Fluid Mechanics, Topology, Cusp Singularities, and Related Matters

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Abstract

This lecture provides an informal discussion of 1) the role of the helicity invariant in relation to the Euler equations of classical fluid mechanics, and the manner in which an understanding of this invariant may be used to obtain steady solutions of the Euler equations of arbitrarily complex streamline topology, and 2) the recently discovered phenomenon of free surface cusps in flows at low Reynolds numbers and the potential relevance of the exact cusp solution to the famous problem of the moving contact line. The lecture concludes with some comments on the role of the computer in relation to both types of investigation.

1. Introduction

In this lecture, I shall discuss two contrasting areas of work in theoretical fluid mechanics in both of which I have been involved over the last 30 years. The first concerns the role of *topological invariants* in certain problems that arise both in the magnetohydrodynamics (MHD) of perfectly conducting fluids and in relation to the Euler equations of classical fluid mechanics; in this area it is global properties of flow and/or magnetic fields with which we are primarily concerned, and in particular the topological structure of these fields. I shall try to indicate some of the fascinating problems that arise at the interface between fluid mechanics and topology, an interface where bridges are already being built, and where I expect to see important developments over the next decade.

The second area relates to Stokes flows in a highly viscous fluid, and in particular to the nature of *singularities* in these flows and the manner in which these singularities dictate the flow structure. Here, the emphasis is obviously on *local*, as opposed to *global*, behaviour; and yet I would still propose to categorise this type of work as topological in spirit, because an understanding of singularities in any flow field is an essential preliminary to development of an understanding of global structure. Incidentally it frequently happens that singularities of the Stokes equations are also singularities of the parent *Navier-Stokes equations*, so that the results may be of wide generality.

In the pedagogical and forward-looking spirit of this *Seminaire International*, I shall emphasise three aspects of the type of work I propose to describe: analogy, generality and simplicity. Analogy, sometimes perfect and sometimes only partial, can provide a powerful weapon in transferring techniques developed in one field to an entirely different field to which the application of these techniques may not have been *a priori* obvious. Effective use of this weapon presupposes a certain familiarity with both fields, and the weapon will simply not be available to anyone who specialises in one narrow field at a too-early stage of his or her career; conversely, the broader the platform of scientific education, the more likely it is that a researcher will be able to make the mental leaps, often through inspired analogy, that characterises truly creative research.

Secondly, generality: by this, I mean the ability to strip away 'irrelevant' features of a given problem; to focus on the essential features without which there would simply be no problem; and to recognize the general circumstances in which these essential features arise.

Thirdly, simplicity: and here I mean the ability not only to solve a problem in mathematical terms, but then to struggle to understand the meaning of the solution in physical terms, and to simplify the presentation of the mathematical solution to the point at which its correspondence with physical reality is so clear that the mathematics can really be said to illumine (rather than, as so often the case, to obscure) the physics.

2. The role of topological invariants

The intrusion of topological ideas in fluid mechanics goes back to the seminal work of Helmholtz (1858) and Kelvin (1869) who established that in inviscid flow governed by the Euler equation, 'vortex lines move with the fluid', or as we might now say are 'frozen in the fluid'. Kelvin recognized that any links or knots in a vortex tube would persist as the flow evolved but it was not until nearly 100 years later, and after the development of appropriate techniques in MHD, that this rather vague concept of 'conservation of knottedness' could be put on a firm mathematical foundation.

In MHD, it is the lines of force of the magnetic field $\mathbf{B}(\mathbf{x}, t)$ which, in the perfect conductivity limit, are 'frozen in the fluid'. In 1958, the American astrophysicist Woltjer proved that if $\mathbf{A}(\mathbf{x}, t)$ is a vector potential for $\mathbf{B}$, i.e. $\mathbf{B} = \text{curl } \mathbf{A}$, and if $\mathbf{B}.\mathbf{n} = 0$ on the boundary of the domain-considered, then the quantity now known as *magnetic helicity*, viz

$$\mathcal{H}_M = \int \mathbf{A}.\mathbf{B} \, dV \tag{2.1}$$

is invariant. He then proved that minimisation of magnetic energy subject to the single constraint $\mathcal{H}_M$ = const. led to magnetic fields satisfying the Beltrami (or force-free) condition

$$\text{curl}\mathbf{B} = \alpha \mathbf{B} \quad , \tag{2.2}$$

where α is a constant proportional to $\mathcal{H}_M$. (Note that both $\mathcal{H}_M$ and α are *pseudo-scalar* quantities: they change sign under parity transformations from right-handed to left-handed frame of reference.)

