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Electromagnetic stirring

H. K. Moffatt

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, England

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It is well known that an alternating magnetic field (either single phase or multiphase) applied to a conductor, whether solid or fluid, will induce electric currents in the conductor, and hence a Lorentz force distribution. This Lorentz force is in general rotational, and if the conductor is fluid, it is set in motion. Thus the magnetic field acts as a nonintrusive stirring device and it can, in principle, be engineered to provide any desired pattern of stirring. Stirring may also be effected through the interaction of a steady current distribution driven through a fluid and the associated magnetic field. When the field frequency is high, the Lorentz force is confined to a thin electromagnetic boundary layer, and the net effect of the magnetic field is to induce either a tangential velocity or a tangential stress just inside the boundary layer. The distribution of velocity or stress is related to the structure of the applied field. Symmetric configurations may lead to patterns of stirring in which the streamlines lie on toroidal surfaces; more generally, however, the streamline pattern is chaotic. Examples of both types are presented, and applications are discussed. A somewhat related problem arises if electrically conducting particles are suspended in a nonconducting fluid, the whole being subjected to the same type of alternating magnetic field. Currents are now induced in the particles, and each particle experiences a force and a couple. When several particles are in suspension, the movement of each is influenced by the presence of the others, and a problem somewhat analogous to the problem of interacting point vortices arises. For a suspension of conducting particles, inhomogeneity of the field leads to inhomogeneity of the concentration of particles, and a bulk flow is, in general, induced. The general principles underlying this behavior are briefly summarized.

I. INTRODUCTION

When an electric current $\mathbf{j}(\mathbf{x},t)$ flows through a conducting body, whether solid or fluid, in the presence of a magnetic field $\mathbf{B}(\mathbf{x},t)$, there is a force per unit volume (the Lorentz force)

$$\mathbf{F} = \mathbf{j} \times \mathbf{B} \quad (1)$$

that acts upon the conductor. In general, this force is rotational, i.e.,

$$\text{curl } \mathbf{F} \neq 0, \quad (2)$$

and, if the conductor is fluid, it cannot be compensated by a pressure gradient. In these circumstances, the fluid must move in response to the force. This, in its simplest terms, is the principle of electromagnetic stirring.

We consider an incompressible liquid confined to a bounded volume V with surface S , and let V^+ be the exterior region. Within the conductor, \mathbf{B} and \mathbf{j} are related by Ampère's law

$$\mu_0 \mathbf{j} = \text{curl } \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0, \quad (3)$$

where $\mu_0 = 4\pi \times 10^{-7}$ (in SI units). The magnetic field may also have external sources, e.g., currents (ac or dc) in coils in the exterior domain V^+ . The general situation that may be considered is sketched in Fig. 1. Currents in the external coils C, C' will, through Faraday's law, induce a current distribution in the conductor. This current may be augmented by the direct application of potential differences between electrodes E, E' imbedded in the boundary S . Thus the current may be induced through application of a time-dependent magnetic field, or electrically, or both. A very

wide range of physical conditions, and an equally wide range of applications particularly in the field of metallurgical processing may be considered. Despite the great practical importance of some of these applications (e.g., the major industrial process of aluminum smelting), and the fact that the fundamental principles of electromagnetic stirring are well understood, understanding of the flows generated by electromagnetic stirring in all but the most idealized circumstances is still at a fairly primitive level.

In this paper, we shall review the principles of the subject, focusing particularly on high-frequency inductive stirring which is amenable to a general analysis that is not too dependent on particular geometrical considerations. Some new results will be obtained in the course of this review, which focuses on the pattern, or topology, of stirring that

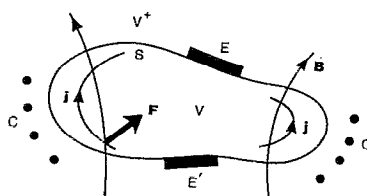


FIG. 1. The general configuration by which electromagnetic stirring may be applied to electrically conducting fluid in a region V . Induction coils C, C' exterior to V provide an ac magnetic field which induces a current distribution in the fluid. In addition, current may be supplied from electrodes E, E' embedded in the surface of V .

may be achieved, rather than on detailed numerical calculations which are in any case greatly hampered by lack of an adequate theory of turbulent flow in confined regions.

In the penultimate section of the paper (Sec. IX), a different type of stirring problem will be discussed, namely, that which arises when electrically conducting particles are suspended in a *nonconducting* fluid. If this system is subjected to a high-frequency field, currents are induced in the particles, which then in general experience both a force and a couple. Interesting interaction problems arise when two or more particles are considered, there being strong similarities with the classical problem of interaction of point vortices. A suspension of conducting particles subjected to an ac field will in general develop inhomogeneities of concentration, and a bulk flow will develop. This bulk flow may have chaotic mixing properties if, again, the magnetic field is suitably engineered.

It may be noted that two previous IUTAM Symposia have been devoted to metallurgical applications of magnetohydrodynamics (Cambridge, 1982, Ref. 1) and liquid metal magnetohydrodynamics (Riga, 1988, Ref. 2). There has also been a triennial series of Bat-Sheva International Seminars on MHD flow and turbulence (see, for example, Ref. 3). The recent book (Ref. 4) by Bojarevičs *et al.* summarizes much of the important work on electrically induced vortical flows, and particularly the work at the Riga Laboratory of Magnetohydrodynamics (Latvian SSR Academy of Sciences).

