$\mathrm{D}G_i/\mathrm{D}t = -G_j\,\partial u_j/\partial x_i, \tag{1.10}$$

and it is interesting to note (from 1.9) that this equation has the lagrangian invariant

$$I_\mathrm{c} = \int_{S_\mathrm{c}} (\boldsymbol{n}\cdot\boldsymbol{G})^{-1}\,\mathrm{d}S. \tag{1.11}$$

Suppose now that $\boldsymbol{B}(\boldsymbol{x}, t)$ is a solenoidal vector field ($\nabla\cdot\boldsymbol{B} = 0$) which is convected by the flow $\boldsymbol{u}(\boldsymbol{x}, t)$ with conservation of the flux of $\boldsymbol{B}$ through every closed material curve C. This is the fundamental property of a magnetic field in a perfectly conducting fluid medium, and we shall use the language of magnetohydrodynamics, although the results are more generally applicable. We shall suppose that $\boldsymbol{B}$ has a (single-valued) vector potential $\boldsymbol{A}$ satisfying

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{A}, \quad \nabla \cdot \boldsymbol{A} = 0. \tag{1.12}$$

The equations satisfied by $\boldsymbol{A}$ and $\boldsymbol{B}$ are

$$\left.\begin{aligned} \partial \boldsymbol{A}/\partial t &= \boldsymbol{u} \wedge (\nabla \wedge \boldsymbol{A}) - \nabla \varphi, \\ \partial \boldsymbol{B}/\partial t &= \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}), \end{aligned}\right\} \tag{1.13}$$

where φ is a scalar potential field, which we assume also to be single-valued. The equivalent lagrangian form for these equations is

$$\frac{\mathrm{D}A_i}{\mathrm{D}t} = u_j \frac{\partial A_j}{\partial x_i} - \frac{\partial \varphi}{\partial x_i} = A_j \frac{\partial u_j}{\partial x_i} - \frac{\partial}{\partial x_i}(\varphi - \boldsymbol{u} \cdot \boldsymbol{A}), \tag{1.14}$$

and

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{B_i}{\rho}\right) = \frac{B_j}{\rho} \frac{\partial}{\partial x_j} u_i. \tag{1.15}$$

For any unknotted closed material curve C spanned by an orientable surface Σ we have

$$I_{\mathrm{c}} = \oint_C \boldsymbol{A} \cdot \boldsymbol{dx} = \int_\Sigma \boldsymbol{B} \cdot \boldsymbol{n} \, \mathrm{d}S = \text{const.} \tag{1.16}$$

This is the frozen flux theorem of Alfven (1942); for an arbitrary material closed curve C, I_{c} is a lagrangian invariant.

The solution of (1.15) may be expressed in terms of the lagrangian particle path $\boldsymbol{x}(\boldsymbol{a}, t)$ which starts from position $\boldsymbol{a}$ at time $t = 0$. Writing $\hat{\boldsymbol{B}} = \boldsymbol{B}/\rho$, this solution (in a form anticipated by Cauchy) is

$$\hat{\boldsymbol{B}}_i(\boldsymbol{x}, t) = \hat{\boldsymbol{B}}_j(\boldsymbol{a}, 0) \, \partial x_i/\partial a_j, \tag{1.17}$$

a form that makes evident the rotation and stretching of the field by the deformation tensor $\partial x_i/\partial a_j$, during convection from $(\boldsymbol{a}, 0)$ to $(\boldsymbol{x}, t)$. The equivalent solution of (1.14) is

$$\boldsymbol{A}_i(\boldsymbol{x}, t) = \boldsymbol{A}_j(\boldsymbol{a}, 0) \, \partial a_j/\partial x_i - \partial \chi/\partial x_i \tag{1.18}$$

for some scalar field χ, constrained by our adoption of the gauge condition $\nabla \cdot \boldsymbol{A} = 0$. Note the appearance of the inverse deformation tensor $\partial a_j/\partial x_i$ in (1.18), with the consequence that

$$\boldsymbol{A}(\boldsymbol{x}, t) \cdot \hat{\boldsymbol{B}}(\boldsymbol{x}, t) = \boldsymbol{A}(\boldsymbol{a}, 0) \cdot \hat{\boldsymbol{B}}(\boldsymbol{a}, 0) - \hat{\boldsymbol{B}}(\boldsymbol{x}, t) \cdot \nabla \chi \tag{1.19}$$

a result obtained by Elsasser (1946), although with a different, and rather special, choice of gauge for which $\chi \equiv 0$.

If we now multiply (1.19) by ρ, and integrate over a material domain V using

$$\int_V \hat{\boldsymbol{B}} \cdot \nabla \chi \rho \, \mathrm{d}V = \int_V \boldsymbol{B} \cdot \nabla \chi \, \mathrm{d}V = \int_S (\boldsymbol{n} \cdot \boldsymbol{B}) \, \chi \, \mathrm{d}S \tag{1.20}$$

we obtain

$$\int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = \int_{V_0} \boldsymbol{A}_0 \cdot \boldsymbol{B}_0 \, \mathrm{d}V - \int_S (\boldsymbol{n} \cdot \boldsymbol{B}) \, \chi \, \mathrm{d}S. \tag{1.21}$$

In particular, if S is a 'magnetic surface' on which $\boldsymbol{n}\cdot\boldsymbol{B}=0$ (a condition that persists since $\boldsymbol{B}$ is a convected field) then

$$H_M = \int_V \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V = \text{const.} \tag{1.22}$$

This type of invariant was first obtained by Woltjer (1958). H_M is the helicity of the magnetic field within the material volume V (Moffatt 1969) and it is clearly a lagrangian invariant.

The lines of force of $\boldsymbol{B}(\boldsymbol{x}, t)$ (or '$\boldsymbol{B}$-lines') at any instant t are given by solution of the differential system

$$\mathrm{d}x/B_x = \mathrm{d}y/B_y = \mathrm{d}z/B_z. \tag{1.23}$$

In general, this system is non-integrable, and the $\boldsymbol{B}$-lines then wander chaotically throughout the fluid domain; exceptionally, however, the $\boldsymbol{B}$-lines may be closed curves or may lie on a family of closed surfaces. The following examples illustrate these possibilities; in each case the domain of definition of $\boldsymbol{B}$ is the sphere $r<1$, and (r, θ, φ) are spherical polar coordinates.

Example 1

$$\boldsymbol{B} = (0, 0, B_\varphi(r, \theta)). \tag{1.24}$$

For this simple field, the $\boldsymbol{B}$-lines are circles about the polar axis; the topology of the field is 'trivial' in the sense that every $\boldsymbol{B}$-line is an unknotted closed curve, and any two $\boldsymbol{B}$-lines are unlinked; the helicity is obviously zero.

Example 2

$$\boldsymbol{B} = \left(\frac{1}{r^2\sin\theta}\frac{\partial\chi}{\partial\theta}, \frac{-1}{r\sin\theta}\frac{\partial\chi}{\partial r}, B_\varphi(r, \theta)\right), \tag{1.25}$$

where the function $\chi(r, \theta)$ is described as the flux function of the meridional (or 'poloidal') part of the field. In this case, $\boldsymbol{B}$-lines lie on tori $\chi(r, \theta) = \text{const.}$ and in general cover these tori, although exceptionally they may be closed curves (torus knots). The field is contained in the sphere provided $\chi(1, \theta) = 0$, and the helicity is given by

$$H_M = 4\pi \iint \chi B_\varphi\, r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{1.26}$$

Example 3

In cartesian coordinates,

$$\boldsymbol{B} = (\alpha z - 8xy, 11x^2 + 5y^2 + z^2 + xy - 3, -\alpha x + 2yz - xy), \tag{1.27}$$

where $\alpha(\neq 0)$ is constant. This is an example of a non-integrable field, whose properties have been studied by Bajer & Moffatt (1990) and Bajer *et al.* (1990). Figure 1*a* shows a portion of a single $\boldsymbol{B}$-line (which is confined to the sphere $r<1$) and figure 1*b* shows a section of this streamline by the diametral plane $x = z$, i.e. a 'Poincaré section'). About 8000 successive points of section for this single $\boldsymbol{B}$-line are shown, and the chaotic wandering is quite evident, although some structure within the chaos is

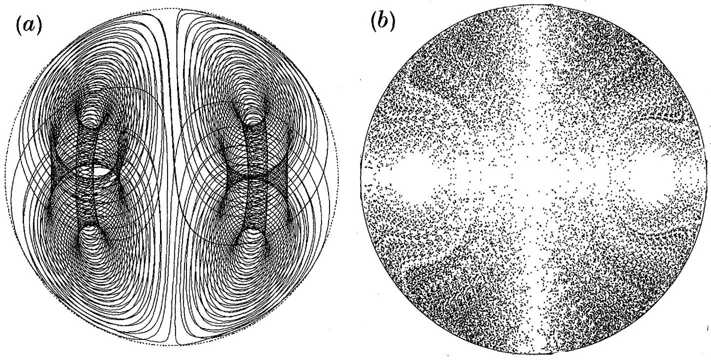

Figure 1. (a) A single chaotic **B**-line of the field (1.26). (b) Poincaré section of this **B**-line (see Bajer *et al.* 1990).

