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Lectures presented at the Pedagogical Workshop
ITP, Santa Barbara, August 1991

RELAXATION UNDER TOPOLOGICAL CONSTRAINTS

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ABSTRACT. This contribution provides an informal introduction to the technique of magnetic relaxation, whereby an extremely wide family of solutions of the equations of magnetostatics, and of analogous steady solutions of the Euler equations, may be obtained, and their stability investigated. We approach this problem through the simpler, and physically more transparent, problem of gravitational relaxation of an incompressible medium of non-uniform density. We then describe the magnetic relaxation technique which yields solutions of nontrivial field topology, and we discuss the contrasting stability criteria for these magnetostatic states and for the analogous Euler flows. Applications to the theory of vortons (*i.e.*, blobs of propagating vorticity) and to the problem of determining 'energy' invariants of knots and links are then discussed. The chapter concludes with a discussion of alternative relaxation procedures involving artificial modification of the Euler equations in a manner that conserves vorticity topology.

1. Gravitational Relaxation

1.1. SOLID ELASTIC MEDIUM

In order to grasp some elementary concepts, consider first the simple problem depicted in figure 1: a block $\mathcal{D}$ of unstrained elastic material of non-uniform density $\rho_0(\mathbf{x})$ is placed in a uniform gravitational field $\mathbf{g}$, and its boundary $\partial\mathcal{D}$ is fixed. What happens? Evidently the denser parts of the medium will sag slightly under the influence of gravity and the block will reach equilibrium in a strained state. In order to reach the equilibrium, we must suppose that, when the medium is in motion, energy is dissipated (otherwise elastic oscillations would persist forever). We may also assume (for simplicity) that the medium is incompressible, so that volume elements can be distorted but not dilated or compressed during the motion.

Let $\mathbf{v}(\mathbf{x},t)$ be the velocity field during the relaxation process, and $\rho(\mathbf{x},t)$ the density field. Then under the assumption of incompressibility, $\nabla\cdot\mathbf{v}=0$

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Published in "Topological Aspects of the Dynamics of Fluids and Plasmas" Ed. H.K. Moffatt et al
Kluwer NATO ASI Series E: Applied Sciences - Vol. 218
pp 3-28.

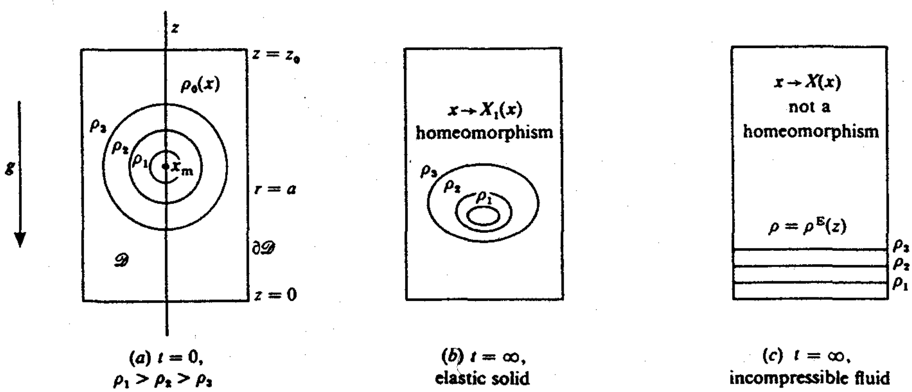

Fig. 1. Gravitational relaxation of an incompressible medium with non-uniform density distribution. When the medium is an elastic solid, the equilibrium state is related by homeomorphism to the initial state. When the medium is fluid, the limit map (as $t \to \infty$) is not a homeomorphism. [From Moffatt, 1985.]

and

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0. \tag{1.1}$$

This equation simply expresses the fact that the density of each material element is conserved; equivalently the isodensity surfaces $\rho = cst.$ are 'frozen in the medium', so that their topology is conserved, *i.e.*, the picture looks qualitatively the same at time t as it does at time zero, although there is obviously a distortion of each surface. Equation (1.1) however embodies the essential topological constraint that characterizes this problem. Note that the volume $V(\rho)$ inside any closed surface $\rho = cst.$ is conserved during relaxation. Thus the function $V(\rho)$ is a topological invariant of the density field; this function may be described as the *signature* of the field $\rho(\mathbf{x}, t)$ for this problem.

Under the action of the velocity field $\mathbf{v}(\mathbf{x}, t)$, the particle initially at position $\mathbf{x}$ moves to position $\mathbf{X}(\mathbf{x}, t)$ at time t where

$$\frac{d\mathbf{X}}{dt} = \mathbf{v}(\mathbf{X}, t) \quad , \quad \mathbf{X}(\mathbf{x}, 0) = \mathbf{x}. \tag{1.2}$$

If $\mathbf{v}(\mathbf{x}, t)$ is known (which of course it isn't until the dynamical equations governing the relaxation process are prescribed and solved) then (1.2) is a third order nonlinear dynamical system which determines the particle paths $\mathbf{X}(\mathbf{x}, t)$. The particle displacement $\mathbf{x} \to \mathbf{X}(\mathbf{x}, t)$ constitutes a mapping which depends continuously on the parameter t. If $\rho_0(\mathbf{x})$ is a nice smooth function, and we have no reason at this stage to suppose otherwise, and if the elastic medium has uniform and physically sensible elastic properties, then obviously for any t, $\mathbf{X}(\mathbf{x},t)$ is a continuous function of $\mathbf{x}$ and its inverse $\mathbf{x}(\mathbf{X},t)$ is also continuous (neighboring particles at time t came from neighboring sites at time $t=0$). The mapping $\mathbf{x} \to \mathbf{X}(\mathbf{x},t)$ is thus a *homeomorphism* for each t, and the family of mappings $\mathbf{x} \to \mathbf{X}(\mathbf{x},t)(0 \le t \le T)$ constitutes an *isotopy* (or flow) for each T. Moreover, for the solution considered, the relaxation process is clearly well under control, and convergent, as $t \to \infty$; we can thus define the limit map

$$\mathbf{x} \to \mathbf{X}^E(\mathbf{x}) = \lim_{t \to \infty} \mathbf{X}(\mathbf{x},t), \tag{1.3}$$

which is also a homeomorphism representing the net displacement of particles under the complete relaxation process.

The solution of (1.1) is

$$\rho(\mathbf{x},t) = \rho(\mathbf{x},0) = \rho_0(\mathbf{x}), \tag{1.4}$$

an equation which establishes a *topological equivalence* between the density fields at time t and at time zero. Note that the Jacobian of the volume-preserving mapping $\mathbf{x} \to \mathbf{X}(\mathbf{x},t)$ has value $+1$:

$$J \equiv \frac{\partial(X_1,X_2,X_3)}{\partial(x_1,x_2,x_3)} = +1. \tag{1.5}$$

The relaxed state is evidently one in which the total energy E of the system (gravitational plus elastic strain energy) is a minimum. What does this mean? Simply that, if we consider a virtual volume-preserving displacement $\mathbf{x} \to \mathbf{x} + \xi(\mathbf{x})$ of the medium from its equilibrium position, with $\xi \cdot \mathbf{n} = 0$ on ∂D, and with

$$\rho(\mathbf{x} + \xi(\mathbf{x})) = \rho^E(\mathbf{x}), \tag{1.6}$$

where $\rho^E(\mathbf{x})$ is the density distribution in the relaxed state, and if we expand E in powers of ξ in the form

$$E = E_0 + \delta^1 E + \delta^2 E + O(\xi^3), \tag{1.7}$$

then $\delta^1 E = 0$ and $\delta^2 E > 0$ for all admissible ξ (*i.e.*, all $\xi(\mathbf{x})$ satisfying the above conditions).