In 1961, J.-J. Moreau, who I am glad to see is a leading participant in this seminar, discovered an analogous result for the helicity

$$\mathcal{H} = \int \mathbf{u}.\boldsymbol{\omega}\, dV \quad , \tag{2.3}$$

in an ideal fluid, where $\mathbf{u}(\mathbf{x}, t)$ is the velocity field and $\boldsymbol{\omega}(\mathbf{x}, t) = \text{curl}\ \mathbf{u}$ is the vorticity field. I would be very interested to know by what process of reasoning Moreau was led to consider this quantity – I don't think he was aware of the analogy with MHD, although he was clearly aware of the topological significance of the result; indeed, in a pregnant footnote, Moreau evaluates $\mathcal{H}$ for two vortex tubes of circulations K_1, K_2, and shows that $\mathcal{H} = \pm 2K_1K_2$ or 0 according as the tubes are linked or not – clearly a topological criterion.

For many years, I was not aware of the work of Moreau, published in a brief communication of the Comptes Rendus de l'Académie des Sciences – and I suspect that I was not alone in this state of ignorance. Throughout the 1960's, I lectured to Cambridge students on magnetohydrodynamics, and each year I presented Woltjer's result; each year, equally, I struggled to understand its meaning (my enthusiasm for the fruitful interplay of teaching and research dates from this period!) and eventually I independently hit upon the 'two linked flux tubes' argument: if $\mathbf{B}$ is confined to two flux tubes containing fluxes Φ_1, Φ_2, then $\mathcal{H}_M$ may be evaluated in the form

$$\mathcal{H}_M = \pm 2n\Phi_1\Phi_2 \quad , \tag{2.4}$$

where n is the (Gauss) linking number of the two tubes, a number that is clearly conserved under frozen field evolution. It was only then that I realised that, since the invariance of $\mathcal{H}_M$ is a consequence of the frozen-in character of the $\mathbf{B}$-field, there must be an analogous invariant in relation to the vorticity field of ideal fluid mechanics.

It is worth lingering a little over the nature of this analogy. The frozen-in (induction) equation for $\mathbf{B}$ is

$$\frac{\partial \mathbf{B}}{\partial t} = \text{curl } (\mathbf{u} \wedge \mathbf{B}) \quad , \tag{2.5}$$

and the vorticity equation for (ideal) flow is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \text{curl } (\mathbf{u} \wedge \boldsymbol{\omega}) \quad , \tag{2.6}$$

where, in addition, $\boldsymbol{\omega} = \text{curl } \mathbf{u}$. Thus (2.5) is a *linear* equation for the evolution of $\mathbf{B}$ (if $\mathbf{u}$ is given), whereas (2.6) is a *nonlinear* equation for the evolution of $\boldsymbol{\omega}$ (and hence of $\mathbf{u}$). The analogy is clearly at best partial; nevertheless it is sufficient, because in the proof of conservation of helicity $\mathcal{H}$, no use is made of the fact that $\boldsymbol{\omega} = \text{curl } \mathbf{u}$. Equation (2.5), in which $\mathbf{B}$ is unrestricted by any such instantaneous dependence on $\mathbf{u}$, is more general than (2.6), and in going from (2.5) to (2.6), we go from the general to the particular; the fact that we embrace a nonlinear system in the process is a happy accident. If $\boldsymbol{\omega}$ were related to $\mathbf{u}$ by a different equation, e.g. $\boldsymbol{\omega} = \text{curl } \mathbf{F}(\mathbf{u})$, then we would be equally fortunate!

In 1974, Arnol'd extended the 'flux-tube' argument to embrace fields whose field lines are not simple closed curves, and identified helicity as the 'asymptotic Hopf invariant' – i.e. asymptotic linking number – for such fields, thus reinforcing the topological interpretation of Moffatt (1969). In the same paper (which incidentally was published in Russian in the proceedings of a Summer School of the Armenian Academy of Sciences, and was for more than a decade completely inaccessible in the West), Arnol'd posed the 'magnetic relaxation' problem: what happens to a magnetic field evolving according to the frozen-field equation (2.5) in a fluid medium in which energy is dissipated by viscosity? Arnol'd showed that when $\mathcal{H}_M \neq 0$, the magnetic energy (and so also the total energy) of the system has a positive lower bound. It follows that the magnetic energy must tend to a constant as $t \to \infty$, and that this must be a local minimum of energy. Now it is known that minimum energy states are stable solutions of the *magnetostatic equations*