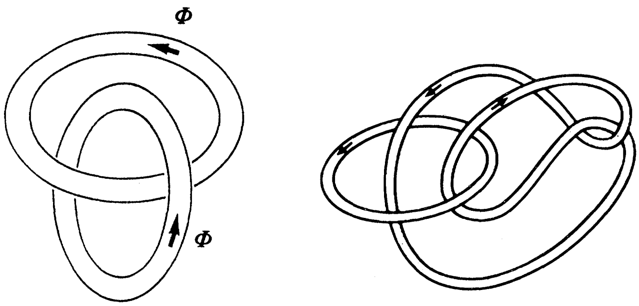

Figure 2. Possible configurations of linked flux tubes. (a) Simple linkage that gives non-zero helicity. (b) Non-trivial linkage with zero linking number and therefore zero helicity.

evident also. The topology is clearly nontrivial; and the helicity in $r<1$, may be calculated to be

$$H = -16\pi\alpha/35. \tag{1.28}$$

For a chaotic field of this kind, there is but a single helicity invariant of the form (1.22) for each subdomain V within which a **B**-line is space-filling. If, however, the **B**-lines lie on a family of surfaces $\theta = \text{const.}$, then, as for the invariant (1.4), we may construct associated surface invariants:

$$H_c = \int_{S_c} \frac{(\boldsymbol{A}\cdot\boldsymbol{B})}{(\boldsymbol{n}\cdot\nabla)(\theta/\rho)}\,\mathrm{d}S. \tag{1.29}$$

There is a further generalization of the helicity invariant (1.22), which may be constructed as follows (Moffatt 1981): let $\boldsymbol{B}_1 = \nabla\wedge\boldsymbol{A}_1$ and $\boldsymbol{B}_2 = \nabla\wedge\boldsymbol{A}_2$ be two independent solenoidal vector fields convected by a flow $\boldsymbol{u}(\boldsymbol{x}, t)$, and let S be a closed material surface, on which

$$\boldsymbol{n}\cdot\boldsymbol{B}_1 = 0, \quad \boldsymbol{n}\cdot\boldsymbol{B}_2 = 0, \tag{1.30}$$

and containing the material volume V. Then

$$H_{12} = \int_V \boldsymbol{B}_1\cdot\boldsymbol{A}_2\,\mathrm{d}V = \int_V \boldsymbol{B}_2\cdot\boldsymbol{A}_1\,\mathrm{d}V = \text{const.} \tag{1.31}$$

This is the 'cross-helicity' between the fields $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.

2. Helicity invariants and topological structure

The relation between the helicity invariant (1.22) and the topological structure of the field $\boldsymbol{B}$ is made transparently clear (Moreau 1961; Moffatt 1969) through consideration of the particular situation in which $\boldsymbol{B}$ is identically zero except in two flux tubes of small cross-section whose axes are the unknotted closed curves C_1, C_2 which may be linked (figure 2*a*). Obviously, the linked configuration is topologically distinct from the unlinked configuration, and the degree of linkage is conserved under frozen-field distortion. We suppose that the field lines have no net twist within either flux tube; this means that either tube may be continuously deformed to a circular tube (like a bicycle tube) within which the $\boldsymbol{B}$-lines are all circular and unlinked; i.e. the topology of the field within either flux tube in isolation is trivial.

In the limit as the tube cross-sections tend to zero, the volume integral (1.22) (with V the whole space) degenerates to the sum of two line integrals round C_1 and C_2 via the substitutions $\boldsymbol{B}\,\mathrm{d}V \rightarrow \Phi_1\,\boldsymbol{dx}$ on C_1, $\Phi_2\,\boldsymbol{dx}$ on C_2, where Φ_1 and Φ_2 are the magnetic fluxes in the tubes; and since for example

$$\oint_{C_1} \boldsymbol{A}\cdot\boldsymbol{dx} = \int_{S_1} \boldsymbol{B}\cdot\boldsymbol{dx},$$

where S_1 spans C_1, and the latter integral equals $\pm\Phi_2$ it follows that

$$H_M = \pm 2n\Phi_1\Phi_2, \tag{2.1}$$

where n is the relative winding number of C_1 and C_2, and the + or − depends on the field directions in the two tubes. Helicity therefore clearly has a topological character, and the invariance of helicity is a consequence of the topological invariance intrinsic to frozen-field distortion.

The argument in this simple form is of course restricted to fields having closed $\boldsymbol{B}$-lines. The argument has however been adapted to more complex topologies by Arnol'd (1974), who has shown how to give meaning to the double limit

$$\lim_{\substack{n\to\infty \\ \Phi_1\to 0}} n\Phi_1\Phi_2 \tag{2.2}$$

when the curve C_1 winds infinitely around the (still closed) curve C_2. Arnol'd describes this limit as the 'asymptotic Hopf invariant', the relation between integrals of the form (1.22) and the Hopf (1931) invariant of topological mappings having been established earlier by Whitehead (1948).

Helicity thus plays a fundamental role in the topological classification of solenoidal vector fields. However, the complete set of helicity invariants is not nearly sufficient to provide a complete classification. This may be easily seen by considering the case when C_1 and C_2 have the more complex linkage shown in figure 2*b*. Here the toroidal flux across the surface S_1 spanning C_1 is zero, and so the helicity integral is zero as for the case when C_1 and C_2 are unlinked. And yet the two cases are clearly topologically distinct. The topology of figure 1*b* is invariant under evolution governed by (1.13*b*), and so any topological invariant that distinguishes this configuration from the unlinked configuration must be somehow contained in (1.13*b*). How can we construct such an invariant? We develop one possible approach in the following sections.

3. Magnetic relaxation

The style of argument presented in this section was suggested by Arnol'd (1974) and developed by Moffatt (1985), and is an essential preliminary to the consideration of steady Euler flows.

We consider a fluid that is incompressible ($\nabla \cdot \boldsymbol{v} = 0$) and contained in a domain D with boundary ∂D on which $\boldsymbol{v} \cdot \boldsymbol{n} = 0$. We consider again a magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ satisfying

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial D, \tag{3.1}$$

a condition that persists under evolution governed by the frozen-field equation (1.13*b*). We now focus attention on the energy of this field given by

$$M(t) = \frac{1}{2}\int_D \boldsymbol{B}^2 \,\mathrm{d}V, \tag{3.2}$$

which by elementary manipulations satisfies the equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_D \boldsymbol{B} \cdot \nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{B}) \,\mathrm{d}V = -\int \boldsymbol{v} \cdot (\boldsymbol{j} \wedge \boldsymbol{B}) \,\mathrm{d}V, \tag{3.3}$$

where $\boldsymbol{j} = \nabla \wedge \boldsymbol{B}$, the current associated with $\boldsymbol{B}$. (Note that a current sheet $\boldsymbol{j}_s = -\boldsymbol{n} \wedge \boldsymbol{B}|_{\partial D}$ may flow on the boundary.)