Finally, we note that there is no guarantee that the minimum energy state is unique. We could vary the problem by first imposing a (volume preserving) displacement $\mathbf{x} \to \mathbf{X}_1(\mathbf{x})$ (thus straining the medium), and then release from rest allowing relaxation to proceed along a different path. By this means, different minimum energy configurations may conceivably be attained, characteristic by a spectrum of minimum energies

$$E_0 \le E_1 \le E_2 \le \ldots \tag{1.8}$$

1.2. INCOMPRESSIBLE FLUID MEDIUM

Consider now the same relaxation problem but in an incompressible (Boussinesq) fluid medium, with the same initial non-uniform density $\rho_0(\mathbf{x})$. Again, for $t > 0$, the denser fluid moves downwards, displacing lighter fluid upwards. For any finite time t, all of the previous considerations apply without change; in particular the density surfaces $\rho = cst.$ move with the flow, and the signature function $V(\rho)$ is invariant. For finite t, $\rho(\mathbf{x}, t)$ is topologically equivalent to $\rho_0(\mathbf{x})$. However this nice state of affairs does not persist in the limit $t \to \infty$. It is physically obvious that, no matter what the initial topology of $\rho_0(\mathbf{x})$ may be, the ultimate state is one in which $\rho = \rho^E(z)$ where z is the vertical coordinate where, for minimum energy (and therefore stability)

$$d\rho^E/dz \leq 0 \quad (\text{all } z). \tag{1.9}$$

The topology of the field $\rho^E(z)$ is very simple: the surfaces $\rho^E = cst.$ are horizontal planes intersecting the boundary $\partial\mathcal{D}$ (figure 1c).

The apparent change in topology as $t \to \infty$ arises because the mapping $\mathbf{x} \to \mathbf{X}(\mathbf{x}, t)$ develops discontinuities for various values of $\mathbf{x}$, *i.e.*, the limit map $\mathbf{x} \to \mathbf{X}^E(\mathbf{x})$ (which exists in the sense that every fluid particle goes somewhere!) is not a homeomorphism. For the simple topology illustrated in figure 1, the limit mapping is obviously discontinuous for $\mathbf{x} = \mathbf{x}_m$ (where $\rho_0(\mathbf{x})$ is maximal) since the fluid particles in an arbitrarily small sphere surrounding $\mathbf{x}_m$ are ultimately spread over the base of the container. A moment's consideration will convince you that the limit mapping is likewise discontinuous for every point $\mathbf{x}$ vertically below $\mathbf{x}_m$. This lack of continuity is present also in the inverse limit map: surfaces $\rho = cst.$ are squeezed together on the fluid boundary as $t \to \infty$ and ultimately coincide in part with the boundary, so that particles ultimately contiguous have inverse images that were widely separated at $t = 0$.

The fluid problem differs from the elastic problem in that there is no counterpart of the elastic strain energy; this lack of constraint is what permits the formation of discontinuities in the limit mapping and the associated simplification in topology of the density field.

The magnetic relaxation problem, to which we now turn, is in some respects intermediate between these two situations. On the one hand, we shall be concerned with minimisation of a single form of energy, namely magnetic energy. On the other hand, this magnetic energy, reflecting as it does the Maxwell tension in the magnetic lines of force in the medium, resembles elastic strain energy, and minimisation of magnetic energy is subject to constraints that are not present for the fluid gravitational relaxation problem. Thus, we shall find that, although discontinuities can appear as $t \to \infty$, nevertheless a nontrivial end-state having topology closely related to that of the critical field, is in general attained.

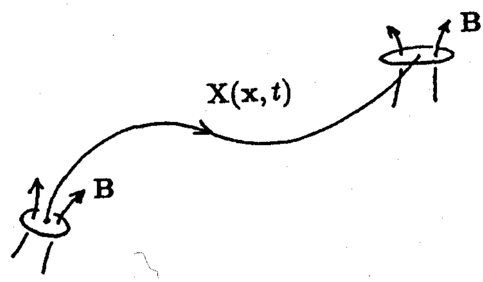

Fig. 2. Particle path and associated deformation of elemental circuit, through which the flux of **B** is conserved.

2. Magnetic Relaxation

2.1. THE MASTER EQUATION

Consider a magnetic field $\mathbf{B}(\mathbf{x}, t)$ in a perfectly conducting incompressible fluid satisfying the solenoidal condition $\nabla \cdot \mathbf{B} = 0$ and the 'frozen field' equation (derivable from Faraday's law of induction)

$$\frac{\partial \mathbf{B}}{\partial t} = \text{curl}\,(\mathbf{v} \times \mathbf{B}), \tag{2.1}$$

where, as before, $\mathbf{v}(\mathbf{x}, t)$ is the velocity field in the medium. If $\mathbf{X}(\mathbf{x}, t)$ is the mapping given by (1.2), then the (Cauchy) solution of (2.1) is given by

$$B_i(\mathbf{X}, t) = B_j(\mathbf{x}, 0)\frac{\partial X_i}{\partial x_j}, \tag{2.2}$$

an equation which encapsulates the 'frozen-in' character of the field (figure 2). Under this evolution, the flux of **B** through any elemental surface element moving with the fluid is conserved.

Since the **B**-lines move with the fluid, we note immediately that the topology of the **B**-field is conserved by equation (2.1). In particular, any knots or links in the **B**-lines are conserved. Any topological invariant of a knot or link must likewise be an invariant of equation (2.1), if we simply adopt an initial condition that conforms to the knot or link under consideration (see §4 below). In this sense, (2.1) may be described as a 'MASTER equation': all topological invariants of knots and links and indeed of more complex chaotic structures are somehow contained within this equation. Unfortunately this statement does not make it any easier to find such invariants!

2.2. THE HELICITY INVARIANT

There is however one topological invariant which is easily obtained, and which plays a central role in the magnetic relaxation problem, namely the magnetic helicity invariant, defined as follows. Let S be any closed surface moving with the fluid on which the condition $\mathbf{n}\cdot\mathbf{B}=0$ is satisfied; note that by the frozen field properly, if this condition is satisfied at time zero, then it is satisfied for all t. Let $\mathbf{A}(\mathbf{x},t)$ be a vector potential for $\mathbf{B}$, *i.e.*, $\mathbf{B} = \text{curl } \mathbf{A}$, and let us for definiteness adopt the Coulomb gauge $\nabla\cdot\mathbf{A}=0$. We define the magnetic helicity in the volume V inside S by the integral

$$\mathcal{H}_M = \int_V \mathbf{A}\cdot\mathbf{B}dV. \tag{2.3}$$

Now, by 'uncurling' (2.1), we have

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v}\times(\nabla\times\mathbf{A}) - \nabla\varphi \tag{2.4}$$

for some scalar field $\varphi(\mathbf{x},t)$. Equations (2.1) and (2.4) may be written in the equivalent Lagrangian forms

$$\frac{DB_i}{Dt} = B_j\frac{\partial v_i}{\partial x_j}, \tag{2.5}$$

$$\frac{DA_i}{Dt} = A_j\frac{\partial v_j}{\partial x_i} - \frac{\partial \chi}{\partial x_i}, \tag{2.6}$$

where $\chi = \varphi - \mathbf{v}\cdot\mathbf{A}$. Hence

$$\frac{D}{Dt}(\mathbf{A}\cdot\mathbf{B}) = -(\mathbf{B}\cdot\nabla)\chi = -\nabla\cdot(\mathbf{B}\chi) \tag{2.7}$$

and so

$$\frac{d\mathcal{H}_M}{dt} = \int_V \frac{D}{Dt}(\mathbf{A}\cdot\mathbf{B})dV = -\int_S(\mathbf{n}\cdot\mathbf{B})\chi dV = 0 \tag{2.8}$$

since $\mathbf{n}\cdot\mathbf{B}=0$ on S. Hence $\mathcal{H}_M$ is indeed invariant.

Note three points: (i) $\mathcal{H}_M$ is in fact gauge-invariant, since replacement of $\mathbf{A}$ by $\mathbf{A}+\nabla\psi$ does not change its value; (ii) the result $\mathcal{H}_M = cst.$ holds also for compressible fluid motion (for which (2.6) hold unchanged while (2.5) holds with $\mathbf{B}$ replaced by $\mathbf{B}/\rho$; (iii) $\mathcal{H}_M$ is a pseudo-scalar (changing sign under a parity transformation from right-handed to left-handed frame of reference).