II. TWO HISTORIC EXAMPLES

A. The 1907 experiment of Northrup⁵

The recent work of Bojarevičs *et al.*⁴ has drawn attention to the extremely interesting early paper of Northrup⁵ who, following an observation of Hering, conducted an experiment which exhibited what would now be described as both the shaping and stirring properties of an ac magnetic field—and this long before the birth of magnetohydrodynamics! Northrup's experiment is depicted in both plan and elevation in Fig. 2: ac current is passed through liquid metal (NaK alloy) contained in two rectangular cavities connected by a narrow channel. The liquid surface is covered by a

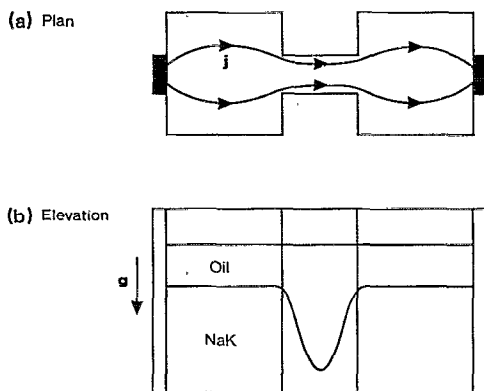


FIG. 2. The 1907 experiment of Northrup (Ref. 5). (a) Plan; (b) elevation; current flows through liquid metal as shown, and the free surface is depressed in the region where the channel is constricted.

layer of oil. When the current flows, the NaK surface in the narrow channel is depressed—a manifestation of the *shaping* effect of an ac field. Northrup attributes this discovery to Hering who, recognizing that the action of the Lorentz force on the conductor was to squeeze it, “jocosely called it the ‘pinch phenomenon.’” The pinch effect is of course fundamental to the operation of thermonuclear plasma containment devices; it is not generally recognized that the terminology dates back to 1907; perhaps it should be renamed the “Northrup–Hering effect.”

Northrup also comments that the liquid on the inclined surface of the depressed liquid metal “showed great agitation.” This is a manifestation of the *stirring* effect, which, as will be shown, necessarily accompanies the shaping function of the magnetic field. Northrup's paper is of seminal importance in the prehistory of magnetohydrodynamics, and it is remarkable that it has for so long been neglected.

B. The 1932 experiment of Braunbeck

This appears to be the earliest experiment in which the torque-producing action of a rotating magnetic field acting on a liquid metal was demonstrated.⁶ The essence of Braunbeck's experiment is shown in Fig. 3: a capsule containing liquid metal and suspended on a torsion wire is subjected to the action of a horizontal magnetic field rotating (through the use of multiphase windings) about a vertical axis. Braunbeck noted that the capsule rotated through a finite angle in the same sense as the rotation of the magnetic field, and proposed this as a device, when suitably calibrated, for measurement of the conductivity of liquid metals. Actually, the relationship between torque and conductivity is linear, at least when the field frequency is not too large; but it is important to operate in the laminar regime if reliable and repeatable results are to be obtained.⁷ In the steady state, the fluid rotates within the capsule and transmits to the capsule a torque equal and opposite to that conveyed through the torsion wire. This experiment is also seminal, in indicating the stirring action engendered by a rotating magnetic field.

III. INDUCTIVE STIRRING

Consider the situation of Fig. 1, in which ac currents in external coils produce an ac magnetic field of the form

$$\mathbf{B}(\mathbf{x}, t) = \text{Re}[\hat{\mathbf{B}}(\mathbf{x})e^{-i\omega t}] \quad (4)$$

applied to fluid in a domain of length scale L . This field may be single phase (if the external currents are all in phase) or

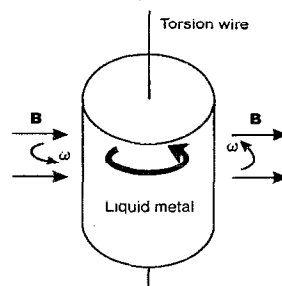


FIG. 3. The 1932 experiment of Braunbeck (Ref. 6): a capsule suspended on a torsion wire, and containing liquid metal, is subjected to a rotating magnetic field.

multiphase. In the more general case, $\hat{\mathbf{B}}(\mathbf{x})$ is complex, i.e.,

$$\hat{\mathbf{B}}(\mathbf{x}) = \mathbf{B}^{(r)}(\mathbf{x}) + i\mathbf{B}^{(i)}(\mathbf{x}) \quad (5)$$

when $\mathbf{B}^{(r)}(\mathbf{x})$ and $\mathbf{B}^{(i)}(\mathbf{x})$ are both real but not necessarily parallel.

We anticipate a stirring action and suppose that a typical fluid velocity is U . An important measure of the back-reaction, or distorting effect, of the velocity field on the magnetic field is the magnetic Reynolds number

$$\text{Re}_m = UL\mu_0\sigma = UL/\eta, \quad (6)$$

where σ is the electrical conductivity of the fluid, and $\eta = (\mu_0\sigma)^{-1}$ is its magnetic diffusivity. In most circumstances of practical importance, this number is small,⁸ i.e.,

$$\text{Re}_m \ll 1. \quad (7)$$

This means that the field may be calculated neglecting the fact that the fluid is in motion, i.e., as if the fluid was in fact a solid conductor. This is a very considerable simplification, which effectively decouples the electromagnetic problem from the (subsequent) fluid-mechanical problem.