Let us adopt the simplest possible equation of motion for the fluid which (i) incorporates the Lorentz force $\boldsymbol{j} \wedge \boldsymbol{B}$ per unit volume appearing naturally in (3.3); (ii) allows $\boldsymbol{v}(\boldsymbol{x}, t)$ to remain solenoidal for all t; and (iii) includes a term that dissipates energy. This is

$$\partial \boldsymbol{v}/\partial t = -\nabla p + \boldsymbol{j} \wedge \boldsymbol{B} - k\boldsymbol{v} \tag{3.4}$$

where $k > 0$ and $p(\boldsymbol{x}, t)$ is a (pressure) field satisfying

$$\left.\begin{aligned} \nabla^2 p &= \nabla \cdot (\boldsymbol{j} \wedge \boldsymbol{B}) \quad \text{in} \quad D, \\ \partial p/\partial n &= \boldsymbol{n} \cdot (\boldsymbol{j} \wedge \boldsymbol{B}) \quad \text{on} \quad \partial D. \end{aligned}\right\} \tag{3.5}$$

Let $K(t)$ be the kinetic energy of the flow:

$$K(t) = \frac{1}{2}\int_D \boldsymbol{v}^2 \,\mathrm{d}V. \tag{3.6}$$

From (3.4), this satisfies

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \int \boldsymbol{v} \cdot (\boldsymbol{j} \wedge \boldsymbol{B}) \,\mathrm{d}V - 2kK, \tag{3.7}$$

so that, from (3.3) and (3.7),

$$\mathrm{d}/\mathrm{d}t(M(t) + K(t)) = -2kK. \tag{3.8}$$

Hence, for so long as $K(t) > 0$, the total energy $M + K$ is monotonic decreasing, and being positive must tend to a limit. Hence, no matter what the initial conditions may be,

$$\left.\begin{aligned} &K(t) \to 0 \\ &M(t) \to M^{(E)} \quad (\text{const.}) \end{aligned}\right\} \quad \text{as} \quad t \to \infty. \tag{3.9}$$

Suppose now that these initial conditions are

$$\boldsymbol{B}(\boldsymbol{x}, 0) = \boldsymbol{B}_0(\boldsymbol{x}), \quad \boldsymbol{v}(\boldsymbol{x}, 0) = 0, \tag{3.10}$$

where $\boldsymbol{B}_0(\boldsymbol{x})$ is an arbitrary solenoidal field of finite energy. In general, the initial Lorentz force $(\nabla \wedge \boldsymbol{B}_0) \wedge \boldsymbol{B}_0$ is not irrotational and so cannot be compensated by the pressure term in (3.4). The fluid must therefore move, and as it does so, it convects the field $\boldsymbol{B}$ as a frozen-in field. Thus the kinetic energy $K(t)$ initially increases from zero (at the expense of magnetic energy) but ultimately tends to zero again according to (3.9). For all finite t, the flow $\boldsymbol{v}(\boldsymbol{x}, t)$ remains smooth, and induces a continuous volume-preserving mapping $\boldsymbol{x} \rightarrow \boldsymbol{X}(\boldsymbol{x}, t)$ associated with the particle paths. For all finite t, the field $\boldsymbol{B}$ evolves through topologically equivalent states, but as $t \rightarrow \infty$, the mapping may develop discontinuities and equally the convected field may develop singularities, although the field energy is strictly under control and bounded from above by its initial value.

For any non-trivial field topology, the magnetic energy is also bounded below and away from zero (Arnol'd 1974; Freedman 1988). In order to understand this, consider a portion of a flux tube of length L and cross-section A carrying uniform field B; the flux $\Phi = BA$ and the volume $V = LA$ are conserved under frozen-field distortion of the kind considered. The contribution to magnetic energy from this portion of tube is $\frac{1}{2}B^2V = \frac{1}{2}V^{-1}\Phi^2L^2$. Hence the energy decreases through reduction of L, i.e. contraction of the tube, together with corresponding increase of its cross-section. The energy can tend to zero only if this process is carried to the limit in which every closed $\boldsymbol{B}$-line contracts to a point, as is possible (figure 3*a*) if the topology is trivial. It cannot happen however if the topology is nontrivial (figure 3*b*); for then the contraction of a flux tube is inevitably impeded by the growth of cross-section of any other flux tube with which it is linked. This argument is given expression in the formal language of topology by Freedman (1988).

It is clear from this description of the process of magnetic relaxation that, as $t \rightarrow \infty$, tangential discontinuities of $\boldsymbol{B}$ must form wherever linked flux tubes are ultimately brought into contact, and that the asymptotic equilibrium state $\boldsymbol{B}^{(E)}(\boldsymbol{x})$ characterized by the magnetic energy $M^{(E)}(>0)$ will generally contain tangential discontinuities (current sheets) imbedded within the domain D. This appears to be the case even if the initial field $\boldsymbol{B}_0(\boldsymbol{x})$ is C^∞, because the rearrangement of $\boldsymbol{B}$-lines during relaxation will still generally tend to produce tangential discontinuities. An example may make this clear: let D be the cylinder $s < a$ (in cylindrical polar coordinates (s, φ, z)) and let $\boldsymbol{B}_0(\boldsymbol{x}) = (0, B_{0\varphi}(s), B_{0z}(s))$, where $B_{0\varphi}(s)$ and $B_{0z}(s)$ are C^∞ functions of non-overlapping bounded supports, as indicated in figure 4. Relaxation can proceed through rearrangement of the circular $\boldsymbol{B}$-lines in the region $s_0 < s < a$, the 'stronger' field lines from the outer region near $s = a$ displacing the 'weaker' field lines in the inner region near $s = s_0$. Minimum energy is achieved by a field of the form

$$\boldsymbol{B}^E(\boldsymbol{x}) = (0, B^E(s), B_{0z}(s)) \tag{3.11}$$

where $s^{-1}B^E(s)$ is the 'rearrangement' of $s^{-1}B_{0\varphi}(s)$ that makes

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{B^E(s)}{s}\right)^2 \leqslant 0 \quad (s_0 \leqslant s \leqslant a). \tag{3.12}$$

In this state, there is clearly a tangential discontinuity of B^E of magnitude $s_0 \max |B_{0\varphi}(s)/s|$ across $s = s_0$. Note that the rearrangement is achieved by a flow of the form

$$\boldsymbol{v}(\boldsymbol{x}, t) = (v_s(s, z, t), \quad 0, \quad v_z(s, z, t)) \quad (t > 0), \tag{3.13}$$

but the asymptotic state is z-independent.

In general, therefore, we are driven to the conclusion that $\boldsymbol{B}$ relaxes to a magnetostatic equilibrium state $\boldsymbol{B}^E(\boldsymbol{x})$ which is topologically accessible from $\boldsymbol{B}_0(\boldsymbol{x})$ in

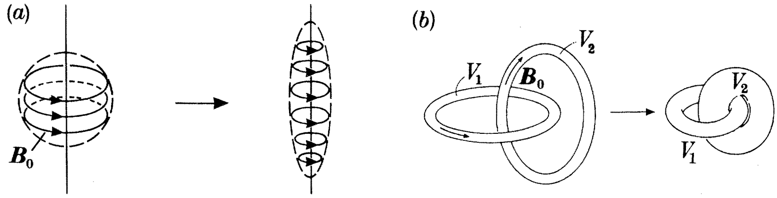

Figure 3. The process of magnetic relaxation. (*a*) Trivial topology, for which magnetic energy can decrease to zero. (*b*) Non-trivial topology providing an impediment to relaxation.

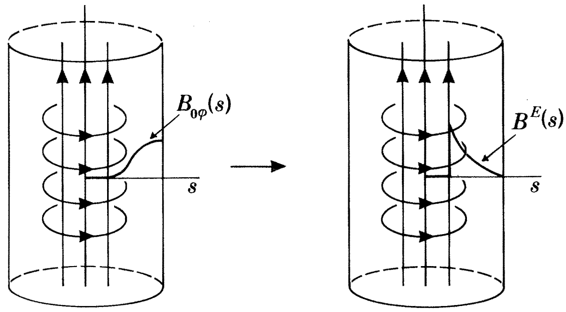

Figure 4. Relaxation of a cylindrically symmetric field to a state of minimum magnetic energy.

the sense that it is obtained from $\boldsymbol{B}_0(\boldsymbol{x})$ through convection and distortion by a flow $\boldsymbol{v}(\boldsymbol{x}, t)$ $(0 \leqslant t < \infty)$ which dissipates a finite fraction of the initial field energy. $\boldsymbol{B}^E(\boldsymbol{x})$ may have tangential discontinuities, or possibly more awkward singularities (e.g. accumulation points of tangential discontinuities); it would be nice to characterise the properties of the function space in which the relaxed fields $\boldsymbol{B}^E(\boldsymbol{x})$ reside; this is as yet an unsolved problem.