Let us evaluate $\mathcal{H}_M$ for the very simple 'prototype' linked field shown in figure 3. Here $\mathbf{B}\equiv 0$ except in two linked flux tubes $\mathcal{T}_1$, $\mathcal{T}_2$ centred on unknotted curves C_1 and C_2. We suppose that within each tube, $\mathbf{B}$ has trivial topology, *i.e.*, the $\mathbf{B}$-lines are unlinked closed curves 'parallel' to C_1 and C_2 respectively. If the cross-section of each tube is small, then in the volume integral (2.3) we may make the replacements

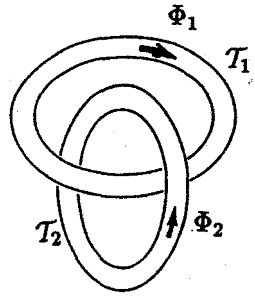

Fig. 3. Prototype linkage of flux tubes $\mathcal{T}_1, \mathcal{T}_2$.

$$\mathbf{B}dV \to \left.\begin{array}{l} \Phi_1 d\mathbf{x}_1 \text{ in } \mathcal{T}_1 \\ \Phi_2 d\mathbf{x}_2 \text{ in } \mathcal{T}_2 \end{array}\right\} \tag{2.9}$$

and the integral becomes

$$\mathcal{H}_M \sim \Phi_1 \oint_{C_1} \mathbf{A} \cdot d\mathbf{x}_1 + \Phi_2 \oint_{C_2} \mathbf{A} \cdot d\mathbf{x}_2. \tag{2.10}$$

Now

$$\oint_{C_1} \mathbf{A} \cdot d\mathbf{x}_1 = \int_{\Sigma_1} \mathbf{B} \cdot \mathbf{n} dS = \Phi_2,$$

where Σ_1 is an open surface spanning C_1; similarly

$$\oint_{C_2} \mathbf{A} \cdot d\mathbf{x}_2 = \Phi_1.$$

Hence, for the oriented linkage of figure 3,

$$\mathcal{H}_M = 2\Phi_1\Phi_2. \tag{2.11}$$

More generally

$$\mathcal{H}_M = \pm 2n\Phi_1\Phi_2, \tag{2.12}$$

where n is the (Gauss) linking number of C_1 and C_2, and the + or − is chosen according as the linkage is right-handed or left-handed.

Equation (2.12) establishes a bridge between an invariant of the partial differential equation (2.1) and the primitive topological invariant n of the two linked curves. This bridge has been reinforced by Arnold (1974, 1986) who has extended the topological interpretation of $\mathcal{H}_M$ to situations in which the $\mathbf{B}$ lines do not form conveniently closed curves, but which may wander chaotically in the domain $\mathcal{D}$.

2.3. MINIMUM ENERGY STATES

Now suppose that the fluid is confined to a domain $\mathcal{D}$ and that $\mathbf{B}$ satisfies the condition $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial\mathcal{D}$. We define the energy of the field $\mathbf{B}$ as

$$M(t) = \frac{1}{2} \int \mathbf{B}^2 dV, \tag{2.13}$$

and we investigate the nature of minimum energy states that can be arrived at under evolution controlled by equation (2.1). That there is in general a positive minimum energy is ensured by the Schwarz inequality

$$\int_D \mathbf{A}^2 dV \int_D \mathbf{B}^2 dV \geq \mathcal{H}_M^2, \tag{2.14}$$

and the Poincaré inequality

$$\frac{\int \mathbf{B}^2 dV}{\int \mathbf{A}^2 dV} \geq q_0^2 > 0. \tag{2.15}$$

[This latter inequality is more familiar in the analogous context in which a current $\mathbf{j}(\mathbf{x})$ confined to a domain $\mathcal{D}$ gives rise to a field $\mathbf{B}$ satisfying $\nabla \times \mathbf{B} = \mathbf{j}$ inside $\mathcal{D}$ and $\nabla \times \mathbf{B} = 0$ outside $\mathcal{D}$ with $\mathbf{B} = O(r^{-3})$ at infinity. The inequality then reads

$$\int \mathbf{j}^2 dV \geq q_0^2 \int \mathbf{B}^2 dV.] \tag{2.16}$$

Multiplying (2.14) and (2.15), we have

$$\int \mathbf{B}^2 dV \geq q_0 |\mathcal{H}_M|, \tag{2.17}$$

and since $\mathcal{H}_M$ is invariant, this clearly provides a lower bound for $M(t)$. As will be clear below, it is the topological barrier implied by nonzero helicity that guarantees this lower bound. As shown by Freedman (1988), even if $\mathcal{H}_M = 0$, a positive lower bound on $M(t)$ exists provided merely that the topology is nontrivial. A simple example of a nontrivial topology having zero helicity is provided by the Whitehead link (figure 4).

2.4. CONSTRUCTION OF A DYNAMICAL PROCESS BY WHICH THE LOWER BOUND MAY BE ATTAINED

Note first that from (2.1), the rate of change of magnetic energy density is given by

$$\frac{d}{dt}\frac{1}{2}\mathbf{B}^2 = \mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} = \mathbf{B}\cdot\nabla\times(\mathbf{v}\times\mathbf{B}) = (\nabla\times\mathbf{B})\cdot(\mathbf{v}\times\mathbf{B}) + \nabla\cdot[\mathbf{B}\times(\mathbf{v}\times\mathbf{B})] \tag{2.18}$$

so that, integrating over the domain,

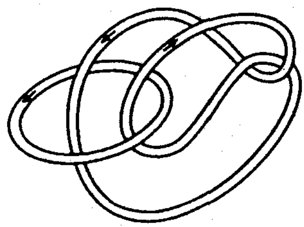

Fig. 4. A simple example of a non-trivial linkage, for which the linking number is zero and the associated helicity is also zero.

$$\frac{d}{dt}M(t) = -\int \mathbf{v}\cdot(\mathbf{j}\times\mathbf{B})dV, \tag{2.19}$$

where $\mathbf{j} = \nabla\times\mathbf{B}$, since $\mathbf{v}\cdot\mathbf{n} = \mathbf{B}\cdot\mathbf{n} = 0$ on $\partial\mathcal{D}$. The right-hand side represents the rate of work of the field on the medium through the action of the Lorentz force $\mathbf{j}\times\mathbf{B}$. We can ensure that $M(t)$ decreases by adopting any dynamical model in which $\mathbf{v}$ 'yields' to this force so that magnetic energy is converted to kinetic energy which can then be dissipated by internal friction. The Navier-Stokes equations for a viscous fluid have this property. A simpler model however, on which we shall focus here, is provided by the instantaneous prescription

$$k\mathbf{v} = \mathbf{j}\times\mathbf{B} - \nabla p, \tag{2.20}$$

where p is a pressure field chosen to ensure that $\nabla\cdot\mathbf{v} = 0$ and $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\mathcal{D}$. Thus p satisfies the Neumann problem

$$\left.\begin{aligned}\nabla^2 p &= \nabla\cdot(\mathbf{j}\times\mathbf{B}) \text{ in } \mathcal{D}\\ \mathbf{n}\cdot\nabla p &= \mathbf{n}\cdot(\mathbf{j}\times\mathbf{B}) \text{ on } \partial\mathcal{D}\end{aligned}\right\} \tag{2.21}$$

Substituting the solution back in (2.20), we may write the result in the form

$$k\mathbf{v} = (\mathbf{j}\times\mathbf{B})_s \tag{2.22}$$

where the suffix s means 'solenoidal projection'.

Substituting in (2.19) we now have

$$\frac{d}{dt}M(t) = -k\int \mathbf{v}^2 dV \tag{2.23}$$

(since $\int \mathbf{v}\cdot\nabla p dV = 0$). Thus $M(t)$ decreases monotonically for so long as $v \neq 0$, and, being bounded below, must tend to a limit. Note the importance of the lower bound for the force of this argument.

Since $M(t) \to M^E(cst)$ as $t \to \infty$, it follows from (2.23) that

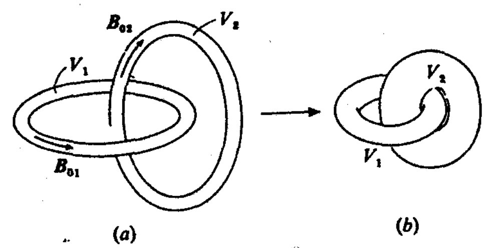

Fig. 5. Magnetic relaxation for the simplest non-trivial topology. (a) the initial state; (b) the stage at which the topological constraint begins to impede the relaxation process. [From Moffatt, 1985.]

$$\int_{\mathcal{D}} \mathbf{v}^2 dV \to 0 \text{ as } t \to \infty. \tag{2.24}$$

Hence, setting aside the extremely unlikely possibility that singularities of $\mathbf{v}$ develop on a set of measure zero in $\mathcal{D}$ (a possibility that is however as difficult to disprove as it is unlikely!) we may conclude that $\mathbf{v} \to 0$ everywhere in $\mathcal{D}$ and that in the asymptotic state the field $\mathbf{B}^E(\mathbf{x})$ satisfies the magnetostatic equation

$$\mathbf{j}^E \times \mathbf{B}^E = \nabla p^E. \tag{2.25}$$

This relaxation process may be best understood with regard to the prototype linked flux tube configuration shown again in figure 5. Just as stretching of B-lines is associated with field intensification (the essence of the dynamo process), so reduction of field energy can occur only through contraction of B-lines. Thus, both flux tubes in figure 5 tend to contract as they release magnetic energy; since the process is volume preserving, the cross sections of the tubes must simultaneously expand, and it is obvious that the process cannot continue indefinitely, since eventually the tubes must make contact with each other. It is the linkage (reflected in this case by a nonzero value of $\mathcal{H}_M$) that provides this topological barrier that guarantees that the energy does indeed have a lower bound.