$$\mathbf{j} \wedge \mathbf{B} = \nabla p \quad , \quad \mathbf{j} = \text{curl } \mathbf{B} \quad , \tag{2.7}$$

provided the current $\mathbf{j}$ and field $\mathbf{B}$ are everywhere finite. Isolated current sheets (tangential discontinuities of $\mathbf{B}$) can easily be incorporated, but Arnol'd hinted that more unpleasant singularities might in fact form during the relaxation process – a possibility that still presents a challenge for functional analysts: the question is, given that $\mathbf{B}(\mathbf{x}, 0)$ is square-integrable and reasonably smooth, what is the natural function space within which $\mathbf{B}(\mathbf{x}, t)$ remains for all t and in the limit $t \to \infty$? I hope that this problem will be solved within the next decade, but I suspect it will take longer!

Why is this important? Here there is a second analogy which brings us back to the fundamental problem of the Euler equations, and this time the analogy is perfect: steady solutions of the Euler equations for incompressible flow satisfy the equations

$$\mathbf{u} \wedge \boldsymbol{\omega} = \nabla h \quad , \quad \boldsymbol{\omega} = \text{curl } \mathbf{u} \quad , \tag{2.8}$$

where $h = p/\rho + \frac{1}{2}\mathbf{u}^2$, with p pressure and ρ density. The analogous variables are here

$$\mathbf{u} \leftrightarrow \mathbf{B} \, , \quad \boldsymbol{\omega} \leftrightarrow \mathbf{j} \quad , \quad h \leftrightarrow p_0 - p \quad , \tag{2.9}$$

where p_0 is an arbitrary constant. This means that if, by any means, we can find a solution of (2.7), then via the replacements (2.9), we have simultaneously found a solution of (2.8). Now the method of magnetic relaxation implies the existence of solutions (possibly 'generalised' solutions) of (2.7) of arbitrary topological structure (since the topology of $\mathbf{B}$, which is conserved, may be arbitrarily prescribed at time $t = 0$). Hence (Moffatt 1985) we may infer the existence of steady Euler flows of arbitrary streamline topology – an extraordinarily wide and rich family!

These steady Euler flows may be regarded as the fixed points in the function space in which unsteady solutions of the Euler equations evolve. For any dynamical system, it is desirable first to locate the fixed points, then to consider their stability. Even if unstable, they provide vital information about the structure of flow trajectories, and indeed a time-average of a highly energetic system which spends long periods in the neighbourhood of unstable fixed points will carry the imprint of these unstable points. It is therefore of great fundamental interest to have a means of locating solutions of (2.8) whether stable or unstable; the method of magnetic relaxation coupled with the analogy (2.9) provides such a means.

It will be evident that this sort of insight into the Euler equations could never have been achieved without the help of magnetohydrodynamics. Indeed, although the Euler equations have been with us for well over 200 years, and the frozen-in character of vorticity has been recognized for well over 100 years, yet it is only in the past decade, with the help of techniques imported from magnetohydrodynamics, that we have begun to recognize the incredibly rich structure of these Euler equations, even as regards their steady solutions.

The analogy (2.9) does not extend to time-dependent perturbations about the (analogous) equilibrium states, the reason being that perturbations of the system (2.7) evolve on an 'isomagnetic' manifold on which the topology of the **B**-field is invariant, whereas perturbations of the system (2.8) evolve on an 'isovortical' manifold on which the topology of the $\boldsymbol{\omega}$-field is invariant. Thus to some extent, the two analogies ($\mathbf{B} \leftrightarrow \boldsymbol{\omega}$ and $\mathbf{B} \leftrightarrow \mathbf{u}$) are brought into conflict in the stability problem (Moffatt 1986). This is why solutions of the magnetostatic problems that are stable do not necessarily yield (via the analogy (2.9)) stable solutions of the Euler equations; indeed the indications are that Euler flows of any significant three-dimensional complexity obtained via the method of magnetic relaxation are invariably unstable; for the reasons indicated above, this does not detract from their significance.

