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Fixed Points of Turbulent Dynamical Systems and Suppression of Nonlinearity

Comment 1.

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1. Introduction

I see my task as first to comment on the approach to turbulence outlined by Philip Holmes, and secondly to broaden the discussion, to introduce some complementary ideas, and perhaps to be a bit provocative at the same time.

What Holmes has described is a decomposition of the turbulent velocity field in a statistically stationary but inhomogeneous flow in the form

$$\mathbf{u}(\mathbf{x},t) = \mathbf{U}(\mathbf{x}) + \sum_{\lambda} a_{\lambda}(t)\mathbf{u}_{\lambda}(\mathbf{x}), \tag{1}$$

where $\mathbf{U}(\mathbf{x})$ is the mean flow and the $\mathbf{u}_{\lambda}(\mathbf{x})$ are a set of structures, the eigenfunctions of the two-point velocity correlation tensor, with the convenient property that the energy of the turbulence is largely concentrated in a small number of leading terms of the series (1). By their construction, the $\mathbf{u}_{\lambda}(\mathbf{x})$ are orthogonal in the sense that

$$\int \mathbf{u}_{\lambda}(\mathbf{x}) \cdot \mathbf{u}_{\lambda'}(\mathbf{x}) d^3\mathbf{x} = \delta_{\lambda\lambda'}, \tag{2}$$

so that substitution of (1) in the Navier-Stokes equations, followed by Galerkin projection onto each $\mathbf{u}_{\lambda}(\mathbf{x})$ in turn, leads to a set of equations for the amplitudes $a_{\lambda}(t)$ of the form

$$\frac{da_{\lambda}}{dt} = b_{\lambda\mu}\, a_{\mu} + c_{\lambda\mu\nu}\, a_{\mu}\, a_{\nu} + d_{\lambda\mu\nu\sigma}\, a_{\mu}\, a_{\nu}\, a_{\sigma}. \tag{3}$$

Here, the linear term $b_{\lambda\mu}\, a_{\mu}$ arises from viscous damping and from a primary part of the interaction of the mean flow with the turbulence; the quadratic term $c_{\lambda\mu\nu}\, a_{\mu}\, a_{\nu}$ arises primarily from the quadratic nonlinearity of the Navier-Stokes equation; and the cubic term $d_{\lambda\mu\nu\sigma}\, a_{\mu}\, a_{\nu}\, a_{\sigma}$ arises from the interaction with the turbulence of the part of the mean flow that is driven by quadratic Reynolds stresses. The set of equations (3) is truncated at a finite level, and the neglected (or 'unresolved') terms are represented by an eddy viscosity $\alpha\nu_T$, where α is a dimensionless parameter of order unity. The set of equations (3) then constitutes a dynamical system of order N (the level of truncation) containing a parameter α which can be varied (within reason !). A bridge is thus constructed between the Navier-Stokes equations and the theory of dynamical systems, from which a rich harvest of nonlinear phenomena may be expected, and is indeed found.

The procedure appears, in the abstract, to be extremely attractive and to hold great potential as a tool for investigation of the interaction of characteristic structures, not only for flows which are inhomogeneous with respect to only one space-coordinate, but for more general situations - e.g. turbulent flow in a pipe of varying cross-section. The procedure is a general one, but construction of the $\mathbf{u}_\lambda(\mathbf{x})$ does require detailed input concerning the measured correlation tensor for a given geometry which may take months, if not years, of painstaking experimental effort, to accumulate. Moreover the minimum reasonable order of truncation N is likely to rise rapidly with geometrical complexity, so that a procedure that appears attractive in the abstract may turn out to be prohibitively cumbersome in practice; there are already signs that this is so even for the standard turbulent channel or pipe flow problems.