2.5. THE IDEA OF TOPOLOGICAL ACCESSIBILITY

It is evident from the above example that tangential discontinuities of $\mathbf{B}$ may form in regions where two flux tubes are pressed together by the relaxation process. Hence the limit map $\mathbf{x} \to \mathbf{X}^E(\mathbf{x})$ will not in general be a homeomorphism, and the limit field $\mathbf{B}^E(\mathbf{x})$ will not therefore be topologically equivalent in a strict sense to the original fields. Nevertheless all links and knots are conserved throughout the relaxation process, and the formation of discontinuities simply brings different flux tubes into contact over

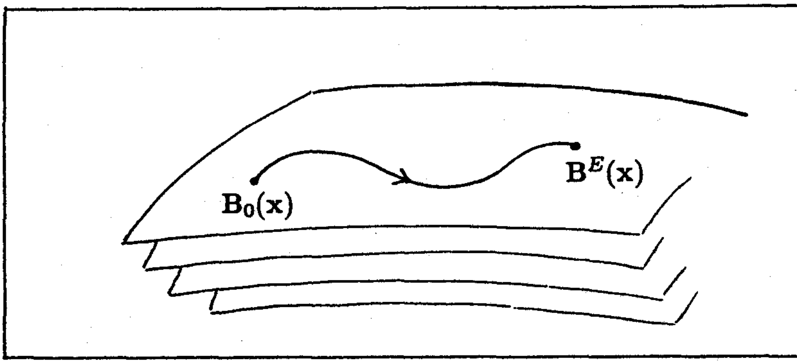

Fig. 6. Isomagnetic foliation of the function-space of solenoidal fields of finite energy. Magnetic relaxation is represented by a trajectory on one sheet of this foliation.

a finite area, without permitting any cutting and reconnecting of B-lines. Since the field $\mathbf{B}^E(\mathbf{x})$ is obtained through the convective action of a smooth velocity field $\mathbf{v}(\mathbf{x},t)$ on the initial field $\mathbf{B}_0(\mathbf{x})$, and since moreover the field $\mathbf{v}$ dissipates a total energy on the time interval $0 > t > \infty$ which is finite, *i.e.*,

$$\int_0^\infty dt \int_{\mathcal{D}} \mathbf{v}^2 dV < \infty, \tag{2.26}$$

we may say that $\mathbf{B}^E(\mathbf{x})$ is *topologically accessible* from $\mathbf{B}_0(\mathbf{x})$.

It is helpful to picture this process in the function space of all bounded square-integrable solenoidal fields (figure 6). This space is foliated through the frozen field property, *i.e.*, folia (or sheets) are distinguished by the property that fields on distinct sheets are topologically distinct. We describe these sheets as 'isomagnetic'. All fields on a given isomagnetic sheet are topologically accessible one from another through frozen field distortion. The magnetic relaxation process may thus be pictured as a trajectory on the isomagnetic sheet determined by the initial field $\mathbf{B}_0(\mathbf{x})$.

2.6. THE STRUCTURE OF EQUILIBRIUM FIELDS $\mathbf{B}^E(\mathbf{X})$

From equation (2.25), it is evident that

$$\mathbf{j}^E \cdot \nabla p^E = 0 \quad \text{and} \quad \mathbf{B}^E \cdot \nabla p^E = 0, \tag{2.27}$$

i.e., both $\mathbf{B}^E$-lines and $\mathbf{j}^E$-lines lie on the family of surfaces $p^E = cst$. It may of course happen that $\nabla p^E \equiv 0$ in some subdomain $\mathcal{D}^p$ of $\mathcal{D}$; then $\mathbf{j}^E \times \mathbf{B}^E = 0$ in $\mathcal{D}^p$, so that

$$\mathbf{j}^E = \alpha(\mathbf{x})\mathbf{B}^E \quad , \quad \mathbf{B}^E \cdot \nabla\alpha = 0 \quad \text{in} \quad \mathcal{D}^p. \tag{2.28}$$

Again, the $\mathbf{B}^E$-lines (which now coincide with the $\mathbf{j}^E$-lines) lie on the family of surfaces $\alpha = \text{cst}$. If, however $\nabla\alpha \equiv 0$ in some further nested subdomain $\mathcal{D}^\alpha \subset \mathcal{D}^p$, then

$$\mathbf{j}^E = \nabla \times \mathbf{B}^E = \alpha \mathbf{B}^E \text{ in } \mathcal{D}^\alpha, \tag{2.29}$$

where α is constant. Fields satisfying the condition (2.28) are described as Beltrami fields. We shall describe the stronger condition (2.29) as the 'strong Beltrami condition', and corresponding fields as strong Beltrami fields. Only if the field is strong Beltrami is it released from the topological constraint of lying on surfaces. The prototype example of a strong Beltrami field is the 'ABC-field'

$$\mathbf{B}^E = (C\sin\alpha z + B\cos\alpha y, A\sin\alpha x + C\cos\alpha z, B\sin\alpha y + A\cos\alpha x) \tag{2.30}$$

for which the condition (2.29) may be easily verified. It is known from the work of Dombre *et al.* (1985) that, when $ABC \neq 0$, at least some of the $\mathbf{B}^E$-lines of this field exhibit chaotic wandering. There are also however islands of regularity within the chaotic sea, within which $\mathbf{B}^E$-lines do lie on surfaces. More general space-periodic fields of the form

$$\mathbf{B}^E = \sum_n \mathbf{B}_n e^{i\boldsymbol{\alpha}_n\cdot\mathbf{x}} \quad , \quad |\boldsymbol{\alpha}_n| = \alpha, \tag{2.31}$$

where the set of vector $\boldsymbol{\alpha}_n$ are symmetrically distributed on the sphere of radius α, have been extensively studied by Zaslavsky *et al.* (1991), and these flows generally exhibit a 'chaotic web' of $\mathbf{B}^E$ lines again containing islands of regularity. It seems likely that the most general space-periodic strong Beltrami field can be expressed in the form (2.31).

Suppose now that the initial field $\mathbf{B}_0(\mathbf{x})$ for our relaxation problem is chaotic in some subdomain $\mathcal{D}^C$ of the fluid domain $\mathcal{D}$. The volume V_C of $\mathcal{D}^C$ (more strictly its measure if it has fractal properties) is conserved under (volume-preserving) relaxation; hence the $\mathbf{B}^E$-lines of the relaxed field are also chaotic within the image domain $\bar{\mathcal{D}}^C$ (under the mapping $\mathbf{x} \to \mathbf{X}^E(\mathbf{x})$); hence $\mathbf{B}^E$ must be a strong Beltrami field in $\bar{\mathcal{D}}^C$.

This result appears quite surprising, because it implies that a strong Beltrami field is always topologically accessible from a chaotic field $\mathbf{B}_0(\mathbf{x})$, no matter what the 'degree of chaos' may be. However it must be recognized that the boundary of $\bar{\mathcal{D}}^C$ may be very highly convoluted, and that complexity of structure of the field $\mathbf{B}_0(\mathbf{x})$ may carry over during relaxation to corresponding complexity of structure of $\bar{\mathcal{D}}^C$, and hence of the strong Beltrami field in $\bar{\mathcal{D}}^C$.

2.7. STABILITY OF MAGNETOSTATIC EQUILIBRIUM

Minimum energy magnetostatic states are, by the nature of the relaxation process, clearly stable: if an equilibrium field $\mathbf{B}^E(\mathbf{x})$ that has been arrived at by magnetic relaxation on an isomagnetic sheet S_0 is perturbed on this sheet, then such perturbation must increase its energy, and within the framework of *any* dynamical model that dissipates kinetic energy, the field will tend to return to its minimum energy state, *i.e.*, to $\mathbf{B}^E(\mathbf{x})$.