There is a second magnetic Reynolds number based on the field frequency, namely

$$\text{Re}_\omega = \omega L^2/\eta = 2(L/\delta)^2, \quad (8)$$

where

$$\delta = (2\eta/\omega)^{1/2}. \quad (9)$$

Here, δ is the familiar "skin depth" of electromagnetism, i.e., the order of magnitude of the depth of penetration of the field into the conductor. If $\delta \ll L$ (i.e., $\text{Re}_\omega \gg 1$), then the field and current, and hence the driving force \mathbf{F} , are confined to a layer of thickness $\sim \delta$ near the surface S . If $\delta \gtrsim L$ (i.e., $\text{Re}_\omega \lesssim 1$), then the conductor is effectively transparent to the field, and the stirring action is much more uniformly distributed throughout the interior of V . When $\text{Re}_\omega \rightarrow 0$, however (or equivalently $\sigma \rightarrow 0$), we have the limit of a nonconducting fluid in which no current is induced and the stirring force tends to zero.

If the frequency ω is increased from zero to infinity (keeping all other parameters constant) it is therefore evident that the stirring efficiency (measured, for example, in terms of the kinetic energy of the stirred fluid) will vary as indicated qualitatively in Fig. 4, with a maximum at some

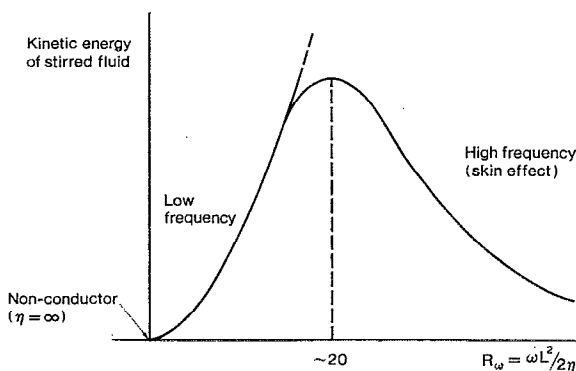


FIG. 4. A qualitative sketch of the variation of kinetic energy of electromagnetically stirred fluid, as a function of the dimensionless frequency parameter Re_ω . Regular perturbation theory is applicable in the low-frequency regime, and skin-effect analysis is applicable in the high-frequency regime.

value of Re_ω of order unity. By assuming a plausible pattern of stirring and adopting a global balance of forces, Tir⁹ obtained a curve of this kind, with a maximum at $\text{Re}_\omega \sim 20$. For values of Re_ω greater than this, a skin-effect analysis may be expected to provide a reasonable description of the stirring effect. For small values of Re_ω , a regular perturbation procedure is available. In the following section, we pursue the former description which provides valuable insights concerning the flow patterns associated with particular classes of ac fields. It may be helpful to note that for a typical liquid metal (aluminum), $\delta \approx 20$ mm when $\omega = 50$ Hz and $\delta \approx 140$ mm when $\omega = 1$ Hz. If the scale L of the domain considered is much larger than δ , then obviously the high frequency approach is appropriate.

IV. HIGH-FREQUENCY INDUCTIVE STIRRING

The high-frequency description as depicted in Fig. 5, was developed in Ref. 10. Here we recapitulate the salient features of this description. At leading order (i.e., in the very high-frequency limit when the field is totally expelled from the conductor) the field in V^+ is a potential field with $\mathbf{B} \cdot \mathbf{n} = 0$ on S , i.e.,

$$\hat{\mathbf{B}}(\mathbf{x}) = \nabla\Phi \quad \text{in } V^+, \quad (10)$$

where

$$\nabla^2\Phi = 0 \quad \text{in } V^+, \quad (11)$$

and

$$\frac{\partial\Phi}{\partial n} = 0 \quad \text{on } S. \quad (12)$$

Moreover, $\hat{\mathbf{B}}(\mathbf{x})$ has prescribed singularities at its sources in the current-carrying coils. We may suppose that this exterior leading-order problem is solved either by analytical or computational techniques. We may then evaluate the tangential field

$$\hat{\mathbf{B}}_S(\mathbf{x}) = \nabla\Phi|_S \quad (13)$$

on the surface S . Note that the surface divergence of this field is

$$\nabla_S \cdot \mathbf{B}_S = -\frac{\partial B_n}{\partial n} = -\frac{\partial^2\Phi}{\partial n^2} \Big|_S \quad (14)$$

by virtue of the condition $\nabla \cdot \mathbf{B} = 0$.

Now since the field frequency is large, but not infinite, this surface field in fact diffuses into the conductor, and its distribution in the electromagnetic boundary layer (i.e., the skin) may be obtained by elementary skin-effect analysis, based on the equation

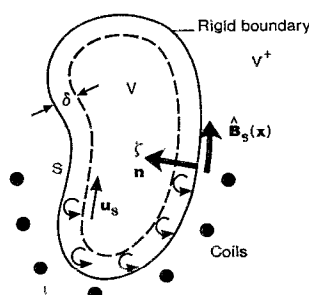


FIG. 5. Notation used in the high-frequency regime. Here u_s denotes the effective tangential velocity that is "induced" by the stirring that generates vorticity in the electromagnetic boundary layer.

$$-i\omega\hat{\mathbf{B}} = \eta \frac{\partial^2 \hat{\mathbf{B}}}{\partial n^2}. \quad (15)$$

Thereafter $\hat{\mathbf{j}}$ is determined from (3) and hence $\mathbf{F}(\mathbf{x}, t)$ may be calculated. This has a mean (i.e., time averaged) part and a fluctuating part with period 2ω . The latter may be of potential importance in stimulating turbulent flow with a dominant frequency; but most attention has been devoted to the mean part, given by

$$\mathbf{F}(\mathbf{x}) = \frac{1}{2} \text{Re}(\hat{\mathbf{j}}^* \wedge \hat{\mathbf{B}}), \quad (16)$$

where the asterisk denotes the complex conjugate; we shall restrict attention to this part here also.