2. The quasi-two-dimensional truncation

When the flow is statistically homogeneous in the streamwise (x) and spanwise (z) directions, the structure functions $\mathbf{u}_\lambda(\mathbf{x})$ take the form

$$\mathbf{u}_\lambda(\mathbf{x}) = e^{i(k_1 x + k_3 z)}\, \mathbf{u}(k_1,\, k_3,\, n,\, y)\,, \tag{4}$$

where λ now represents the triple $(k_1,\, k_3,\, n)$, and $n = 1,\, 2,\, 3, \ldots$. The particular truncation described by Holmes, whose consequences have been explored by Aubry *et al* (1988), retains only structures for which $k_1 = 0$, $n = 1$ and $k_3 = mk$, $m = \pm 1, \pm 2, \ldots, \pm 5$. Within the limits of this truncation, the velocity field (1) has the two-dimensional form

$$\mathbf{v}(\mathbf{x},t) = \mathbf{U}(y,\, t) + \mathbf{u}(y,\, z,\, t) \tag{5}$$

where

$$\mathbf{u}(y,\, z,\, t) = \sum_m a_m(t) e^{imkz} \mathbf{u}^{(m)}(y). \tag{6}$$

Reality of $\mathbf{u}$ for all $(y,\, z,\, t)$ implies that

$$a_{-m}(t) = a_m^*(t)\ ,\ \mathbf{u}^{(-m)}(y) = \mathbf{u}^{(m)*}(y), \tag{7}$$

so that the system (3) is of fifth order in complex amplitudes, i.e. of tenth order in real variables. Each structure function $e^{imkz}\mathbf{u}^{(m)}(y)$ has a non-zero x-component, as well as components in the y- and z-directions. The velocity field (5) is two-dimensional only in the restricted sense that

$$\frac{\partial \mathbf{v}}{\partial x} = 0. \tag{8}$$

Even so, there are implications that are hard to reconcile with the detailed conclusions of Aubry *et al* (1988).

For, from the incompressibility condition $\nabla \cdot \mathbf{u} = 0$, we may introduce a stream-function $\psi(y, z, t)$ for the flow in the (y, z) plane: writing $\mathbf{u} = (u, v, w)$, this is defined by

$$v = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial y}, \tag{9}$$

and the vorticity field is given by

$$\boldsymbol{\omega} = \nabla \wedge \mathbf{u} = \left(-\nabla^2 \psi, \ \frac{\partial u}{\partial z}, \ -U'(y) - \frac{\partial u}{\partial y} \right). \tag{10}$$

With $D/Dt = \partial/\partial t + v\partial/\partial y + w\partial/\partial z$, the x-component of the Navier-Stokes equation is

$$\frac{D}{Dt}(U + u) = \nu \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (U + u), \tag{11}$$

and the x-component of the vorticity equation is

$$\frac{D}{Dt}\omega_x = \nu \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \omega_x. \tag{12}$$

Hence the flow in the cross-stream plane is totally decoupled from the mean flow, and evolves as for freely decaying two-dimensional turbulence. In particular, using only

$$v = 0 \text{ on } y = 0 \text{ and } pv \longrightarrow 0 \text{ as } y \longrightarrow \infty, \tag{13}$$

we find that

$$\frac{d}{dt} \int \int \left(v^2 + w^2 \right) dy\, dz = -2\nu \int \int \omega_x^2 \, dy\, dz \tag{14}$$

Hence the energy associated with the flow in the (y, z) plane necessarily decays to zero, and there is apparently no possibility of a steady state other than that in which $v = w = 0$, and consequently, from (11) , $u = 0$ also.

It is hard to reconcile this elementary result with the conclusion of Aubry *et al* (1988) that, for a certain range of values of the eddy viscosity parameter α, non-trivial fixed points of the dynamical system (3) do exist, representing streamwise rolls having no variation whatsoever in the streamwise direction. There is a paradox here that is difficult to track down, because the 'one-way' coupling between the cross-stream flow (v, w) and the streamwise flow $(U + u)$, represented by equations (11) and (12) (i.e. (v, w) obviously affects $U + u$, but $U + u$ does not affect (v, w)), becomes a two-way coupling between each pair $\{u_\lambda(\mathbf{x}), \mathbf{u}_{\lambda'}(\mathbf{x})\}$ when the representation (4) is used, since each $\mathbf{u}_\lambda(\mathbf{x})$ simultaneously involves streamwise and cross-stream components.

It is obviously important to resolve this paradox before attempting to incorporate modes for which $k_1 \neq 0$. The 'invariant subspace' with $k_1 = 0$ provides, as Holmes has said, the backbone on which higher-order systems, incorporating realistic streamwise variation, must be constructed; for the reasons stated above, I am not yet convinced that this backbone is in a sufficiently sound state to support such constructions, but I hope that the paradox to which I have drawn attention here can be swiftly resolved.