Not all magnetostatic equilibria however are the result of a magnetic relaxation process, and it is useful to have an explicit criterion for minimality of magnetic energy with respect to frozen field distortions. Let $\xi(\mathbf{x})$ be a virtual volume-preserving displacement of the medium, which we regard as the mapping associated with a steady velocity field $\mathbf{v}(\mathbf{x})$ acting for a short time interval $0 < t < \tau$, with $\nabla \cdot \mathbf{v} = 0$. It is easily shown that

$$\xi(\mathbf{x}) = \eta(\mathbf{x}) + \frac{1}{2}(\eta \cdot \nabla)\eta + O(\eta^3), \tag{2.32}$$

where $\eta(\mathbf{x}) = \tau\mathbf{v}(\mathbf{x})$, and that the first and second order variation of $\mathbf{B}^E(\mathbf{x})$ under frozen field distortion (*i.e.*, on the isomagnetic sheet S_0) are

$$\delta^1\mathbf{B} = \nabla \times (\eta \times \mathbf{B}^E) \quad , \quad \delta^2\mathbf{B} = \frac{1}{2}\nabla \times (\eta \times \delta^1\mathbf{B}) \tag{2.33}$$

(and generally, $\delta^n\mathbf{B} = \frac{1}{n}\nabla \times (\eta \times \delta^{(n-1)}\mathbf{B})$).

The magnetic energy in the disturbed state is then given by

$$M = M^E + \delta^1 M + \delta^2 M + O(\eta^3), \tag{2.34}$$

where

$$\delta^1 M = \int \mathbf{B}^E \cdot \delta^1\mathbf{B}dV, \tag{2.35}$$

$$\delta^2 M = \frac{1}{2}\int [(\delta^1\mathbf{B})^2 + 2\mathbf{B}^E \cdot \delta^2\mathbf{B}]dV. \tag{2.36}$$

It is not difficult to prove that, under the equilibrium condition (2.25) and the boundary conditions

$$\mathbf{B}^E \cdot \mathbf{n} = 0 \quad , \quad \eta \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{D}, \tag{2.37}$$

$\delta^1 M = 0$, as is to be expected.

A word of caution is needed here concerning a certain class of displacements $\eta(\mathbf{x})$ for which $\delta^1\mathbf{B} = 0$ (and therefore $\delta^n\mathbf{B} = 0$ for all $n \geq 2$ also); an example of such displacements are those everywhere parallel to $\mathbf{B}^E$ so that $\eta \times \mathbf{B}^E = 0$. The field is clearly uninfluenced by such displacements, and it seems appropriate to exclude them in a consideration of stability. We therefore define an 'admissible' displacement function $\eta(\mathbf{x})$ by the three conditions

$$\nabla \cdot \eta = 0, \quad \mathbf{n} \cdot \eta = 0 \text{ on } \partial\mathcal{D}, \quad \nabla \times (\eta \times \mathbf{B}^E) \not\equiv 0, \tag{2.38}$$

and we assert that the field $\mathbf{B}^E$ is stable if

$$\delta^2 M > 0 \text{ for all admissible } \boldsymbol{\eta}. \tag{2.39}$$

The *ABC* field provides an interesting and nontrivial illustration of the application of this procedure. With

$$\boldsymbol{\eta}(\mathbf{x}) = \sum_m \boldsymbol{\eta}_m e^{i\mathbf{k}_m\cdot\mathbf{x}} \quad , \boldsymbol{\eta}_m \cdot \mathbf{k}_m = 0, \tag{2.40}$$

and

$$\mathbf{B}^E = \sum_n \mathbf{B}_n e^{i\boldsymbol{\alpha}_n\cdot\mathbf{x}}, \quad |\boldsymbol{\alpha}_n| = |\alpha|, \tag{2.41}$$

direct calculation (Moffatt, 1986) gives

$$\delta^2 M = \frac{1}{2}\sum_{n,m} |\boldsymbol{\eta}_m|^2 |\mathbf{k}_m \cdot \mathbf{B}_n|^2, \tag{2.42}$$

and since this is positive for all admissible $\boldsymbol{\eta}$, the equilibrium is stable. This conclusion still holds when $\boldsymbol{\eta}(\mathbf{x})$ is unrestricted by the incompressibility condition (see Sero-Guillaume and Moffatt, 1992, ~~this volume~~ in preparation).

3. Analogous Euler Flows

3.1. TWO USEFUL ANALOGIES

Let us now consider the corresponding situation in relation to solutions of the Euler equations of classical hydrodynamics of an incompressible inviscid fluid. In terms of the velocity field $\mathbf{u}(\mathbf{x}, t)$ and the pressure field $p(\mathbf{x}, t)$ these equation are simply

$$\frac{D\mathbf{u}}{Dt} \equiv \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p \quad , \quad \nabla\cdot\mathbf{u} = 0, \tag{3.1}$$

where ρ is the fluid density, assumed uniform. Using the vector identity $\mathbf{u}\times(\nabla\times\mathbf{u}) = \frac{1}{2}\nabla\mathbf{u}^2 - \mathbf{u}\cdot\nabla\mathbf{u}$, the evolution equation may be written in the alternative form

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u}\times\boldsymbol{\omega} - \nabla h, \tag{3.2}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the vorticity field and $h = p/\rho + \frac{1}{2}\mathbf{u}^2$. The curl of this equation gives the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{u}\times\boldsymbol{\omega}). \tag{3.3}$$

We now note two analogies which are quite different in nature and which must not be confused.

Analogy A

First note the analogy between equation (2.1) and (3.3) with the identifications

$$\mathbf{v} \longleftrightarrow \mathbf{u} \quad , \quad \mathbf{B} \longleftrightarrow \boldsymbol{\omega}, \tag{3.4}$$

all four fields being solenoidal. The analogy is not complete, because whereas $\mathbf{u}$ in (3.3) is constrained to be the velocity field associated with $\boldsymbol{\omega}$ through $\boldsymbol{\omega} = \mathrm{curl}\ \mathbf{u}$, there is no corresponding constraint on $\mathbf{v}$ in (2.1). This means that a much wider range of initial conditions can be considered in relation to (2.1); it is this freedom that allows the growth of magnetic modes (*i.e.* dynamo action) that have no obvious counterpart in the Euler flow context.

Despite the imperfection of the analogy however, certain properties of (2.1) do carry over to (3.3). Most important among these is the fact that (3.3) is a 'frozen field' equation so that all topological properties of the vorticity field $\boldsymbol{\omega}$ are conserved under Euler evolution. In particular, there is a helicity invariant analogous to (2.3) namely (Moffatt, 1969)

$$\mathcal{H} = \int_V \mathbf{u} \cdot \boldsymbol{\omega} dV \quad , \quad \boldsymbol{\omega} \cdot \mathbf{n} = 0 \text{ on } \partial V. \tag{3.5}$$

This (kinetic) helicity is conserved for so long as the velocity and vorticity fields remain smooth (C^1) functions of position $\mathbf{x}$. [Other papers in this volume will address the possibility that singularities may develop under Euler evolution within a finite time; if this occurs, then viscous effects must presumably cause reconnection of vortex lines in the neighborhood of such singularities, and helicity (and likewise energy) will then no longer be conserved.]

Similarly, any theorem that is dependent only on the frozen-in character of the field $\mathbf{B}$, will apply equally to the field $\boldsymbol{\omega}$ determined by (3.3). Some such theorems (*e.g.* Kelvin's circulation theorem) were discovered first in the Euler equation context!

However, some statements do *not* carry over: *e.g.* stable magnetostatic equilibria are characterised by minimum magnetic energy, but stable solutions of the Euler equations are *not* characterised by minimisation of enstrophy $\int \boldsymbol{\omega}^2 dV$.