As noted in the Introduction, it is only the curl of this body force that is responsible for fluid motion; this is given by

$$\text{curl } \mathbf{F} = -(\rho/\delta)(\mathbf{n} \times \mathbf{Q}_S)e^{-2\zeta/\delta}, \quad (17)$$

where

$$\mathbf{Q}_S = (1/\mu_0\rho)(\nabla_S \cdot |\hat{\mathbf{B}}_S|^2 - \text{Im } \hat{\mathbf{B}}_S^* \nabla_S \cdot \hat{\mathbf{B}}_S), \quad (18)$$

is a vector field defined on the surface S , with dimensions of an acceleration; ρ is the fluid density, \mathbf{n} is a unit normal on S directed into V , and ζ is a boundary-layer coordinate in the direction \mathbf{n} . The first of the two terms on the right-hand side of (18) is nonzero if $|\hat{\mathbf{B}}_S|^2$ is nonuniform on S , and is directed from points where $|\hat{\mathbf{B}}_S|^2$ is minimal toward points where it is maximal. The second term is nonzero only for multiphase fields and represents a tangential drag effect associated with traveling or rotating fields.

It is natural to use a lubrication approximation in the boundary layer to determine the flow associated with this force (and torque) distribution. The lubrication approximation (which neglects inertia effects in the layer) is valid only if $\text{Re } \delta/L \ll 1$, where Re is the Reynolds number of the flow generated. This condition is not satisfied if the stirring is at all vigorous; nevertheless, the approximation gives a useful qualitative indication of the effect of the torque distribution (17). A boundary-layer flow is generated with asymptotic value (as $\zeta/\delta \rightarrow \infty$) just inside the layer given by

$$\mathbf{u}_S(\mathbf{x}) = (\delta^2/8\nu)\mathbf{Q}_S(\mathbf{x}). \quad (19)$$

Thus the effect of the electromagnetic field is to replace the no-slip boundary condition at a rigid surface by the effective slip condition

$$\mathbf{u} = \mathbf{u}_S(\mathbf{x}), \quad (20)$$

where $\mathbf{u}_S(\mathbf{x})$ is given by (19). It would be useful if this result could be extended to circumstances where $\text{Re } \delta/L$ is not small.

V. ROTATION INDUCED BY A ROTATING MAGNETIC FIELD

An illuminating and simple example of the application of the above theory is given by the prototype problem of the effect of a rotating field on fluid contained in a cylinder of circular cross section (radius a). Figure 6(a) (from Ref. 11) shows the instantaneous magnetic field distribution (which is the same as that when a conducting cylinder rotates with uniform angular velocity $-\omega$ in a field that is uniform far

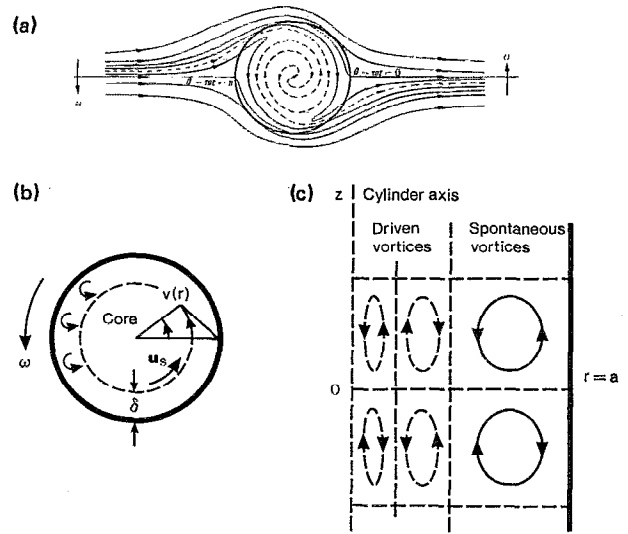


FIG. 6. (a) (Reproduced from Ref. 11.) This figure shows the magnetic field distribution when an electrically conducting cylinder is placed in a field rotating at high frequency. (b) The velocity profile of the electrically driven rotating flow inside the cylinder. (c) The structure of the Taylor vortex instability to which this flow is subject when the Taylor number is large (derived from Ref. 12).

from the cylinder. The slip velocity $\mathbf{u}_S(\mathbf{x})$ as defined above is in this case entirely associated with the second term of (18), and is given by

$$\mathbf{u}_S(\mathbf{x}) = \boldsymbol{\Omega} \times \mathbf{x}, \quad \text{where } \boldsymbol{\Omega} = (2B_0^2\eta/\mu_0\rho\nu a\omega)\hat{\mathbf{e}}_z, \quad (21)$$

where $\hat{\mathbf{e}}_z$ is a unit vector parallel to the axis of the cylinder. The associated flow in the interior is therefore the rigid body rotation

$$\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{x}. \quad (22)$$

The velocity decreases to zero on the cylinder boundary as sketched in Fig. 6(b). This problem can be solved for arbitrary values of Re_ω , not necessarily large, and of course the boundary layer thickens as Re_ω decreases. In the outer part where the circulation is a decreasing function of radius, the flow is potentially unstable. It was shown by Richardson¹² that the flow is indeed unstable when the Taylor number based on the thickness of the unstable region exceeds about 3×10^3 ; spontaneous Taylor vortices appear in this region together with much weaker vortices, driven by viscous shear stresses, in the inner stable region [Fig. 6(c)].¹³

This type of electromagnetic-induced rotation has many important applications, notably in the continuous casting process. Figure 7 shows this process in schematic form for steel production. Molten steel is supplied to the mold, usually of circular or rectangular cross section, and it solidifies from the boundary inward as it is drawn downward. Stirring of the liquid steel is found to lead to a better end product, partly because it impedes the growth of dendrites at the solidification front which leads to an undesirable grainy structure in the solidified steel. Low-frequency fields have to be used in order to penetrate to the liquid region. A rotating field induces a swirling flow by the mechanism described above; equally, a traveling field, or a combination of fields, can be used to generate more complex patterns of stirring.