3. The role of fixed points

If the Navier-Stokes equations can be reduced, by proper orthogonal decomposition or otherwise, to a finite-dimensional dynamical system, then a battery of techniques is available, as Holmes has described, for analysis of this system. The natural first step is to locate the fixed points of the system in the N-dimensional space of the variables $a_\lambda(t)$, and to classify these as stable or unstable. If the decomposition is sound, then each such fixed point should correspond to a fixed point of the Navier-Stokes equation regarded as an evolutionary dynamical system in an infinite-dimensional space of, say, square-integrable solenoidal fields; such fixed points correspond to *steady* solutions $\mathbf{u}(\mathbf{x})$ of the Navier-Stokes equations, satisfying

$$(\mathbf{u}\wedge\boldsymbol{\omega})_S \equiv \mathbf{u}\wedge\boldsymbol{\omega} - \nabla h = -\nu\nabla\wedge\boldsymbol{\omega}, \tag{15}$$

where $\boldsymbol{\omega} = \nabla\wedge\mathbf{u}$, for some scalar field $h\left(= p/\rho + \frac{1}{2}\mathbf{u}^2\right)$ satisfying

$$\nabla^2 h = \nabla\cdot(\mathbf{u}\wedge\boldsymbol{\omega}). \tag{16}$$

We use the symbol $(..)_S$ to denote the 'solenoidal projection' of the field, obtained *via* solution of the Poisson equation (16) for h. Equations (15) and (16) must of course be coupled with non-zero boundary conditions on $\mathbf{u}$ and/or p, since otherwise there can be no non-trivial steady state.

In the case of the Euler equations ($\nu = 0$), it has been shown (Moffatt 1985) that steady solutions $\mathbf{u}^E(\mathbf{x})$ exist having arbitrarily prescribed streamline topology, these flows being characterised by subdomains $D_n (n = 1, 2, ...)$ in which streamlines are chaotic and $\boldsymbol{\omega}^E = \alpha_n\,\mathbf{u}^E$, (i.e. the flow is a Beltrami flow in each D_n), and by the presence of vortex sheets of finite extent, randomly distributed in the spaces between the D_n. The presence of these vortex sheets suggests that these flows will in general be unstable within the framework of the Euler equations.

If, nevertheless, the phase space trajectory representing a turbulent flow spends a large proportion of its time in a neighbourhood of such fixed points (as is not untypical behaviour for low-order dynamical systems in which heteroclinic orbits are known to play a key role) then this behaviour should be recognizable in the statistics of $\mathbf{u}$.

This sort of consideration has led Kraichnan & Panda (1988) to examine the evolution of the quantities

$$J \equiv \frac{<(\mathbf{u}\wedge\boldsymbol{\omega})^2>}{<\mathbf{u}^2><\boldsymbol{\omega}^2>}, \qquad Q \equiv \frac{<(\mathbf{u}\wedge\boldsymbol{\omega})_S^2>}{<\mathbf{u}^2><\boldsymbol{\omega}^2>}, \tag{17}$$

in a direct numerical simulation of decaying isotropic turbulence, and to compare these with the values J_G, Q_G that pertain to a Gaussian velocity field with (at each t) the same velocity spectrum as the dynamically evolving field. Note that Q would be zero if the flow were a steady Euler flow and that any reduction of Q/Q_G from its initial value of unity may be interpreted in terms of an intrinsic tendency of the system to spend more time near the fixed points (in some sense) than far from them. Similarily, any reduction in J/J_G from unity not only indicates a similar 'hovering' near fixed points, but also provides an estimate of the proportion of the volume that is (typically)

occupied by the Beltrami domains D_n in the Euler flows corresponding to the fixed points.
Kraichnan & Panda in fact found the interesting result (corroborated by Shtilman & Polifke 1989) that Q/Q_G decreases to 0.57 and J/J_G decreases to 0.87, the latter decrease being associated with a simultaneous *increase* of the normalised mean-square helicity to 1.20, an effect foreshadowed in previous studies (Pelz *et al* 1985, 1986; Kit *et al* 1987; Rogers & Moin 1987; Levich 1987). These values indeed suggest a significant 'Eulerization' - i.e. suppression of nonlinearity - and a relatively weak 'Beltramization' - i.e. alignment of vorticity with velocity.
Kraichnan & Panda (1988) found a similar suppression of nonlinearity in a system with quadratic nonlinearity but random coupling coefficients, in an ensemble of 1000 realizations, i.e. a system like (3), but with $d_{\lambda\mu\nu\sigma} = 0$, and conjectured that this is "a generic effect associated with broad features of the dynamics". If this is true, then it provides a glimmer of hope that techniques based on weak nonlinearity may yet have some value in systems that are ostensibly strongly nonlinear! We argue this point further in the following section.