The minimum energy state takes a particularly simple form when the initial magnetic field is confined to a single magnetic flux tube knotted in the form of an arbitrary knot K. The field within the knot may have an arbitrary twist h about the axis of the tube, and the associated helicity of the field is then

$$\mathcal{H}_M = h\Phi^2 \quad . \tag{2.10}$$

If h is rational, then the **B**-lines are closed within the knot tube, being torus knot satellites of K. If this field is allowed to relax to a minimum energy state, the fluid being assumed incompressible ($\nabla.\mathbf{u} = 0$), then on dimensional grounds the minimum energy is given by

$$M_{\min} = m(h)\Phi^2 V^{-1/3} \quad , \tag{2.11}$$

where Φ is the (conserved) flux in the tube and V is its (conserved) volume. The function $m(h)$ is determined (in principle) by the topology of the knot K, different knots being characterised by different minimum ('ground state') energy functions (Moffatt 1990). Freedman

& He (1991) have succeeded in placing a lower bound on M_{min} (using a 'three-halves' energy function $|\mathbf{B}|^{3/2}$ rather than the usual $|\mathbf{B}|^2$) in terms of the minimum 'crossing number' of K – thus providing a first bridge between knot energy and more conventional measures of knot complexity.

There is great potential interest in the concept of 'knot energy' which arises in many disparate fields, ranging from molecular biology (structure of DNA) to cosmology (the structure of cosmic strings). Fluid mechanics provides a natural 'physical' setting for the abstract study of knots, the 'isotopies' of topologists being simply the flow fields and associated particle paths familiar to fluid dynamicists. Use of different vocabularies has for too long obscured the natural affinity between these two fields of study!

There is one further intriguing link between the concept of helicity, and invariants familiar to knot theoreticians. If the flux tube considered above is allowed to shrink onto the knot curve C, then one may reasonably ask: what is the asymptotic form of the helicity integral? The answer is that

$$\mathcal{H}_M/\Phi^2 = \mathcal{W} + \mathcal{T} + \mathcal{N} \quad , \tag{2.12}$$

where $\mathcal{W}$ is the *writhe* of C, $\mathcal{T}$ is the total *torsion* of C divided by 2π, and $\mathcal{N}$ is the intrinsic *twist* of the field $\mathbf{B}$ within the knot tube. If the $\mathbf{B}$-lines are closed after one passage around C, then $\mathcal{N}$ and $(\mathcal{W} + \mathcal{T})$ are both integers which are invariant under deformations of C which do not pass through any inflexional configuration; but if C does pass through an inflexion, then $\mathcal{W} + \mathcal{T}$ jumps by ± 1, and $\mathcal{N}$ has a compensating jump ∓ 1. This type of behaviour was perceived by Călugăreanu (1961), but has only recently been fully clarified (Moffatt & Ricca 1992). The discontinuity can be realised physically as illustrated in figure 1.

3. Singularities of Stokes flows

Stokes flows are slow viscous flows for which inertia forces are (in a first approximation) negligible. They are described by the Stokes equation (the low Reynolds number limit of the Navier-Stokes equation)

$$\mu\nabla^2\mathbf{u} = \nabla p \quad , \tag{3.1}$$

where p is the pressure field and μ the coefficient of viscosity. It follows that $\nabla^2\boldsymbol{\omega} = 0$ where $\boldsymbol{\omega}$ is, as before, the vorticity. In particular, for two-dimensional flows with $\boldsymbol{\omega} = (0, 0, -\nabla^2\psi)$, the stream function $\psi(x, y)$ satisfies the biharmonic equation

$$\nabla^4\psi = 0 \quad . \tag{3.2}$$

There is here an analogy, which has proved useful with problems in elasticity in which ψ becomes the Airy stress function. As shown first by Muskhelishvili (1953) in the elasticity context, solutions of the biharmonic equation can be expressed in terms of two functions $f(z)$ and $g(z)$ of the complex variable $z = x + iy$, in the form

$$\psi = \mathrm{Im}(f(z) + \bar{z}g(z)) \quad , \tag{3.3}$$

where $\bar{z} = x - iy$. Richardson (1968) demonstrated the potential usefulness of this representation for slow viscous flow with a free surface Γ, in which nonlinearity enters through the free surface boundary conditions. Using dimensionless variables, these boundary conditions reduce to

$$\left.\begin{aligned} f(z) + \bar{z}g(z) &= 0 \\ \text{and} \quad \mathrm{Im}[g(z)\overline{(dz/ds)}] &= (4C)^{-1} \end{aligned}\right\} \quad , \tag{3.4}$$

where C is the capillary number (essentially the ratio of viscous forces to forces induced by surface tension).