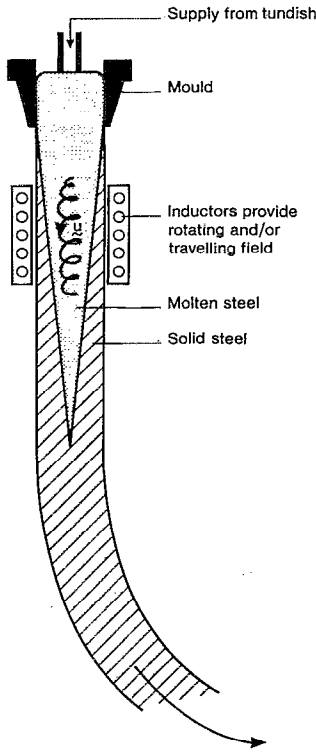


FIG. 7. Qualitative sketch of the continuous casting process for steel. Fluid is supplied from the tundish to the mold where it is cooled and solidified from the boundary inward. The solidified steel below the mold is stirred by means of a low-frequency rotating and/or traveling magnetic field.

These techniques are widely used in the steel industry, particularly in USSR, where many theoretical developments have been reported by Kapusta¹⁴ and others, and in Japan.¹⁵

Other applications of the technique of electromagnetic rotation include the removal of impurities by centrifugal action¹⁶ and the production of fine droplets in flow from a nozzle.¹⁷

$$\begin{aligned} \mathbf{u} = & \{2x(x^2 + 2y^2 + 2z^2 - 1) + \lambda(\lambda x + \mu y)(5r^2 - 3) - x[(\lambda^2 + \mu^2)(r^2 - 1) + 2(\lambda x + \mu y)^2] + 4\mu y \\ & - y(3x^2 + y^2 + z^2 - 1) + \mu(\lambda x + \mu y)(5r^2 - 3) - y[(\lambda^2 + \mu^2)(r^2 - 1) + 2(\lambda x + \mu y)^2] - 4\mu x, \\ & - z(3x^2 + y^2 + z^2 - 1) - z[(\lambda^2 + \mu^2)(r^2 - 1) + 2(\lambda x + \mu y)^2]\}. \end{aligned} \quad (25)$$

The particle paths of this steady flow (which of course coincide with the streamlines) are inevitably chaotic,¹⁸ and strong mixing can be achieved by a suitable choice of λ and μ [see Fig. 8(c)]. In practical contexts, the driven flow will not be a Stokes flow and inertia forces may be expected to modify the pattern of flow, but not its chaotic character. This conclusion that electromagnetic stirring can quite easily yield chaotic advection is believed to be new.

An important application of single-phase stirring is the induction furnace, whose primary purpose is the melting and heating of metals for processing purposes. The Joule heating is here inevitably accompanied by electromagnetic stirring [Fig. 9(a)], which is beneficial in transporting heat to the interior of the furnace. Figure 9(b) shows the alternative pattern of flow that can be achieved with a traveling magnet-

VI. STIRRING IN A SPHERE

Consider now the situation when V is a sphere and when the field \mathbf{B} far from the sphere has the form

$$\mathbf{B} \sim \mathbf{p} \cos \omega t + \mathbf{q} \sin \omega t, \quad (23)$$

i.e., a superposition of two uniform fields in quadrature. The potential problem in V^+ is then trivial, and the boundary-layer problem outlined in Sec. IV is easily solved. Calculation then gives for the surface vector field $\mathbf{Q}_S(\mathbf{x})$,

$$\begin{aligned} \mathbf{Q}_S(\mathbf{x}) = & (9/4a^4\mu_0\rho) [(\mathbf{p}\cdot\mathbf{x})(\mathbf{p}\times\mathbf{x})\times\mathbf{x} \\ & + (\mathbf{q}\cdot\mathbf{x})(\mathbf{q}\times\mathbf{x})\times\mathbf{x} + 2a^2\mathbf{x}\times(\mathbf{p}\times\mathbf{q})]. \end{aligned} \quad (24)$$

The last term here is the multiphase ingredient tending to induce rotation about the direction of $\mathbf{p}\times\mathbf{q}$. We consider three possibilities in turn.

(i) $\mathbf{q}=0$: *axisymmetric flow*. In this case, $\mathbf{Q}_S(\mathbf{x})$ is directed from the two points on the sphere where $\mathbf{B}_S = 0$ to the equator where $|\mathbf{B}_S|$ is maximal [Fig. 8(a)]. The flow that is induced is axisymmetric about the direction of \mathbf{p} with a double vortex in the meridian plane.

(ii) $\mathbf{p}=(b,0,0)$, $\mathbf{q}=(0,b,0)$: *axisymmetric flow with swirl*. This is the case of a field in the (x,y) plane rotating about the z axis. The first two ingredients of (24) combine to give a flow just like that of Fig. 8(a), but now axisymmetric about the direction of $\mathbf{p}\times\mathbf{q}$. The third ingredient induces rotation about this direction. Hence the resultant streamlines are helices on the toroidal surfaces whose cross section is shown in Fig. 8(b).