4. Turbulence regarded as a sea of weakly interacting vortons

We here use the term 'vorton' in the sense of Moffatt (1986) to represent a vorticity structure of compact support which propagates without change of shape with its intrinsic self-induced velocity $\mathbf{U}$ relative to the ambient fluid. This is in effect a generalised vortex ring which is a steady solution of the Euler equations in a frame of reference translating with the vorton. In a frame of reference fixed relative to the fluid at infinity, the vorticity field has the form

$$\boldsymbol{\omega}(\mathbf{x},t) = \boldsymbol{\omega}^V(\mathbf{x} - \mathbf{U}t), \tag{18}$$

and if the associated velocity is $\mathbf{u}(\mathbf{x},t)$, then

$$(\mathbf{u} \wedge \boldsymbol{\omega})_S = 0, \tag{19}$$

where the suffix S again denotes 'solenoidal projection'.
Such vortons provide a wide family of relatively stable structures which are associated, albeit indirectly, with fixed points of the Euler dynamical system, and this suggests that they may provide a natural basis for a description of turbulence which exploits to the full any natural tendency to suppress nonlinearity. To this end, let us suppose that a turbulent velocity field $\mathbf{u}(\mathbf{x},t)$ can be expressed as a sum of weakly interacting vortons:

$$\mathbf{u}(\mathbf{x},t) = \sum_n \mathbf{u}^{(n)}(\mathbf{x} - \mathbf{U}_n t) + \mathbf{v}(\mathbf{x},t) \tag{20}$$

where each $\mathbf{u}^{(n)}$ satisfies $\left(\mathbf{u}^n \wedge \boldsymbol{\omega}^{(n)}\right)_S = 0$, and where $\mathbf{v}(\mathbf{x},t)$ represents the residual velocity field resulting from the interaction of vortons. This interaction process is represented schematically in figure 1, where it is conceived as a Kelvin-Helmholtz type of instability associated with grazing incidence of vortons. Substitution of (20) in the Navier-Stokes equation gives

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \sum_{n \neq m}\left(\mathbf{u}^{(n)} \wedge \boldsymbol{\omega}^{(m)}\right)_S + \nu\nabla^2\mathbf{v}. \qquad (21)$$

Note that, although $\mathbf{u}^{(n)} \wedge \boldsymbol{\omega}^{(m)} \equiv 0$ outside the support $D^{(m)}$ of $\boldsymbol{\omega}^{(m)}$, $\left(\mathbf{u}^{(n)} \wedge \boldsymbol{\omega}^{(m)}\right)_S$ includes a pressure contribution which in fact falls off as r^{-4} with distance from $D^{(m)}$.