Analogy B

Secondly, note the analogy between the equations of magnetostatic equilibrium (see (2.25))

$$\mathbf{j} \times \mathbf{B} = \nabla p \quad , \quad \mathbf{j} = \nabla \times \mathbf{B}, \tag{3.6}$$

and the *steady* Euler flow equation (see (3.2))

$$\mathbf{u} \times \boldsymbol{\omega} = \nabla h \quad , \quad \boldsymbol{\omega} = \nabla \times \mathbf{u}, \tag{3.7}$$

Here the analogy is evidently between the variables

$$\mathbf{B} \longleftrightarrow \mathbf{u} \quad , \quad \mathbf{j} \longleftrightarrow \boldsymbol{\omega} \quad , \quad p \longleftrightarrow h_0 - h, \tag{3.8}$$

where h_0 is some positive constant. To every solution of (3.6), obtained by whatever means, there corresponds a solution of (3.7) *via* the correspondences (3.8); thus, to every solution of (3.6) obtained by the magnetic relaxation procedure described in section 2, there corresponds a solution of the Euler equations satisfying corresponding boundary conditions (*e.g.* if $\mathbf{n} \cdot \mathbf{B} = 0$ on $\partial\mathcal{D}$, then $\mathbf{n} \cdot \mathbf{u} = 0$ on $\partial\mathcal{D}$ for the corresponding solution). Tangential discontinuities of $\mathbf{B}$ (*i.e.* current sheets) in the magnetostatic solutions become vortex sheets in the Euler flows.

Magnetic relaxation thus provides an important, albeit indirect, technique for obtaining a very wide family of fully three-dimensional solutions of the steady Euler equations. Indeed, since the topology of the initial field $\mathbf{B}_0(\mathbf{x})$ in the relaxation problem is arbitrary, we have here a technique of finding Euler flows of arbitrary streamline topology. But beware! The analogy does *not* extend to considerations of stability: although magnetostatic equilibria obtained by magnetic relaxation are stable, there is no guarantee that the analogous Euler flows are stable, because *unsteady* Euler dynamics conserves the topology not of the $\mathbf{u}$-field but rather of the $\boldsymbol{\omega}$-field, a vital distinction.

The situation is represented in figure 7 which shows again the function space of all bounded square-integrable solenoidal fields, with its isomagnetic foliation that constrains the magnetic relaxation process. To the equilibrium field $\mathbf{B}^E(\mathbf{x})$ there corresponds an Euler flow $\mathbf{u}^E(\mathbf{x})$ with vorticity $\boldsymbol{\omega}^E(\mathbf{x})$. If this flow is perturbed by a weak impulsive pressure field, so that the vortex lines are distorted from the equilibrium configuration, then the vorticity field $\boldsymbol{\omega}(\mathbf{x}, t)$ evolves on the submanifold of fields for which this vorticity topology is conserved. This condition defines a different foliation of the function space, described as the *isovortical* foliation. The stability of the magnetostatic field $\mathbf{B}^E(\mathbf{x})$ is determined by behaviour on the isomagnetic folium; the stability of the analogous Euler flow $\mathbf{u}^E(\mathbf{x})$ by the behaviour on the isovortical folium.

3.2. ARNOLD'S STABILITY CRITERION

The unsteady Euler equations (3.1) conserve kinetic energy

$$K = \frac{1}{2}\rho \int \mathbf{u}^2 dV, \tag{3.9}$$

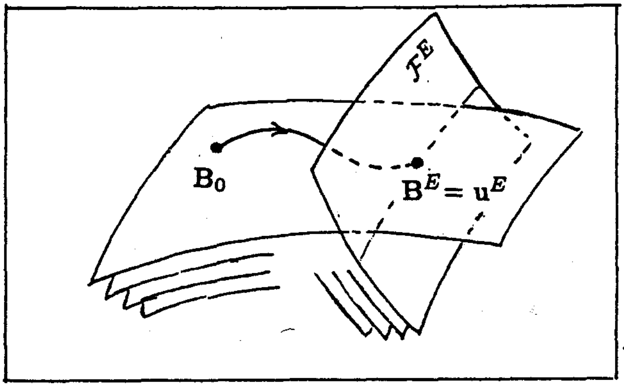

Fig. 7. Isovortical and isomagnetic foliations. Stability of the analogous Euler flow $\mathbf{u}^E(\mathbf{x})$ is determined by the variation of energy on the isovortical folium $\mathcal{F}^E$.

and this further constrains the otherwise free evolution on an isovortical folium. If K has an extremum (maximum or minimum) at the 'point' $\mathbf{u} = \mathbf{u}^E$ in function space, with respect to perturbation on the isovortical folium $\mathcal{F}^E$ through $\mathbf{u}^E$, then the 'curves' of constant K are closed on $\mathcal{F}^E$, so that in this sense (*i.e.* with respect to the energy norm) the flow remains permanently in a neighborhood of $\mathbf{u}^E$, and is therefore stable. This consideration yields a criterion for stability (Arnold, 1966). First note that, with the notation of section (2.7) above, the first and second order variations of vorticity under isovortical perturbation are

$$\delta^1\omega = \nabla \times (\eta \times \omega^E) \quad , \quad \delta^2\omega = \frac{1}{2}\nabla \times (\eta \times \delta^1\omega) \tag{3.10}$$

Hence, using the solenoidal projection as in (2.22), the corresponding velocity perturbations are

$$\delta^1\mathbf{u} = (\eta \times \omega^E)_s \quad , \quad \delta^2\mathbf{u} = \frac{1}{2}(\eta \times \delta^1\omega)_s, \tag{3.11}$$

The first order variation in kinetic energy is

$$\delta^1 K = \rho \int_{\mathcal{D}} \mathbf{u}^E \cdot \delta^1\mathbf{u} dV, \tag{3.12}$$

and it is easily shown, under the conditions $\mathbf{u}^E \cdot \mathbf{n} = \eta \cdot \mathbf{n} = 0$ on $\partial\mathcal{D}$, that $\delta^1 K = 0$, a result due to Kelvin. The second order variation is

$$\begin{aligned} \delta^2 K &= \frac{1}{2}\rho \int \left[(\delta^1\mathbf{u})^2 + 2\mathbf{u}^E \cdot \delta^2\mathbf{u}\right] dV \\ &= \frac{1}{2}\rho \int \left[(\eta \times \omega^E)_S^2 + \mathbf{u}^E \cdot (\eta \times \delta^1\omega)_S\right] dV, \end{aligned} \tag{3.13}$$

Arnold ?

and the equilibrium is stable if

$$\left.\begin{array}{lll} \text{either} & \delta^2 K > 0 & \text{for all admissible } \eta \\ \text{or} & \delta^2 K < 0 & \text{for all admissible } \eta \end{array}\right\}. \tag{3.14}$$

Two comments may be made in relation to this criterion. First, although there appears to be complete parity between maximum and minimum energy states in this criterion, there is, as pointed out by Arnold, a distinction if weak dissipative (viscous) effects are taken into account. These act in such a way as to destabilize the states of maximum energy, just as for the simpler prototype problem of free rotation of a rigid body about its centre of mass. In this case, steady rotation about the axis of minimum inertia has (for given angular momentum) maximum kinetic energy; if the body has a cavity containing viscous fluid, then this equilibrium state becomes unstable, viscous dissipation in the cavity always tending to drive the system to its state of minimum kinetic energy (*i.e.* rotation about the axis of maximum inertia).

Secondly, the qualification concerning stability 'with respect to the energy norm' is important. There are some situations in which a flow may be stable in this sense, but which nevertheless exhibit unbounded growth of perturbations. An example considered by Moffatt and Moore (1978) is the spherical vortex of Hill discovered nearly one hundred years ago (Hill, 1894). Axisymmetric perturbations of this vortex decay everywhere except in the immediate neighborhood of the rear stagnation point, where they grow in an unbounded manner (see figure 8, from Pozrikidis, 1986, who followed the disturbance numerically well into the nonlinear regime). This flow is stable to axisymmetric disturbances with respect to the energy norm but unstable with respect to a norm that focuses on this singular region.

The Arnold criterion (3.14) may be used (Moffatt, 1986) to examine the stability of the ABC-flow

$$\mathbf{u}^E = (C\cos\alpha z + B\sin\alpha y, A\cos\alpha x + C\sin\alpha z, B\cos\alpha y + A\sin\alpha x). \tag{3.15}$$

With a virtual displacement

$$\eta = \eta_0(\cos kz, \sin kz, 0), \tag{3.16}$$

we find from (3.13), after some calculation,

$$\delta^2 K = -\frac{1}{4}\eta_0^2 \frac{(A^2+B^2)\alpha k^3}{(\alpha^2+k^2)}, \tag{3.17}$$

which is positive or negative according as $\alpha k <$ or > 0. Hence $\delta^2 K$ is neither maximal nor minimal, and we may therefore draw no conclusions as regards stability of the flow (3.15). There is however a rather strong inference that the flow is unstable, and further arguments to this effect have been adduced by Moffatt (1986).