Being a magnetostatic equilibrium (with $\boldsymbol{v} = 0$), $\boldsymbol{B}^E(\boldsymbol{x})$ satisfies the equation

$$\boldsymbol{j}^E \wedge \boldsymbol{B}^E = (\nabla \wedge \boldsymbol{B}^E) \wedge \boldsymbol{B}^E = \nabla p^E \tag{3.14}$$

for some scalar field $p^E(\boldsymbol{x})$. The field is characterised by its magnetic energy M^E which is clearly minimal with respect to small frozen-field perturbations of the medium (for a full discussion see Moffatt 1986*a*). There may be more than one such equilibrium topologically accessible from an initial field $\boldsymbol{B}_0(\boldsymbol{x})$ by different routes in function space (by adopting a non-zero initial condition for $\boldsymbol{v}(\boldsymbol{x}, t)$, or by using a different dissipative mechanism in (3.4) or by varying the relaxation process in some other way). Among these equilibria, however, there is always one whose energy $M^E (> 0)$ is least (or more than one with equal least energy); such a state (or states) are the most stable (or 'ground') states available to the field with its prescribed topology.

These considerations have interesting implications for the theory of topological invariants of knots and links in $\mathbb{R}^3$. Suppose for example that we have an arbitrary knot K, and let $\mathscr{T}_\epsilon(K)$ be a tubular neighbourhood of the knot of cross-section $A = \pi\epsilon^2$. Let $\boldsymbol{B}_0(\boldsymbol{x})$ be a field of magnitude B_0 within $\mathscr{T}_\epsilon(K)$ aligned along the axis of the tube, so that the magnetic flux is $\Phi = B_0 A$. The volume of $\mathscr{T}_\epsilon(K)$ is LA where L is the length of the knot. We now let this field relax as already described. During this process Φ and V are invariant, and so also is the helicity of the field, which, for dimensional reasons, must be of the form

$$H = h\Phi^2 \tag{3.15}$$

where h is a dimensionless number (which may be positive or negative or zero). As shown by Berger & Field (1984), with a particular convention for the field twist within the tube, $h = N_+ - N_-$ where $N_\pm$ are the numbers of positive and negative crossovers in a plane projection of the knot. Even more interestingly, the field relaxes to a minimum energy state compatible with the knot topology, and the energy M^E in this state, again for dimensional reasons, must have the form

$$M^E = m\Phi^2 V^{-\frac{1}{3}}, \tag{3.16}$$

where $m(>0)$ is a real number determined solely by the original knot topology. The number m is a topological invariant, and generally (excluding mirror symmetries) two topologically distinct knots will yield distinct values of m. Of course if there is more than one minimum energy state for a given knot, then we obtain a sequence of numbers $(m_0, m_1, m_2, \ldots)$ with $0 < m_0 \leqslant m_1 \leqslant m_2 \leqslant \ldots$ characterizing these states. Similar considerations apply to links. We may then talk unambiguously of 'the energy spectrum of knots and links', a point of view developed in more detail by Moffatt (1990*b*). Parallel developments on the purely topological side are given in Freedman & He (1990*a*, *b*).

4. Stability of magnetostatic equilibria and of analogous Euler flows

The techniques described in §3 provide a means by which magnetostatic equilibrium states of prescribed magnetic field topology may (at least in principle) be constructed. It is well-known that such states are characterized by a magnetic energy that is stationary with respect to small frozen-field displacements, and that stability is assured if the magnetic energy is in fact minimal with respect to such displacements (Bernstein *et al.* 1958). The magnetic relaxation technique will generally yield such stable states, since the magnetic energy may be expected in general to decrease to a minimum rather than a saddle point within the subspace of fields that are topologically accessible from the initial field.

Let $\boldsymbol{\xi}(\boldsymbol{x})$ be a small volume-preserving virtual displacement of the medium. Then, as described in Moffatt (1986*a*), the first and second order variations of $\boldsymbol{B}$ about the equilibirum state $\boldsymbol{B}^E$ are given by

$$\delta^1 \boldsymbol{B} = \nabla \wedge (\xi \wedge \boldsymbol{B}^E), \quad \delta^2 \boldsymbol{B} = \tfrac{1}{2}\nabla \wedge (\xi \wedge \delta \boldsymbol{B}^1), \tag{4.1}$$

and the corresponding variations of magnetic energy are

$$\delta^1 M = \int_D \boldsymbol{B}^E \cdot \delta^1 \boldsymbol{B}\, \mathrm{d}V \tag{4.2}$$

$$\delta^2 M = \int_D (\boldsymbol{B}^E \cdot \delta^2 \boldsymbol{B} + \tfrac{1}{2}(\delta^1 \boldsymbol{B})^2)\, \mathrm{d}V. \tag{4.3}$$

It is easy to show that $\delta^1 M = 0$ when $\boldsymbol{B}^E$ satisfies the magnetostatic equilibrium conditions, and stability is then guaranteed provided

$$\delta^2 M \geqslant 0 \tag{4.4}$$

for all 'admissible' displacement fields $\boldsymbol{\xi}(\boldsymbol{x})$. For the reasons given above, we expect (4.4) to be satisfied for any magnetostatic equilibrium that is the outcome of a magnetic relaxation process. For example, the condition (3.12) is equivalent to (4.4) for the cylindrically symmetric equilibrium (3.11), provided that only axisymmetric displacement fields are admitted.

The magnetostatic equilibria considered above have a double interest, since to each such equilibrium, there corresponds (by exact analogy) a solution of the steady Euler equations of classical (inviscid) fluid mechanics. Thus, by the substitutions

$$\boldsymbol{B}^E \to \boldsymbol{u}^E, \quad \boldsymbol{j}^E \to \boldsymbol{\omega}^E, \quad p^E \to -h^E, \tag{4.5}$$

the magnetostatic equations

$$\boldsymbol{j}^E \wedge \boldsymbol{B}^E = \nabla p^E, \quad \boldsymbol{j}^E = \nabla \wedge \boldsymbol{B}^E, \quad \nabla \cdot \boldsymbol{B}^E = 0 \tag{4.6}$$

are replaced by the steady Euler equations

$$\boldsymbol{u}^E \wedge \boldsymbol{\omega}^E = \nabla h^E, \quad \boldsymbol{\omega}^E = \nabla \wedge \boldsymbol{u}^E, \quad \nabla \cdot \boldsymbol{u}^E = 0 \tag{4.7}$$

for an incompressible inviscid fluid. Here, h^E is the (Bernoulli) total 'head' in the analogue fluid. The minus sign in the analogue relation $p^E \to -h^E$ is to be particularly noted; physically, this is related to the fact that the Lorentz force associated with a curved magnetic flux tube acts towards the centre of curvature, whereas the centrifugal force in the analogous curved stream tube acts away from the centre of curvature. This change of sign is immaterial as far as the structure of equilibrium states is concerned; but it is of critical importance in relation to the stability of these states.