Even for the simpler (linear) problems involving rigid boundaries, either fixed or moving in a prescribed manner, there are paradoxical situations whose resolution has usually involved decades of endeavour. Among these, the Stokes paradox is perhaps pre-eminent: the 'Stokeslet' flow described by the stream function

$$\psi = \frac{F}{4\pi\mu} r \ln r \sin\theta \quad , \tag{3.5}$$

(in plane polar coordinates (r, θ)) corresponds to the application of a point force F (really a line force in three-dimensions) at $r = 0$. The corresponding velocity components are proportional to $\ln r$, and there is a logarithmic divergence not only at $r = 0$ (where it might be expected) but also at $r = \infty$ (where everything should be under control). The paradox is resolved by regarding the Stokeslet (3.5) as the first term of an 'inner' asymptotic expansion, matched in a suitable way term-by-term with an outer (Oseen) expansion.

Flows bounded by a sharp corner can yield comparable paradoxes. For example, if one plane boundary is 'scraped' over another at constant angle α and constant speed U (figure 2a) then as shown by Taylor (1960) the appropriate local similarity solution has the form

$$\psi = Urf(\theta) \quad , \tag{3.6}$$

for a function f which is easily determined. This is the first term of a regular perturbation series solution of the full Navier-Stokes equation having a finite radius of convergence (Hancock et al 1981). Here the velocity components are of order U, but the stress components are $O(r^{-1})$ and the total (integrated) tangential force on the moving boundary has a logarithmic divergence, clearly unphysical. A similar divergence occurs for the problem of a plate pushed through the free surface of a viscous fluid (figure 2b): on the (untenable) assumption that the free surface remains horizontal the force required to impel the plate downwards would be infinite; hence the frequently quoted remark that not even Hercules could (as alleged) have dipped his arrows in the envenomed blood of the Hydra without truly superhuman strength!

But of course, the free surface does not remain horizontal, but is drawn downwards (figure 2c) in some neighbourhood of the contact line through the agency of viscosity. We shall see below that this effect converts the r^{-1} stress singularity to a milder (integrable) $r^{-1/2}$ singularity so that Hercules could after all have achieved the apparently impossible!

This proposed resolution of what may be described as the Herculean paradox derives from an exact solution of the Stokes equations recently obtained (Jeong & Moffatt 1992). This solution, obtained through the complex variable representation (3.3) and the use of a conformal mapping of the flow domain to the interior of the unit circle, describes a flow involving a *cusp singularity* on a free surface; specifically it is the flow due to a vortex dipole of strength α at distance d below the asymptotic level of the free surface (figure 3a). The brilliant experiments of Joseph et al (1991) demonstrated the extraordinarily razor-sharp character of the cusps that can appear on a free surface when (as in this model problem) the streamlines on the free surface converge towards a stagnation line.

Formation of the cusp involves a battle between viscosity and surface tension. The following simple argument (due to John Hinch) indicates why the apparent cusp forms despite the smoothing effect that is usually associated with surface tension. Flow near the stagnation point (figure 3b) is in part due to a (virtual) point force 2γ upwards located roughly at the centre of curvature of the free surface at the point of symmetry, and in part to a downward velocity U due to the remote forcing. The upward velocity on the plane of symmetry due to the Stokeslet is obtained from (3.5) with $F = 2\gamma$:

$$u = \frac{\gamma}{2\pi\mu}\ln(r_0/r) \tag{3.7}$$

for some r_0 and this balances U at $r = R$ where

$$\frac{R}{r_0} = \exp\{-2\pi\mu U/\gamma\} \quad . \tag{3.8}$$

The dimensional quantities r_0, U are determined by the global geometry and the driving mechanism for the flow. For the model problem of figure 3a, the only possibilities are

$$r_0 = c_1 d \quad , \quad U = c_2\alpha/d^2 \quad , \tag{3.9}$$

where c_1 and c_2 are dimensionless constants of order unity. Hence (3.8) becomes

$$\frac{R}{d} = c_1 \exp\{-2\pi c_2 C\} \quad , \tag{3.10}$$

where $C = \mu\alpha/\gamma d^2$ is the capillary number (the *a priori* estimate of the ratio of viscous forces to surface tension forces). The exact solution of the problem in fact locates the 'cusp' at $(0, -\frac{2}{3}d)$ and yields $c_1 = 256/3$, $c_2 = 16$. Hence

$$\frac{R}{d} = \frac{256}{3}\exp\{-32\pi C\} \quad . \tag{3.11}$$

This is a rather extraordinary result, because even when C is of order unity, it gives an extremely small value for R/d, so small indeed that the continuum approximation must clearly break down in the neighbourhood of the cusp. What resolves the incipient singularity is not surface tension in its continuum manifestation but rather the actual structure of the liquid-air interface determined by intermolecular forces, or possibly dynamic effects associated with air flow above the cusp induced by the no-slip condition.