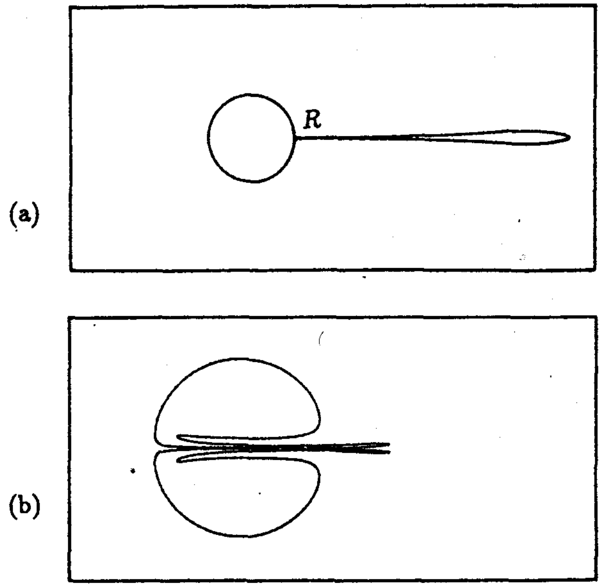

Fig. 8. Nonlinear instability of Hill's vortex. (a) when the initial perturbation of the spherical surface of the vortex is prolate, it sheds a spike from the rear stagnation point R; (b) when the perturbation is oblate, the spike is intrusive, and an irrotational hole is 'drilled' through the vortex. [From Pozrikidis, 1986.]

3.3. VORTONS

We use the word 'vorton' in the sense of Moffatt (1988) to mean a nonlinear rotational disturbance which propagates with constant velocity $\mathbf{U}$ and without change of structure in a fluid of infinite extent. Thus the vorticity field of a vorton has the form

$$\omega(\mathbf{x},t) = \mathbf{f}(\mathbf{x} - \mathbf{U}t), \tag{3.18}$$

where $\mathbf{f}(\xi)$ is a solenoidal field of compact support in $\mathbf{R}^3$. Hill's vortex mentioned above is a classical example of a vorton; further examples are provided by the wide family of spherical vortices with swirl discovered by Hicks (1899).

In a frame of reference moving with this vorton (*i.e.* with velocity $\mathbf{U}$ relative to the fluid at infinity), the flow is steady. We may therefore seek to determine the complete family of vortons by the technique of magnetic relaxation, starting with an initial field of the form

$$\mathbf{B}_0(\mathbf{x}) = \mathbf{B}_{00} + \mathbf{b}_0(\mathbf{x}), \tag{3.19}$$

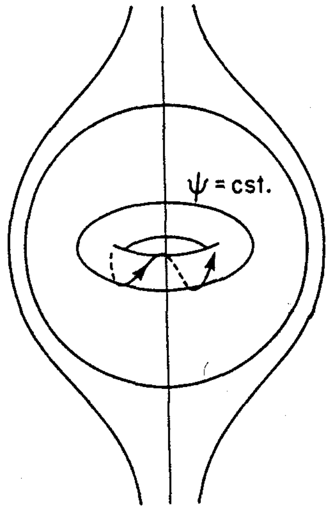

Fig. 9. Axisymmetric vorton topology. The stream-surfaces ψ = cst. are a family of nested tori, and the signature of the flow consists of the pair $\{V(\psi), W(\psi)\}$ where $V(\psi)$ is the volume of the torus with label ψ and $W(\psi)$ is the azimuthal flux within this torus. [From Moffatt, 1988.]

where $\mathbf{B}_{00}$ is uniform (and subsequently to be identified with the velocity $-\mathbf{U}$ in the analogous Euler flow) and $\mathbf{b}(\mathbf{x})$ is a field of compact support of sufficient strength to guarantee the existence of a region $\mathcal{D}$ such that $\mathbf{n}\cdot\mathbf{B}_0$ on $\partial\mathcal{D}$. Inside $\mathcal{D}$, the $\mathbf{B}_0$-lines are either closed curves, or they lie on surfaces (*e.g.* nested tori) or they exhibit chaotic wandering. These (topological) properties are invariant during the relaxation process.

The simplest situation, and probably the most important, is that in which $\mathbf{B}_0(\mathbf{x})$ is axisymmetric. Let $\mathbf{B}_0$ have the poloidal-toroidal decomposition

$$\mathbf{B}_0 = \mathbf{B}_P + \mathbf{B}_T, \tag{3.20}$$

where (in cylindrical polar coordinates (r, ϕ, z))

$$\mathbf{B}_T = (0, B(r,z), 0), \quad \mathbf{B}_P = \left(\frac{1}{r}\frac{\partial\psi}{\partial z}, 0, -\frac{1}{r}\frac{\partial\psi}{\partial r}\right). \tag{3.21}$$

The surfaces ψ = cst. are a family of nested tori within $\mathcal{D}$. Under magnetic relaxation, the volume $V(\psi)$ inside each such torus and the azimuthal flux of field $W(\psi)$ within each such torus are conserved (figure 9). The field relaxes to a magnetostatic equilibrium characterised by the signature $\{V(\psi), W(\psi)\}$, and the analogous Euler flow (or vorton) is likewise characterised by this signature.

A very special situation arises if $\mathbf{B}_{00} = 0$, in which case the analogous Euler flows consists of vortons which do not propagate (*i.e.* $\mathbf{U} = 0$), but persist as stationary rotational 'excitations' in the fluid. Examples of such fields are given in a separate paper in this volume (Chui and Moffatt, 1992).

4. Knots and Links

4.1. THE ENERGY OF A KNOT

The magnetic relaxation procedure lends itself in an interesting way to the theory of knots and links in $\mathbf{R}^3$, and yields the intriguing result that every knot (or link) is characterised by an intrinsic minimum energy which is determined solely by the knot topology (Moffatt, 1990a; Freedman and He, 1991). With each knot K, we associate a magnetic flux tube of volume V carrying flux Φ and such that the total helicity of the field is $\mathcal{H}_M = h\Phi^2$ (see Ricca and Moffatt, 1992, this volume). If this field is allowed to relax to a minimum energy state, then this minimum energy is necessarily given by

$$M_{\min} = m(h)\Phi^2 V^{-1/3}, \tag{4.1}$$

where $m(h)$ is a dimensionless function of the dimensionless variable h (related to the twist of the field within the flux tube). When h is of order unity, the relaxation proceeds essentially through contraction of the axis of the flux tube and expansion of the mean cross-section (thus keeping V constant). The process comes to a halt when the knot 'tightens' on itself (as illustrated for the trefoil knot in figure 10). At this stage, the axial length L is of order $V^{1/3}$ and the mean tube cross-section A is of order $V^{2/3}$.

When $|h| \gg 1$, the opposite occurs! The tube cross-section decreases due to the strong twist component of field around the tube axis; in the case of the unknot (see Chui and Moffatt, 1992), axisymmetric equilibrium is attained when

$$L \sim |h|^{2/3} V^{1/3} \quad , \quad A \sim |h|^{-2/3} V^{2/3}, \tag{4.2}$$

and it seems likely that the same result holds for an arbitrary knot. In this case, the energy function $m(h)$ satisfies

$$m(h) = O(|h|^{4/3}) \quad \text{for} \quad |h| \gg 1. \tag{4.3}$$

For an arbitrary knot, we may then anticipate that the energy function has the form sketched in figure 11, with symmetry about the axis $h = 0$ only for achiral knots. The curve must have a minimum value, m_c say, attained for $h = h_c$, and the real number m_c is determined solely by the knot type. Similar considerations of course apply to links.

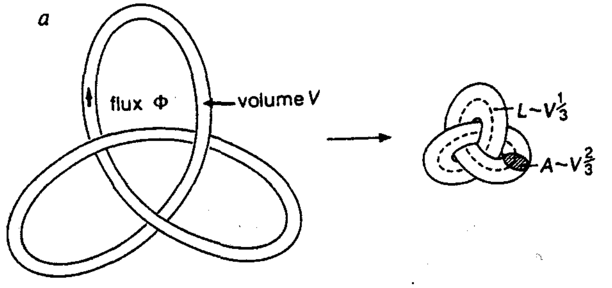

Fig. 10. Relaxation of a flux tube in the form of the trefoil knot. The minimum energy has the form $m(h)\Phi^2 V^{-1/3}$, where Φ and V are the (conserved) flux and volume, and $h = \mathcal{H}_M/\Phi^2$ where $\mathcal{H}_M$ is the conserved helicity. The parameter h can take any value depending on the twist of the field within the tube. The dimensionless function $m(h)$ is determined (in principle) by the topology of the knot. [From Moffatt, 1990].