(iii) $\mathbf{p}=(b,0,0)$, $\mathbf{q}=b(\lambda,\mu,0)$: *chaotic flow*. This is the most general situation and the surface driving represented by (24) is a cubic function of the Cartesian coordinates (x,y,z) , with no apparent symmetry. The *Stokes flow* in the sphere resulting from this cubic driving is likewise cubic: in normalized form

ic field. In practice, the flows generated are turbulent and we are faced with the usual difficulties of modeling turbulent flow compounded by the fact that here the streamlines of the mean flow are evidently closed. Simple closures of the $k\epsilon$ -type are unlikely to be reliable in these circumstances, particularly near corner regions where rapid distortion of the turbulence by the mean strain occurs.¹⁹

VII. FREE SURFACE EFFECTS

If part or all of the boundary of V is a free surface Γ (as in the induction furnace), then the same type of lubrication analysis¹⁰ yields an effective tangential *stress* just inside the skin given by

$$\boldsymbol{\tau}_\Gamma = \frac{1}{4}\rho \delta \mathbf{Q}_S(\mathbf{x}) \quad (26)$$

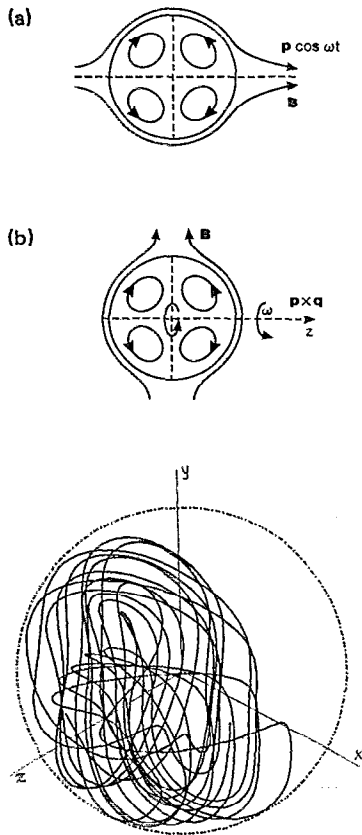


FIG. 8. Structure of flow in a sphere driven by high-frequency electromagnetic stirring. (a) Single-phase field; (b) rotating field, rotating about the z axis; (c) the general case in which the effective slip velocity u_s is a cubic function of the Cartesian coordinates (x, y, z) , with no symmetry; a single streamline of resulting steady flow for the case $\lambda = 2, \mu = 3$.

[see Fig. 10(a)]. It was presumably this stress that was responsible for the “great agitation” on the fluid surface noted by Northrup in Ref. 1.

When a high-frequency magnetic field is used for shaping and forming the boundary of a liquid metal, this effective surface stress and a concomitant stirring effect cannot be avoided. The general sense of the internal circulations developed can again be inferred from the direction of the surface vector $\mathbf{Q}_s(\mathbf{x})$. Note that when a free surface intersects a rigid boundary, the strong local Lorentz force associated with the singularity of the external potential field has a tendency to round off the corner as shown in Fig. 10(b). The following three examples are typical.

(i) *Shaping of the cross section of a cylindrical column.*

Figure 11(a) shows a configuration of coils that can be used to form a “cruciform” section in a falling column of liquid metal,²⁰ prior to solidification. This shape results from a balance of a magnetic pressure over the surface (i.e., $\mathbf{F} \cdot \mathbf{n}$ integrated through the skin) and surface tension which tends to restore a circular section. The shaping effect has been realized experimentally²¹ at the MADYLAM laboratory in Grenoble. Consideration of the surface stress distribution indicates that eight circulation cells must be established within the liquid metal. These cells are driven by the surface stress τ_r and their intensity is limited only by the viscosity of the fluid. Since no rigid boundary is present to retard the flow, a large circulation, and indeed a turbulent interior flow, is the most likely consequence.

(ii) *Levitation melting.* Figure 11(b) shows a distribution of coils that can be used to provide a net upward force

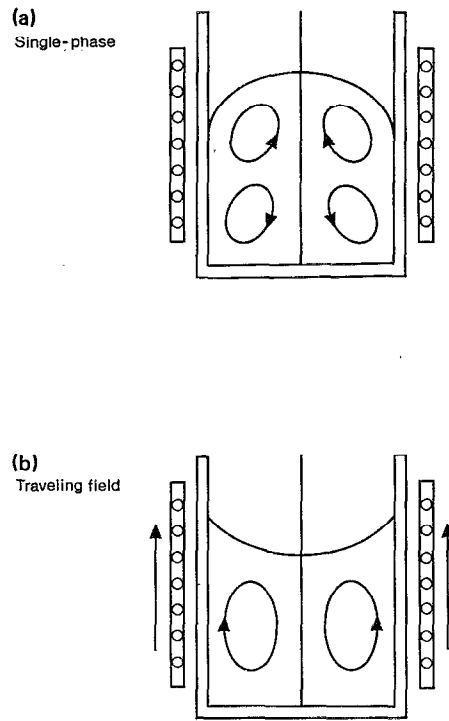


FIG. 9. Sketch showing the principle of the induction furnace. (a) Single-phase windings generating a double (axisymmetric) eddy structure and a concave free surface; (b) alternative traveling field, generating a single eddy structure and a convex free surface.

sufficient to levitate a small globule of liquid metal.^{22,23} This technique of “crucible-free” melting is used for processing rare metals when great purity is required. Here the shape is determined by a balance of magnetic, gravity, and surface tension forces and can be obtained from a minimum energy principle.²⁴ Again the surface stress distribution drives an interior flow as shown in the figure. Estimates and visual observations of the surface disturbance suggest that this flow