The stability problem for the analogous Euler flow $\boldsymbol{u}^E(\boldsymbol{x})$ is different from that for the magnetostatic field $\boldsymbol{B}^E(\boldsymbol{x})$ because, under the unsteady Euler equations

$$\partial \boldsymbol{u}/\partial t = \boldsymbol{u} \wedge \boldsymbol{\omega} - \nabla h, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{4.8}$$

it is the vorticity field, rather than the velocity field, that has a 'frozen-in' character. Thus, under virtual displacements $\boldsymbol{\xi}(\boldsymbol{x})$, the first and second order perturbations of vorticity (cf. (4.11)) are

$$\delta^1 \boldsymbol{\omega} = \nabla \wedge (\boldsymbol{\xi} \wedge \boldsymbol{\omega}^E), \quad \delta^2 \boldsymbol{\omega} = \tfrac{1}{2} \nabla \wedge (\boldsymbol{\xi} \wedge \delta^1 \boldsymbol{\omega}). \tag{4.9}$$

The associated first and second order perturbations of velocity are then

$$\delta^1 \boldsymbol{u} = (\boldsymbol{\xi} \wedge \boldsymbol{\omega}^E)_{\mathrm{s}}, \quad \delta^2 \boldsymbol{u} = \tfrac{1}{2} (\boldsymbol{\xi} \wedge \delta^1 \boldsymbol{\omega})_{\mathrm{s}}, \tag{4.10}$$

where the suffix s denotes the 'solenoidal projection' of the vector, obtained by standard techniques, to guarantee that

$$\nabla \cdot \delta^1 \boldsymbol{u} = \nabla \cdot \delta^2 \boldsymbol{u} = 0. \tag{4.11}$$

The first and second order variations of kinetic energy are then given (cf. (4.2) and (4.3)) by

$$\delta^1 K = \int_D \boldsymbol{u}^E \cdot \delta^1 \boldsymbol{u} \, \mathrm{d}V, \tag{4.12}$$

$$\delta^2 K = \int_D (\boldsymbol{u}^E \cdot \delta^2 \boldsymbol{u} + \tfrac{1}{2} (\delta^1 \boldsymbol{u})^2) \, \mathrm{d}V. \tag{4.13}$$

Again, it is easy to show that, by virtue of (4.7) together with $\boldsymbol{u}^E \cdot \boldsymbol{n} = 0$ on ∂D,

$$\delta^1 K = 0. \tag{4.14}$$

Note now, however, that $\delta^2 K$ is a different functional of $\boldsymbol{\xi}$ from $\delta^2 M$, so that we may make no deduction from (4.4) concerning the sign of $\delta^2 K$. The best that can be deduced in general (Moffatt 1986*a*) is that (on identifying the equilibrium states $\boldsymbol{B}^E$ and $\boldsymbol{u}^E$),

$$\delta^2 K \geqslant -\delta^2 M \tag{4.15}$$

for all volume-preserving displacements ξ. Thus the kinetic energy of an Euler flow that is the analogue of a stable magnetostatic equilibrium may be maximal or minimal or neither (i.e. a saddle) with respect to virtual displacements, the vorticity field being frozen in the fluid.

According to an argument of Arnol'd (1966) the Euler flow is stable if the kinetic energy is maximal or minimal, i.e. if

$$\delta^2 K \geqslant 0 \quad \text{for all admissible } \xi, \tag{4.16}$$

or

$$\delta^2 K \leqslant 0 \quad \text{for all admissible } \xi. \tag{4.17}$$

The reason is that the perturbed flow $\boldsymbol{u}(\boldsymbol{x}, t)$ evolves under the kinetic energy constraint $K = \text{const.}$ If K is maximal when $\boldsymbol{u} = \boldsymbol{u}^E(\boldsymbol{x})$, then the perturbed flow remains permanently in a neighbourhood of $\boldsymbol{u}^E(\boldsymbol{x})$ (at least with respect to an energy norm); similarly if K is minimal. However, if K is neither maximal or minimal, the 'surfaces' $K = \text{const.}$ have hyperbolic structure near the equilibrium point $\boldsymbol{u}^E(\boldsymbol{x})$ in the relevant function space, and the condition $K = \text{const.}$ therefore places no constraint on the magnitude of the perturbation $\boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{u}^E(\boldsymbol{x})$; hence if $\delta^2 K$ is indefinite in sign, the flow $\boldsymbol{u}^E(\boldsymbol{x})$ may (and presumably will) be unstable.

5. The relaxation procedure of Vallis *et al.* (1989)

The magnetic relaxation procedure described in §3 is 'natural' to the problem of locating stable magnetostatic equilibria because it respects the frozen-in character of the magnetic field, which is the essential feature of the stability problem. It is not natural to the problem of locating stable Euler flows, because magnetic relaxation occurs in a subspace that does not span the space of perturbations governed by the unsteady Euler equations in which the vorticity field is frozen-in.

It is important therefore to enquire whether there are alternative relaxation procedures that are natural to the Euler equations in the sense that relaxation to equilibrium occurs in the subspace in which perturbations most naturally evolve, i.e. the subspace of flows $\boldsymbol{\omega}(\boldsymbol{x}, t)$ for which the vorticity field $\boldsymbol{\omega}(\boldsymbol{x}, t)$ is topologically accessible from some initial reference field $\boldsymbol{\omega}_0(\boldsymbol{x})$. One such procedure, which we shall describe as 'VCY relaxation', has been devised by Vallis *et al.* (1989): Suppose that the vorticity $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$ is artificially constrained to evolve under the frozen-field equation

$$\partial \boldsymbol{\omega}/\partial t = \nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{\omega}), \tag{5.1}$$

where

$$\boldsymbol{v} = \boldsymbol{u} + \alpha\, \partial \boldsymbol{u}/\partial t \tag{5.2}$$

and α is a constant. Under evolution determined by (5.1), the topology of the vorticity field is certainly conserved; in particular, the helicity

$$\mathscr{H} = \int_D \boldsymbol{u} \cdot \boldsymbol{\omega}\, \mathrm{d}V \tag{5.3}$$

is invariant, being still a measure of the 'degree of knottedness' of the vorticity field. The kinetic energy K of the flow is not, however, conserved when $\alpha \neq 0$; in fact elementary manipulations yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_D \boldsymbol{u}^2\, \mathrm{d}V = -\alpha \int (\partial \boldsymbol{u}/\partial t)^2\, \mathrm{d}V. \tag{5.4}$$

Hence, if $\alpha > 0 (< 0)$, then K is monotonic decreasing (increasing) for so long as $\partial \boldsymbol{u}/\partial t \neq 0$. Unfortunately, this is not sufficient in general to guarantee the existence of non-trivial steady states. Unless either a non-zero lower bound or an upper bound can be placed on K, there is no guarantee that K does not simply tend to zero when $\alpha > 0$, or that K does not increase without limit when $\alpha < 0$. In either case, no useful conclusion may be drawn.

There are, however, two special circumstances in which an upper bound can be placed on K, and useful conclusions may be drawn.

(*a*) *Two-dimensional flows*

Suppose that

$$\boldsymbol{u} = (\partial\psi/\partial y,\ -\partial\psi/\partial x, 0) \tag{5.5}$$

and

$$\boldsymbol{\omega} = (0,\ 0,\ \omega_z), \tag{5.6}$$

where $\omega_z = -\nabla^2\psi$ and $\psi = \psi(x,\ y,\ t)$. Then (5.1) describes convection of the vortex lines by the flow $\boldsymbol{v}$, and the enstrophy of the flow,

$$\Omega = \int_D \omega_x^2 \,\mathrm{d}x\,\mathrm{d}y, \tag{5.7}$$

is conserved. Moreover the kinetic energy is bounded by a Poincaré inequality of the form

$$K \leqslant kA\Omega = \text{const.}, \tag{5.8}$$

where A is the cross-sectional area of the (two-dimensional) domain D, and k is a dimensionless number of order unity determined solely by the shape of D. Hence, choosing $\alpha < 0$ in (5.2) and (5.4), K is monotonic increasing and bounded above and therefore tends to a constant. Hence from (5.4), $\partial \boldsymbol{u}/\partial t \rightarrow 0$ (at least almost everywhere) and so as described by Vallis *et al.* (1989), $\boldsymbol{u}$ tends to an equilibrium state $\boldsymbol{u}^E(\boldsymbol{x})$ whose vorticity field $\boldsymbol{\omega}^E(\boldsymbol{x})$ is topologically accessible from the initial field $\boldsymbol{\omega}_0(\boldsymbol{x})$. In this two-dimensional context, the vorticity field $\boldsymbol{\omega}^E = (0,\ 0,\ \omega_z^E)$ is obtained simply by rearrangement of the vortex lines of $\boldsymbol{\omega}_0 = (0,\ 0,\ \omega_0)$; the word 'isovortical' is frequently used in this context to describe two vorticity fields $\omega_1(\boldsymbol{x})$, $\omega_2(\boldsymbol{x})$ such that

$$\omega_2(\boldsymbol{X}) = \omega_1(\boldsymbol{x})$$

where $\boldsymbol{x} \rightarrow \boldsymbol{X}$ is an area-preserving orientable continuous mapping in the plane (i.e. a rearrangement in the above sense).