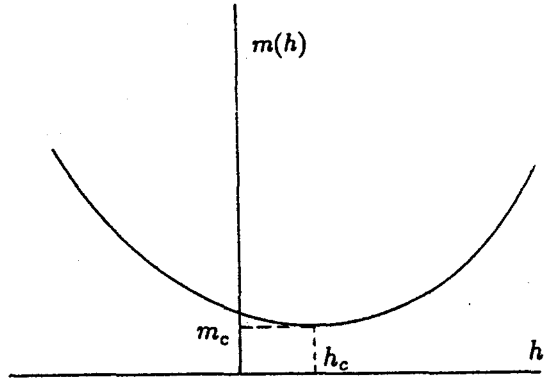

Fig. 11. Expected qualitative form of the energy function $m(h)$ for an arbitrary knot.

The fact that a knot is characterised by a minimum energy, or more correctly a minimum energy function $m(h)$ dependent on the twist (or 'framing') parameter h, is of potential importance in such contexts as polymer physics and molecular biology, where knotted structures may be expected to seek configurations of minimum energy, in the absence of external perturbations. The energy functional $M = \frac{1}{2}\int \mathbf{B}^2 dV$ that we have chosen may however be too specialized for application in these contexts; we adopt a more general approach in the following sub-section.

4.2. ALTERNATIVE ENERGY FUNCTIONALS

The mathematics of the problem is here served by choosing a physical model which is unrealistic but – and this is what matters – self-consistent. Suppose that our conducting fluid is also endowed with magnetic permeability *via* a relationship of the form

$$\mathbf{H} = \mathbf{H}(\mathbf{B}) \quad \text{where} \quad \mathbf{j} = \operatorname{curl} \mathbf{H}. \tag{4.4}$$

The field $\mathbf{B}$ still satisfies the induction equation (2.1) and the Lorentz force is still given by $\mathbf{j} \times \mathbf{B}$. The only change in obtaining the energy equation (2.23) which lies at the heart of the relaxation process is that the magnetic energy is now replaced by

$$M(t) = \int dV \int \mathbf{H} \cdot d\mathbf{B} \tag{4.5}$$

(which yields (2.13) when $\mathbf{B} = \mathbf{H}$). If, for example, $\mathbf{H} = |\mathbf{B}|^{\lambda}\mathbf{B}$ for $\lambda > -1$, then

$$M(t) = \frac{1}{\lambda + 2} \int |\mathbf{B}|^{\lambda+2} dV. \tag{4.6}$$

In this case, relaxation of a knotted flux tube leads to a minimum energy which, on dimensional grounds, has the form

$$M_{\min} = m(h)\Phi^{\lambda+2}V^{-(1+2\lambda)/3}. \tag{4.7}$$

The choice $\lambda = -\frac{1}{2}$ is special in that $M_{\min}$ is then independent of V, a fact exploited by Freedman and He (1991), who have proved the striking result that $\int |\mathbf{B}|^{3/2} dV$ is bounded below by the least value of the crossing number of the knot (*i.e.* the average over all projections of the number of (unsigned) crossings).

5. Relaxation Using Artificial Dynamics

A technique has been suggested by Vallis, Carnevale and Young (1989) whereby relaxation to steady Euler flows by a process that conserves *vorticity* topology may be forced by the introduction of 'artificial' dynamics. The dynamics *has* to be artificial if energy is to be dissipated while the vorticity field remains frozen in the fluid.

Let us here attempt to construct such a process by appropriate adaptation of the magnetic relaxation procedure. Let $\mathbf{j} = \operatorname{curl} \mathbf{B}$, and suppose that $\mathbf{j}$ evolves according to the frozen field equation

$$\frac{\partial \mathbf{j}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{j}), \tag{5.1}$$

so that

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{v} \times \mathbf{j} - \nabla \phi = (\mathbf{v} \times \mathbf{j})_s \tag{5.2}$$

for some scalar field ϕ. This is an artificial evolution equation, which we may however couple with the same choice of velocity as before,

$$k\mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla p = (\mathbf{j} \times \mathbf{B})_s, \tag{5.3}$$

which yields an energy equation for $M = \frac{1}{2}\int \mathbf{B}^2 dV$,

$$\frac{dM}{dt} = \int \mathbf{B} \cdot (\mathbf{v} \times \mathbf{j}) dV = -k \int \mathbf{v}^2 dV. \tag{5.4}$$

If either a lower bound or an upper bound can be placed on M, then (choosing $k > 0$ or $k < 0$ respectively) we are driven to a steady state in which $\mathbf{v} = 0$ and $\mathbf{j} \times \mathbf{B} = \nabla p$.

Now make the replacements $\mathbf{B} \to \mathbf{u}, \mathbf{j} \to \omega$, so that (5.2) and (5.3) become

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{k}\left[(\omega \times \mathbf{u})_s \times \omega\right]_s, \tag{5.5}$$

where the suffix s again represents solenoidal projection. With this equation, we associate the energy equation (the exact counterpart of (5.4))

$$\frac{d}{dt}\frac{1}{2}\int \mathbf{u}^2 dV = -k^{-1}\int (\omega \times \mathbf{u})_s^2 dV. \tag{5.6}$$

Thus, provided a bound on the energy exists, the system is driven to a steady Euler flow for which $(\mathbf{u} \times \omega)_s = 0$. Moreover, the topology of ω (the analogue of $\mathbf{j}$) is conserved during this relaxation, *i.e.* the relaxation occurs on an isovortical folium, so that the final state, being an energy extremum on the isovortical folium, may be expected to be stable, by Arnold's criterion.

Unfortunately, everything hinges on the existence of either a lower or an upper bound on the energy. Recall that previously a lower bound was constructed through a product of Schwarz and Poincaré inequalities. Now however these inequalities are

$$\int \mathbf{B}^2 \int \mathbf{j}^2 \geq \left(\int \mathbf{B} \cdot \mathbf{j}\right)^2 = \text{cst.} \quad , \quad \int \mathbf{j}^2 \Big/ \int \mathbf{B}^2 \geq q_0^2, \tag{5.7}$$

and these cannot be combined to give a lower bound on $\int \mathbf{B}^2 dV$. There is therefore no guarantee that evolution determined by (5.5) will lead to a steady state. If $k > 0$, the energy may decrease to zero, the vorticity field becoming more and more complex although its topology is conserved, while if $k < 0$, the energy may increase to infinity. This difficulty was encountered equally by Vallis *et al.* (1989) within the framework of the particular artificial dynamics that they adopted.

An upper bound can be placed on the energy however when the flow is either two-dimensional (Vallis *et al.*) or axisymmetric without swirl (Moffatt, 1990b). In the two-dimensional case, enstrophy is conserved by (5.5) and the energy has an upper bound (*cf* 5.7b) proportional to this constant enstrophy. The situation for axisymmetric flow with swirl is similar. In these cases, the artificial evolution equation (5.5), with $k < 0$, must drive the system to a *stable* Euler flow for which the topology of the vorticity field is prescribed in advance. What does this mean for the two-dimensional case in which the vortex lines are all parallel to the z-axis? There is still a topology associated with the family of plane curves $\omega(x, y) = \text{cst}$. These isovorticity curves move with the fluid during the evolution process (5.5), and the area $A(\omega)$ inside any such curve is conserved. This is now the appropriate signature which fully describes the topology of the vorticity field, and which characterises the Euler flow that the system approaches as $t \to \infty$.

Since the energy increases during this type of evolution, it is perhaps a misuse of words to describe it as a *relaxation* process; it has more the character of a *spin-up* process which reaches stable equilibrium when the energy is maximal within the family of flows of prescribed signature function $A(\omega)$.

Acknowledgements

I am grateful to Andrew Gilbert who helped to develop the formulation of the 'artificial' relaxation process described in section 5, and who first recognized its applicability to the two-dimensional situation.

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