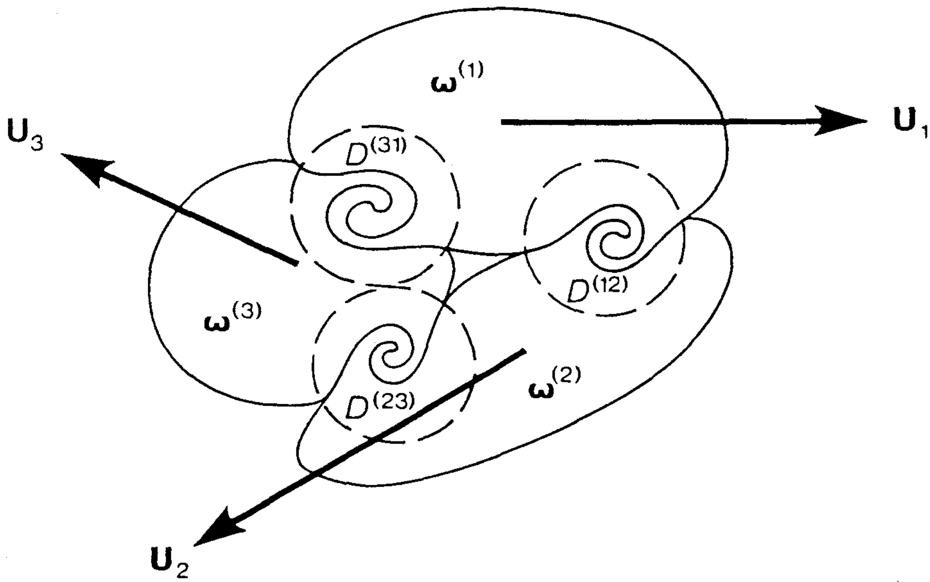

Figure 1. Schematic representation of the interaction of vortons $\omega^{(1)}$, $\omega^{(2)}$, $\omega^{(3)}$ and the production of offspring vortons by Kelvin-Helmholtz instability in the interaction domains $D^{(23)}$, $D^{(31)}$, $D^{(12)}$.

As the perturbation $\mathbf{v}$ grows on the smaller scale of effective vorton interaction, energy is of course extracted from the parent vortons which may either adjust in quasi-steady manner, or may after several such collision processes be destroyed. The field $\mathbf{v}(\mathbf{x},t)$ may be expected to restructure itself as a sum of 'offspring' vortons, convected with the local velocity $\bar{\mathbf{u}}$ associated with larger scales of motion, i.e.

$$\mathbf{v}(\mathbf{x},t) = \sum_n \mathbf{v}^{(n)}\left(\mathbf{x} - (\mathbf{V}_n + \bar{\mathbf{u}})\,t\right) + \mathbf{w}(\mathbf{x},t) \qquad (22)$$

and the process may now continue, with developing intermittency as the cascade to smaller scale vortons proceeds.

The process envisaged here is similar in spirit to that envisaged by Frisch, Sulem & Nelkin (1978) in their proposed 'β-model' of turbulence, but with the additional dynamical feature that at each length scale, solutions of the Euler equations provide the reference point for the next step (vorton interaction) of the cascade process.
The representation (20) provides an appropriate framework for understanding the suppression of Q (eqn. 17) relative to the Gaussian value Q_G. Neglecting $\mathbf{v}$, we have

$$(\mathbf{u} \wedge \boldsymbol{\omega})_S = \sum_{n \neq m} \left(\mathbf{u}^{(n)} \wedge \boldsymbol{\omega}^{(m)} \right)_S \tag{23}$$

and the (n, m) term in the sum is non-zero only in the domain $D^{(nm)}$ of interaction of the vortons $\boldsymbol{\omega}^{(n)}$ and $\boldsymbol{\omega}^{(m)}$. If we suppose further that these domains do not overlap then

$$< (\mathbf{u} \wedge \boldsymbol{\omega})_s^2 > = < \sum_{n \neq m} \left(\mathbf{u}^{(n)} \wedge \boldsymbol{\omega}^{(m)} \right)_S^2 > . \tag{24}$$

This may be expected to be a factor q less than the Gaussian value, where q is the proportion of the fluid volume in which significant vorton interactions occur. The value $q = Q/Q_G \approx 0.57$ found by Kraichnan & Panda (1988) is not implausible from this point of view.

5. Conclusion

Decompositions such as (1) or (20) of a turbulent velocity field seem to hold promise in capturing the dynamics of long-lived coherent structures and their interactions. In both cases, the fixed points of the underlying dynamical systems play an important rôle both in understanding the extent to which nonlinearity may be suppressed, and as a starting point for analysis of nonlinear interactions between the basic structures. The study of Aubry *et al* (1988) provides a valuable starting point, and the computations and analysis initiated by these authors now need to be further developed and extended, and reconciled with more primitive considerations concerning two-dimensional turbulence.
The concept of interacting vortons, and the associated suppression of $< (\mathbf{u} \wedge \boldsymbol{\omega})_S^2 >$ and (weak) relative enhancement of mean-square helicity $< (\mathbf{u} \cdot \boldsymbol{\omega})^2 >$, have already stimulated a number of experiments, both real-fluid and numerical-simulation. This is an area of intense current interest, in which further rapid developments may be expected.

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