Flows obtained by this procedure will in general have maximal energy with respect to isovortical perturbations, and will therefore be stable, by Arnol'd's (1966) criterion.

We note that the procedure described above has close points of contact with the procedure used by Campbell & Kadtke (1987) (see Aref *et al.* 1988) to determine absolutely stationary configurations of systems of point vortices, i.e. steady solutions of the Euler equations of prescribed (and very particular) vorticity topology.

(*b*) *Axisymmetric flows*

Suppose now that, in spherical polar coordinates $(r,\ \theta,\ \phi)$,

$$\boldsymbol{u} = \left(\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\varphi},\ \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r},\ 0\right) \tag{5.9}$$

and
$$\boldsymbol{\omega} = (0,\ 0,\ \omega_\varphi(r,\ \theta,\ t)), \tag{5.10}$$
where $\psi = \psi(r,\ \theta,\ t)$ and
$$\omega_\varphi = -\frac{1}{r\sin\theta}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right). \tag{5.11}$$
In this case (5.1) implies that
$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\omega_\varphi}{r\sin\theta}\right) = 0, \tag{5.12}$$
where
$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla, \tag{5.13}$$
so that, in particular,
$$\int\left(\frac{\omega_\varphi}{r\sin\theta}\right)^4 \mathrm{d}V = \text{const.} \tag{5.14}$$
Hence the enstrophy is bounded above, since
$$\begin{aligned}\Omega = \int_D \omega_\varphi^2\,\mathrm{d}V &= \int\left(\frac{\omega_\varphi}{r\sin\theta}\right)^2\cdot(r\sin\theta)^2\,\mathrm{d}V \\ &\leqslant \left\{\int_D\left(\frac{\omega_\varphi}{r\sin\theta}\right)^4\mathrm{d}V\int_D(r\sin\theta)^4\mathrm{d}V\right\}^{\frac{1}{2}} = \text{const.}\end{aligned} \tag{5.15}$$
Hence K is also bounded above, since an inequality of the form (5.8) still applies, A being now the area of a meridional section of the axisymmetric domain D.

Hence in this case also VCY relaxation with $\alpha < 0$ will in principle yield stable Euler flows for which again the vorticity field $\boldsymbol{\omega}^E(\boldsymbol{x})$ is topologically accessible from the initial field $\boldsymbol{\omega}_0(\boldsymbol{x})$.

The procedure of Vallis *et al.* (1989) has been placed in a more general context by Shepherd (1990) who shows how any hamiltonian dynamical system may be modified in such a way as to drive the system towards an energy maximum or minimum (if such a state exists) while conserving those invariants (known in two-dimensional contexts as Casimirs) that are essentially topological in character. The technique can therefore be applied not only to the Euler flow problem, but also to more complex systems of equations involving effects of stratification and/or compressibility. A more elaborate relaxation procedure has also been advocated (Moffatt 1989) to establish the existence of steady solutions $\{\boldsymbol{u}(\boldsymbol{x}),\ \boldsymbol{B}(\boldsymbol{x})\}$ of the magnetohydrodynamic equations of an ideal fluid, the topology of both fields $\boldsymbol{u}$ and $\boldsymbol{B}$ being prescribed in a compatible manner.

6. Flow signature

Consider again the magnetic relaxation problem in a two dimensional domain D, with magnetic field expressible in terms of a flux function $\chi(x,\ y,\ t)$ by
$$\boldsymbol{B} = (\partial\chi/\partial y,\ -\partial\chi/\partial x,\ 0). \tag{6.1}$$
The $\boldsymbol{B}$-lines are then the contours $\chi = \text{const.}$, and in particular we may suppose that
$$\chi = 0 \quad \text{on} \quad \partial\mathscr{D}. \tag{6.2}$$

Let us suppose first that $\chi_0(x, y) = \chi(x, y, 0)$ has only one stationary point P_0 in the interior of D and that this is a maximum; then $\boldsymbol{B} = 0$ at P_0 and the $\boldsymbol{B}$-lines are (in general) elliptic in a neighbourhood of P_0.

Let $A(\chi_c)$ be the area enclosed by the curve $\chi_0 = \chi_c$ where χ_c is a constant satisfying $0 \leqslant \chi_c \leqslant \chi_{max}$. Obviously, $A(\chi_c)$ is monotonic decreasing in this interval, with

$$A(0) = A_D, \quad A(\chi_{max}) = 0 \tag{6.3}$$

where A_D is the area of D. Moreover, if $\boldsymbol{B}$ is differentiable, then

$$A = O(\chi_{max} - \chi_c) \quad \text{when} \quad \chi_c \rightarrow \chi_{max}. \tag{6.4}$$

During magnetic relaxation, the $\boldsymbol{B}$-lines are frozen in the fluid and the frozen-field equation becomes simply $\mathrm{D}\chi/\mathrm{D}t = 0$. If we focus attention on one $\boldsymbol{B}$-line $\chi(x, y, t) = \chi_c$, then, since the flow is incompressible, the area within this $\boldsymbol{B}$-line is constant, i.e. the function $A(\chi_c)$ is an invariant of the relaxation process. It is therefore appropriate to describe $A(\chi_c)$ as the signature of the field (Moffatt 1986*b*).

Since relaxation proceeds in such a way as to decrease magnetic energy, it is obvious that $|\boldsymbol{B}|$ must remain everywhere bounded and hence $\chi(x, y, t)$ remains differentiable for all t. In the asymptotic equilibrium situation ($t \rightarrow \infty$), $\chi(x, y, t) \rightarrow \chi^E(x, y)$, where

$$\nabla^2 \chi^E = F(\chi^E) \tag{6.5}$$

for some (current) function $F(\chi^E)$; this is the well-known Grad–Shafranov equation describing two-dimensional magnetostatic equilibrium. The nature of the relaxation process allows us to assert that, for every signature function satisfying (6.3), (6.4) and $A'(\chi_c) \leqslant 0$, there exists a magnetostatic equilibrium in D; the function $F(\chi^E)$ characterising this equilibrium is in principle determined by the signature function which may equally be expressed as a function $A(\chi^E)$. Indeed, elimination of χ^E between the equations

$$F = F(\chi^E), \quad A = A(\chi^E) \tag{6.6}$$

in general implies a relation

$$F = F(A). \tag{6.7}$$

A simple example may make this clear. If D is the elliptic domain

$$D: \quad x^2/a^2 + y^2/b^2 < 1 \tag{6.8}$$

then (6.5) is satisfied by

$$\chi^E(x, y) = \chi_{max}(1 - x^2/a^2 - y^2/b^2) \tag{6.9}$$

provided

$$F(\chi^E) = -2\chi_{max}(1/a^2 + 1/b^2) = \text{const.} \tag{6.10}$$

The signature function for this field is

$$A(\chi) = \pi ab(1 - \chi/\chi_{max}). \tag{6.11}$$

Hence only a field $\boldsymbol{B}_0(x, y)$ with this (linear) signature can relax to the equilibrium (6.9) for which all field lines are ellipses.

Suppose now that we start at $t = 0$ with a field $\boldsymbol{B}_0(x, y)$ with elliptic streamlines so that

$$\chi_0(x, y) = G(1 - x^2/a^2 - y^2/b^2) \tag{6.12}$$

for some monotonic function $G(\cdot)$. The signature function is then

$$A(\chi_c) = \pi ab(1 - G_0^{-1}(\chi_c)). \tag{6.13}$$

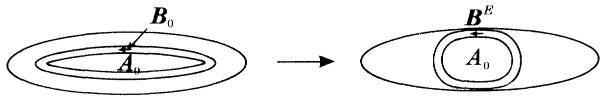

Figure 5. Relaxation of a field initially confined to an elliptic pathway within an elliptic domain; the path contracts to a minimum length, the contained area A_0 being conserved.

Hence an initial field with arbitrary (monotonic) signature can easily be constructed; e.g. if

$$A(\chi_c) = \pi ab(1-\chi_c/\chi_{max})^2, \tag{6.14}$$

then

$$G(Y) = [1-(1-Y)^{\frac{1}{2}}]\chi_{max} \tag{6.15}$$

and the required initial field is

$$\chi_0(x, y) = [1-(x^2/a^2+y^2/b^2)^{\frac{1}{2}}]\chi_{max}. \tag{6.16}$$

This field cannot relax to the field (6.9), but must relax to a smooth magnetostatic equilibrium with non-elliptic $\boldsymbol{B}$-lines, and non-trivial current function $F(\chi^E)$.