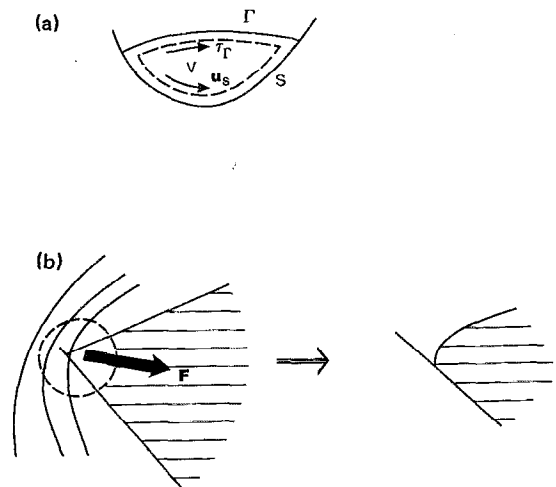


FIG. 10. The general situation when the fluid domain V is bounded partly by a rigid boundary S and partly by a free surface Γ . (a) An effective stress τ_r is induced just inside the electromagnetic boundary layer on the free surface; (b) the large Lorentz force at the sharp corner has a tendency to round off the corner as indicated.

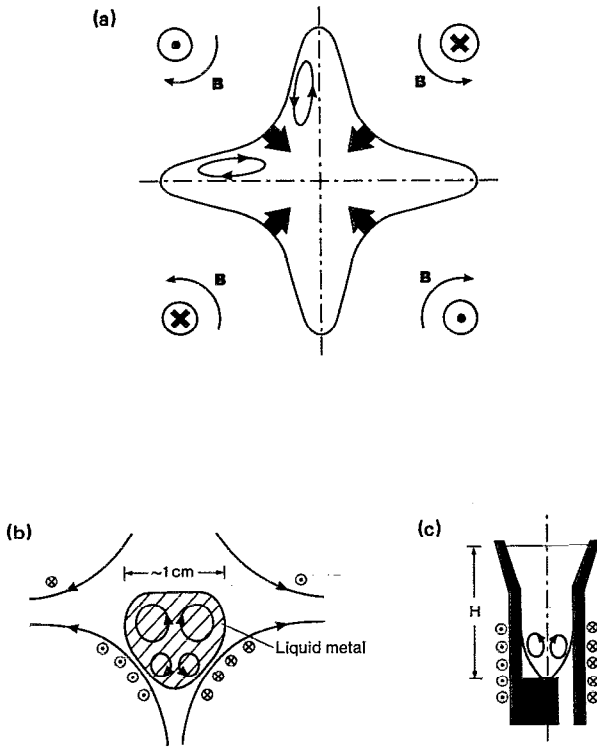


FIG. 11. Configurations in which the role of the high-frequency field is primarily shaping or levitating, but in which a stirring action associated with the induced surface stress must occur. (a) Shaping of a cylindrical column using a quadrupolar field (from Ref. 20). (b) Levitation of a liquid metal globule (Refs. 22–24). (c) Use of partial levitation to provide an electromagnetic valve (from Ref. 26).

is also turbulent, despite the (normally) small dimensions of the globule. The manner in which this flow influences the surface shape is taken into account in Ref. 23. Levitation of a layer of liquid metal by a “meander coil” arrangement has been similarly treated in Ref. 25; again, internal circulation is inevitable.

(iii) *Electromagnetic valve*. Finally, an example of *partial* levitation is shown in Fig. 11(c). This configuration has been suggested by Lillicrap²⁶ as a means of switching off the flow of liquid metal through a vertical nozzle. As the amplitude of the (high-frequency) field is increased, the flow rate decreases and is cut off completely when the parameter $B^2/\mu\rho gH$ reaches a critical value of order unity. Again, there must be an associated circulation within the partially levitated liquid metal with primary cells as indicated in the figure.

VIII. ELECTRICALLY INDUCED STIRRING

The prototype configuration here is that of the “weld-pool problem” sketched in Fig. 12(a). Here, current is supplied at a point electrode on the free surface and spreads out radially. The “self”-magnetic field associated with this current is purely azimuthal, and the resulting Lorentz force drives a flow in the meridian plane, whose properties have been very widely studied.⁴ The configuration has been realized experimentally at the Riga Laboratory of Magnetohydrodynamics; the most remarkable and unexpected property

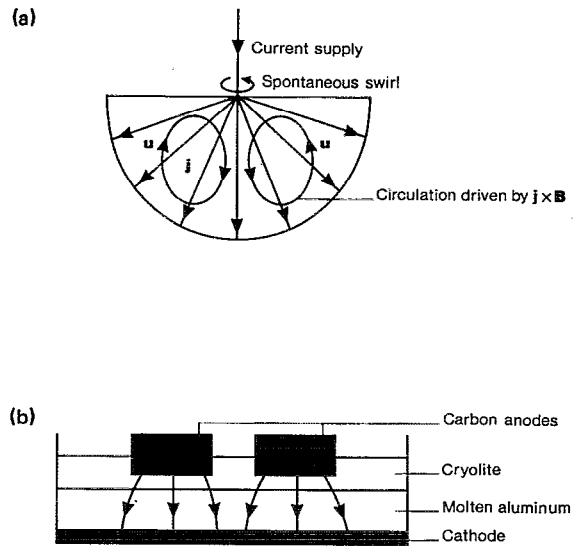


FIG. 12. Electrically driven flows. (a) The prototype “weld-pool” flow in which current is supplied at a boundary point and spreads out radially through the fluid; the primary driven flow is meridional, but a spontaneous swirl can develop (Ref. 27). (b) Sketch of the essential features of the aluminum smelting process; dc current spreads from the anodes to the cathode through the electrolytic salt (cryolite) and through the layer of molten aluminum that forms below the salt. The Lorentz force drives a flow in both conductors, and the interface is subject to instabilities (Ref. 29).

of the flow²⁷ is that it develops a strong symmetry-breaking (or more precisely parity-breaking) instability in which the fluid spontaneously rotates, in one direction or the other, about the vertical axis. This effect has been analyzed in some detail, but a simple explanation for the behavior is elusive.