The reason for the appearance of non-elliptic $\boldsymbol{B}$-lines can be easily understood: in the absence of any boundary constraint, each $\boldsymbol{B}$-line would relax to a configuration of minimum length for prescribed contained area, i.e. to a circle. In the presence of the elliptic boundary, this process is impeded, but field lines on which the field is relatively strong will, as it were, win in the tendency to become circular, and a compromise between this tendency and the boundary constraint will be achieved. An extreme situation is illustrated in figure 5, which shows relaxation of a field initially confined to an elliptic 'pathway' within D; if the area A_0 inside this pathway is less than πb^2, then relaxation to circular $\boldsymbol{B}$-lines is possible; if however $\pi b^2 < A_0 < \pi a^2$, as illustrated, then the relaxed field lies on a pathway that is part circular and part elliptic, with smooth joining at the boundary. This is the path of minimum length containing the prescribed area A_0.

More generally, the problem of determining two-dimensional magnetostatic equilibria (and hence of course of analogous Euler flows) is evidently expressible in the following form: for given D and given signature $A(\chi)$ satisfying (6.3) and (6.4), find the function $\chi(x, y)$ that minimizes the energy $\frac{1}{2}\int_D (\nabla\chi)^2 \,dx\,dy$. Specification of $A(\chi)$ amounts to a topological constraint. The method of magnetic relaxation provides a computational algorithm for the solution of a wide range of problems of this kind.

The signature concept is easily extended to axisymmetric configurations described by a (Stokes) flux function $\chi(r, \theta, \varphi)$. In this case it is the volume $V(\chi)$ of each torus $\chi =$ const. that is conserved during magnetic relaxation, so that $V(\chi)$ is the appropriate signature. If in addition there is a toroidal component of field $B_\varphi(r, \theta, t)$, then the toroidal flux $W(\chi)$ within the torus $\chi =$ const. is also conserved, so that the signature is now the pair $\{V(\chi), W(\chi)\}$. We shall find a use for this in §9 below.

7. The formation of discontinuities near saddle points

If the initial field $\chi_0(x, y)$ in the two-dimensional relaxation considered in §6 has any saddle points, then tangential discontinuities of $\boldsymbol{B}$ may form during relaxation, by the mechanism indicated in figure 6. The $\boldsymbol{B}$ lines are hyperbolic in the

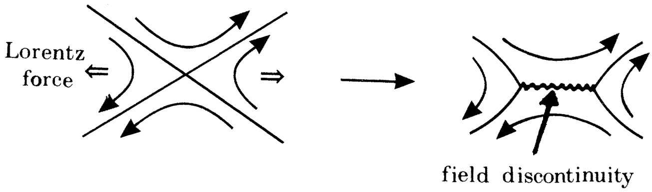

Figure 6. Mechanism by which a tangential discontinuity may form at an intersection of separatrices.

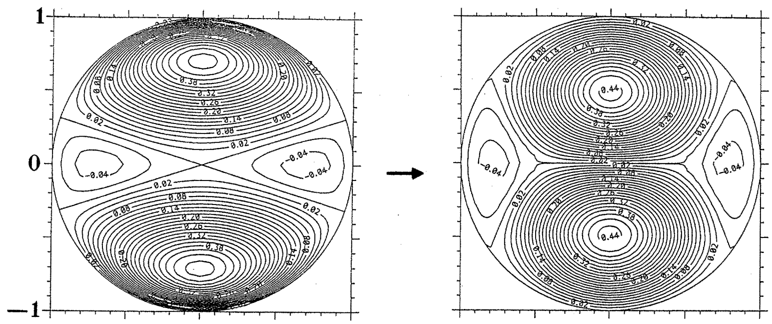

Figure 7. Relaxation of the prototype field (7.1), showing the development of a tangential discontinuity (Bajer 1989).

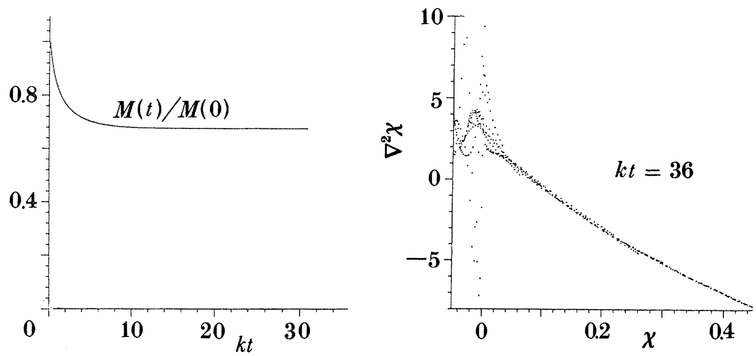

Figure 8. (*a*) Relaxation of energy for the process depicted in figure 7. (*b*) Plot of $\nabla^2\chi$ against χ, showing evidence of a functional relation, when $kt = 36$ (Bajer 1989).

neighbourhood of the saddle point, and the Lorentz force may act in such a way as to cause the angle between the separatrices to collapse to zero. This process has been analysed numerically by Bajer (1989); figure 7 shows the nature of the relaxation process, computed on the basis of (1.13) and (3.4) for initial flux function

$$\chi_0(r, \theta) = r^2(1-r^2)(\cos^2\theta_0 - \cos^2\theta), \tag{7.1}$$

where $2\theta_0$ is the acute angle between the separatrices at $r = 0$. The contours $A =$ const. are shown for $\cos^2\theta_0 = 0.8 (\theta_0 = 27°)$ and $kt = 0, 36$, and the formation of the field discontinuity is clear. There are severe numerical problems in following this process to the asymptotic limit, but the qualitative nature of the process is clear. During this relaxation process, the energy settles down quite rapidly (figure 8*a*) to its asymptotic level (about 67% of its initial value) but fine-scale adjustment to equilibrium in the neighbourhood of the cuspidal singularities is relatively slow as may be seen from the plot of $\nabla^2\chi$ against χ at $kt = 36$ (figure 8*b*). In equilibrium a functional relation $\nabla^2\chi = F(\chi)$ must emerge, and the scatter of points near $\chi = 0$ provides an indication of persisting disequilibrium near the cusps. There seems little doubt however that the equilibrium field structure is nearly attained in figure 7*b* at $kt = 36$, and only numerical problems limit the accuracy with which this equilibrium state may be determined.

Our interest here is in the analogous Euler flow, for which the asymptotic flux function $\chi^E(r,\theta)$ is replaced by an analogous stream function $\psi^E(r, \theta)$. Figure 7*b* then represents the streamlines of a steady Euler flow, with a non-uniform vortex sheet on the segment between the two cusps. This flow is presumably unstable to a Kelvin–Helmholtz type of instability; but its existence is nevertheless of considerable interest, and it is hard to see how this existence could be inferred other than via the magnetostatic analogy and the magnetic relaxation argument.

Note that the separatrices divide the flow domain into four subdomains $D_i (i = 1, 2, 3, 4)$, within each of which a signature function $A_i(\chi)$ may be defined, which is invariant during the relaxation process. For a general domain D, the general initial flux function $\chi_0(r, \theta)$ is characterized topologically (i) by the topology of the network of separatrices which divide the domain into, say n subdomains $D_i (i = 1, 2, \dots, n)$, and (ii) by the set of signature functions $\{A_i(\chi)\}$ for these subdomains. During relaxation, the topology of the network can change only through the collapse of angles and associated formation of current sheets (i.e. tangential discontinuities) as described above; the set $\{A_i(\chi)\}$ remains invariant however and survives to characterise the asymptotic relaxed field.

The general Euler flow obtained by this process may thus be expected to contain a number of tangential discontinuities (i.e. vortex sheets) of finite extent corresponding to all the saddle points of the initial reference function $\chi_0(r, \theta)$. The pressure field p^E in the Euler flow is continuous across each of these sheets, as may be seen by integrating the Euler equation across the discontinuity.

8. Formation of discontinuities under VCY relaxation

Consider again the relaxation process advocated by Vallis *et al.* (1989), applied now to a two dimensional flow in the domain D with initial stream-function $\psi_0(r, \theta)$ satisfying

$$\psi_0 = 0 \quad \text{on} \quad \partial D. \tag{8.1}$$

In general, the initial vorticity field $\omega_0 = -\nabla^2\psi_0$ does not satisfy the condition

$$\omega_0 = \text{const.} \quad \text{on} \quad \partial D. \tag{8.2}$$

This presents an immediate difficulty, because the VCY procedure involves convection of the vorticity ω by a velocity field $\boldsymbol{v}$ satisfying $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on ∂D; hence if (8.2) is not satisfied, then ω can never become constant on ∂D, i.e. equilibrium described by $\omega = F(\psi)$ can never be attained at the boundary.

For this reason, we shall restrict attention in the following discussion to initial fields $\psi_0(r, \theta)$ for which both the conditions (8.1) and (8.2) are satisfied. Relaxation then proceeds according to the equation

$$\mathrm{D}\omega/\mathrm{D}t = 0, \quad \omega(r, \theta, 0) = \omega_0(r, \theta), \tag{8.3}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{v}\cdot\nabla$ and $\boldsymbol{v}$ is defined by (5.2) with $\alpha < 0$. Here it is the topology

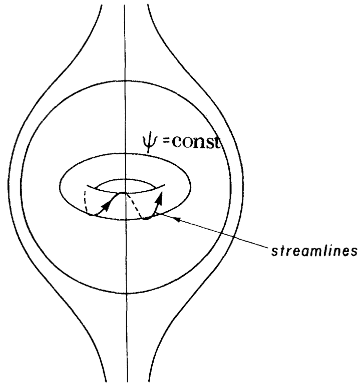

Figure 9. Axisymmetric vorton with swirl. This flow is characterized by a signature $\{V(\psi), W(\psi)\}$, when $V(\psi)$ is the volume within a torus $\psi = \text{const.}$, and $W(\psi)$ is the azimuthal flux within the torus.

of the initial vorticity field $\omega_0(r, \theta)$ that is relevant. Again this field defines a network of separatrices dividing D into subdomains $D_i (i = 1, 2, \ldots, n)$, and again we have a set of signature functions $\{A_i(\omega)\}$ now involving the (conserved) area within any closed contour $\omega = \text{const.}$ Now during VCY relaxation, there is no impediment to the collapse of angles between separatrices with associated formation of discontinuities of ω, but the set $\{A_i(\omega)\}$ remains as the crucial topological invariant of the process.

In the steady state predicted by Vallis *et al.* (1989) the streamlines must of course coincide with the iso-vorticity contours $\omega = \text{const.}$, and so the picture is similar to that described in §7, only now the tangential discontinuities are discontinuities of shear-rate and not of velocity. Numerical simulations of two-dimensional turbulence (e.g. Brachet *et al.* 1986) frequently indicate the presence of layers of very high vorticity gradient, suggesting that the steady Euler flows, whose existence is implied by the VCY procedure, may play some kind of 'attracting' role in the governing unsteady dynamics.

9. Vortons

We use the word vorton, as in Moffatt (1986*b*), to denote a rotational disturbance, the vorticity field being of bounded support, which propagates without change of structure and with constant velocity in an unbounded fluid. Relative to a frame of reference that moves with the vorton, the flow is steady, and as shown by Moffatt (1986*b*, 1988), a very wide family of axisymmetric vortons, both with and without a swirl component of velocity about the axis of symmetry, may be obtained by the magnetic relaxation technique, coupled with the analogy (4.5). These vortons are characterized by a signature $\{V(\psi), W(\psi)\}$ within the rotational region D_R; outside this region, the flow is the unique potential flow settling to a uniform stream at infinity and matching smoothly to the rotational flow across ∂D_R. The configuration is sketched in figure 9; note that, provided the initial field has no saddle points off the axis of symmetry, field discontinuities cannot form during the magnetic relaxation process, and so the vortons obtained by this method are characterized by a continuous velocity field; however, the vorticity field is generally discontinuous across ∂D_R (as for Hill's spherical vortex, or for any of the family of spherical vortices with swirl discovered by Hicks (1899)).

As discussed in §4, there is no guarantee that these vortons are stable, even to axisymmetric disturbances, within the framework of the unsteady Euler equations. If the VCY relaxation procedure could be adapted to the vorton configuration (i.e. to an infinite domain with $\boldsymbol{u}$ tending to a uniform stream at infinity) then this particular difficulty would be overcome. There are difficulties in achieving this, however, because of the need to choose the frame of reference in which the asymptotic flow (as $t \rightarrow \infty$) will be steady, not a straightforward matter since the VCY procedure (involving an artificial modification of the Euler equations) does not conserve momentum. And yet, as argued by Benjamin (1976), vortex rings are characterized by maximality of kinetic energy within the class of axisymmetric flows attainable by rearrangement of the circular vortex lines (cf. (5.12)); so one would expect the VCY procedure, which increases kinetic energy within just such a class of flows, to be well-adapted to the problem. Further developments may be expected.

10. Discussion

This paper has focussed on what are essentially lagrangian techniques for obtaining solutions of the steady Euler equations of non-trivial topological structure, and for certain stability considerations. These techniques rely on identification of appropriate lagrangian invariants, i.e. integrals or functions defined over material domains which remain invariant by virtue of some kind of frozen-field evolution.

The three-dimensional magnetic relaxation problem is important because it yields stable magnetostatic equilibria and hence (by exact analogy) steady Euler flows, which are probably in general unstable. From the standpoint of the Euler equations, the technique is perhaps a little academic for this reason; however it has been argued elsewhere (Moffatt 1990*a*) that these flows are important when regarded as unstable fixed points of the (infinite-dimensional) Euler dynamical system, it being a common feature of dynamical systems that solutions may remain for a high proportion of the time in neighbourhoods of such unstable fixed points. The technique is also of interest in establishing a bridge between mathematical topology and classical fluid mechanics, a bridge dimly perceived by Kelvin (1869), but now seen to be on reasonably firm foundations. The techniques of fluid mechanics are in fact highly relevant to certain problems (e.g. determination of new knot and link invariants) that are purely topological in character (Freedman & He 1990*a*, *b*; Moffatt 1990*b*).

Magnetic relaxation yields magnetostatic equilibria with prescribed magnetic field topology, and hence, by analogy, Euler flows with prescribed streamline topology. An alternative procedure, as devised by Vallis *et al.* (1989) may yield Euler flows with prescribed vorticity topology. However, the procedure works only when the kinetic energy either has a non-zero lower bound or a finite upper bound; a finite upper bound has been established for two-dimensional flows and (in the present paper) for axisymmetric flows without swirl.

For two-dimensional magnetic relaxation with flux function χ, the appropriate lagrangian invariant is a signature (or area) function $A(\chi)$, or a set of such functions $\{A_i(\chi)\}$. For VCY relaxation, this is replaced by $A(\omega)$ (or $\{A_i(\omega)\}$), the area within contours $\omega = \text{const}$. In either case, it is the signature that remains invariant during relaxation, and that characterizes the asymptotic steady state. Similarly, for axisymmetric problems the signature is a volume function $V(\chi)$ or (for VCY relaxation) $V(q)$ where $q = \omega_\varphi / r \sin\theta$. Axisymmetric flows with swirl obtained via magnetic relaxation are characterised by a double signature $\{V(\psi), W(\psi)\}$ where W is the azimuthal flux within the torus $\psi = \text{const}$.

It has been conjectured (Moffatt 1990*a*) that turbulence may usefully be regarded as a 'sea of interacting vortons'. This provides a natural explanation for the apparent 'suppression of nonlinearity' that has been identified in direct numerical simulations of turbulence (Kraichnan & Panda 1988). It also provides a rationale for the coherent structures that are such an all-pervasive feature of many turbulent flows (Hussain 1986). In this picture, the region of interaction of vortons is the seat of Kelvin–Helmholtz instabilities, which, via spiral wind-up, lead to inertial transfer of energy to small scales. This picture is quite different from the traditional Kolmogorov cascade picture, but has the merit of starting from fully nonlinear solutions of the Euler equations (the vorton solutions) which may provide a useful basis for a more deductive theory of turbulence.

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