The same type of (dc) stirring occurs in the enormously important aluminum smelting process [Fig. 12(b)] which involves the passage of nonuniform current through both the molten salt electrolyte (cryolite) and the layer of aluminum that forms beneath it. The stirring associated with the divergence of the current lines is here further complicated by interface effects and the possibility of very undesirable interface instabilities.²⁸ This is a problem of the greatest practical importance, and to which the principles of magnetohydrodynamics have already found fruitful application.²⁹

IX. STIRRING ACTION OF AN ac FIELD ON CONDUCTING PARTICLES SUSPENDED IN A NONCONDUCTING FLUID

Finally, we describe briefly some of the interesting effects that arise³⁰ when an electrically conducting particle is suspended in a nonconducting fluid in a field of the form (4).

Suppose for simplicity that the particle is spherical of radius a . The field then induces a dipole moment

$$\boldsymbol{\mu} = \text{Re}(\hat{\boldsymbol{\mu}}e^{-i\omega t}) \quad (27)$$

associated with currents induced in the particle. The (complex) amplitude $\hat{\boldsymbol{\mu}}$ is linearly related to the local value of $\hat{\mathbf{B}}$:

$$\hat{\boldsymbol{\mu}} = (4\pi/\mu_0)a^3\alpha(\lambda)\hat{\mathbf{B}}(\mathbf{x}), \quad (28)$$

where $\alpha = \alpha^{(r)} + i\alpha^{(i)}$ is a complex function of the frequency parameter $\lambda = a/\delta$. This dipole then experiences a force \mathbf{F}

and a couple \mathbf{G} given by the elementary formulas

$$\mathbf{F} = \langle \boldsymbol{\mu} \cdot \nabla \mathbf{B} \rangle = \frac{1}{2} \text{Re} \langle \hat{\boldsymbol{\mu}} \cdot \nabla \mathbf{B}^* \rangle, \quad (29)$$

$$\mathbf{G} = \langle \boldsymbol{\mu} \wedge \mathbf{B} \rangle = \frac{1}{2} \text{Re} \langle \hat{\boldsymbol{\mu}} \wedge \mathbf{B}^* \rangle. \quad (30)$$

The force may be decomposed into two parts, an irrotational lift-type force $\mathbf{F}^L(\mathbf{x})$ proportional to $\alpha^{(r)}$ and a solenoidal drag-type force $\mathbf{F}^D(\mathbf{x})$ proportional to $\alpha^{(i)}$. Moreover, \mathbf{G} is related to \mathbf{F}^D by the equation³⁰

$$\mathbf{F}^D + \frac{1}{2} \nabla \wedge \mathbf{G} = 0. \quad (31)$$

Under the action of a uniform field rotating about the z axis, the particle experiences just the torque, and it rotates with uniform angular velocity $\boldsymbol{\Omega}$. Under the Stokes approximation,

$$\boldsymbol{\Omega} = (|B|^2 \alpha^{(i)} / 2\mu_0 \rho \nu) \hat{\mathbf{e}}_z, \quad (32)$$

and the associated velocity field is the couplet

$$\mathbf{u}(\mathbf{x}) = (\boldsymbol{\Omega} \times \mathbf{x})(a/r)^3. \quad (33)$$

If several spherical particles are in suspension, then each moves in the resultant velocity field of the others. The movement of the spheres is then governed by a Hamiltonian system, with Hamiltonian

$$H = \Omega a^3 \sum_{m \neq n} \frac{1}{r_{mn}}, \quad (34)$$

where r_{mn} is the distance between the centers of two spheres, labeled m and n . This description is of course valid only when each r_{mn} is large compared with a . As for the problem of N point vortices³¹ the *three-sphere* problem is integrable, but the particle paths in the associated flow are chaotic. Three suspended spheres subjected to a rotating field are therefore sufficient to generate chaotic advection in the surrounding fluid.

A suspension of spheres of (small) volume concentration $c(\mathbf{x}, t)$ is also amenable to analysis. Condition (31) implies that there is zero net force on the fluid if $\nabla c \equiv 0$. However, the lift force \mathbf{F}^L drives particles toward field minima so that inhomogeneities of c , in general, must develop. These then interact with \mathbf{G} through a term $-\frac{1}{2} \mathbf{G} \times \nabla c$ in the Navier–Stokes equation, and this term is in general responsible for generating a bulk flow.

X. CONCLUSIONS

This review has covered a very wide range of problems, and has therefore necessarily been somewhat superficial in its treatment. It is hoped nevertheless that it will at least convey the flavor of an interesting branch of magnetohydro-

dynamics, which is amenable to both theory and experiment, and which is of fundamental importance in numerous industrial contexts. In qualitative terms, the stirring action of the Lorentz force, whether via induction or via direct application of electric fields to the fluid, is reasonably well understood. The major challenge, as in so many other contexts, is still to find an adequate and reliable method of modeling the turbulent flows to which this stirring action most frequently gives rise.

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