When $C = 1$, the formula (3.11) gives $R/d = 1.87 \times 10^{-42}$. From a purely mathematical point of view, it is remarkable that such a small number should emerge from a problem whose statement as a nonlinear boundary value problem itself involves no small parameters. Here, allow me to draw attention to Feynman's (1963) thought-provoking discussion of the extremely small ratio of the gravitational attraction to the electrical repulsion of two electrons (one divided by 4.17×10^{42}). Feynman writes: "Where could such a tremendous number come from? Some say that we shall one day find the 'universal equation', and in it, one of the roots will be this number. It is very difficult to find an equation for which such a fantastic number is a natural root." Well, I don't of course wish to suggest that cusp singularities have any implications for a unified field theory; but merely to point out that huge numbers can (and do) emerge from certain nonlinear boundary-value problems arising in very classical fluid-dynamical contexts.

Returning now to the Herculean paradox (figure 2c), the flow in a neighbourhood of the contact line looks very similar to the cusp flow if we simply place a vertical plate on the plane of symmetry and move it downwards with the velocity at the cusp (actually $16\alpha/d^2$) derived from the exact cusp solution. All the conditions of the problem are then satisfied: the flow satisfies the biharmonic equation, and the required conditions on the free surface and on the vertical plate (at least locally) are satisfied. The cusp solution (Jeong & Moffatt 1992) gives a velocity of order $r^{1/2}$ and stress components of order $r^{-1/2}$ (except in a neighbourhood of the free surface stagnation point which is so small that it can presumably be neglected); hence the tangential stress on the plate is integrable, and the force required to impel it downwards is finite and of order μU (per unit length in the cusp direction), independent of capillary number provided this is of order unity or greater.

Thus, as claimed above, the Herculean paradox is resolved.

4. Discussion

I would like to add some comments concerning the role of the computer in relation to problems of the type discussed above. First, the method of magnetic relaxation (Moffatt 1985) provides a procedure by which Euler flows of prescribed streamline topology may (in principle) be computationally determined. However, in practice, the procedure, involving a three-dimensional time-dependent evolution of coupled vector fields, one of which (the magnetic field) is not subject to diffusion, is beyond the limits of current computer power. Progress is possible however for axisymmetric configurations (Chui & Moffatt 1992) and for two-dimensional configuration (Bajer 1989, Linardatos 1992), for which a variety of new nonlinear steady Euler flows have been found via the magnetic relaxation technique.

It is interesting to note that, although the theoretical arguments leading to the knot energy formula (2.11) are quite simple, the actual computational determination of $m(h)$ for any given nontrivial knot is likely to be a difficult matter that may have to wait for the next (or next-but-one!) generation of computers.

In relation to problems involving singularities, whether temporal or spatial, the limitations of computers are well-known. The free-surface cusp problem discussed in §3 provides a good example: it would be exceedingly difficult for a computer using either finite differences or finite elements to discover a singular behaviour of the type revealed by the formula (3.10) (an attempt using finite elements is described by Joseph et al 1991). It is a different matter when the investigator knows in advance the nature of the structure that is the object of investigation; for then, as for example in the work of Collins & Dennis (1976), repeated rescaling of length and velocity by appropriate amounts can certainly permit computational verification of a behaviour predicted theoretically (Moffatt 1964). In more complicated situations (e.g. investigation of finite-time singularities of the unsteady Euler equations) creative interaction between theoretical argument and computational experimentation (as for example in Pumir & Siggia 1992) is essential in the tentative and halting steps that characterise progress at the fundamental level.

Figure captions

Figure 1 Writhe, torsion and twist; writhe is continuously converted to torsion, and torsion (of the centreline) is discontinuously converted to twist of the ribbon about its centreline (Moffatt & Ricca 1992).

Figure 2 a) The paint-scraper problem (Taylor 1960) which requires an infinite tangential force on the moving plate; b) the Herculean paradox: downward movement of the plate through the free surface, assumed horizontal, requires an infinite force; c) resolution of the Herculean paradox: the stress remains singular at the contact line, but the singularity is integrable.

Figure 3 a) An idealised problem which can be solved exactly in the Stokes limit (Jeong & Moffatt 1992): a free surface cusp forms at $(0, -\frac{2}{3}d)$; b) balance of the Stokeslet velocity due to surface tension and the velocity U due to remote forcing determines the exponentially small radius of curvature R at the stagnation point as given by eqn. (3.8).

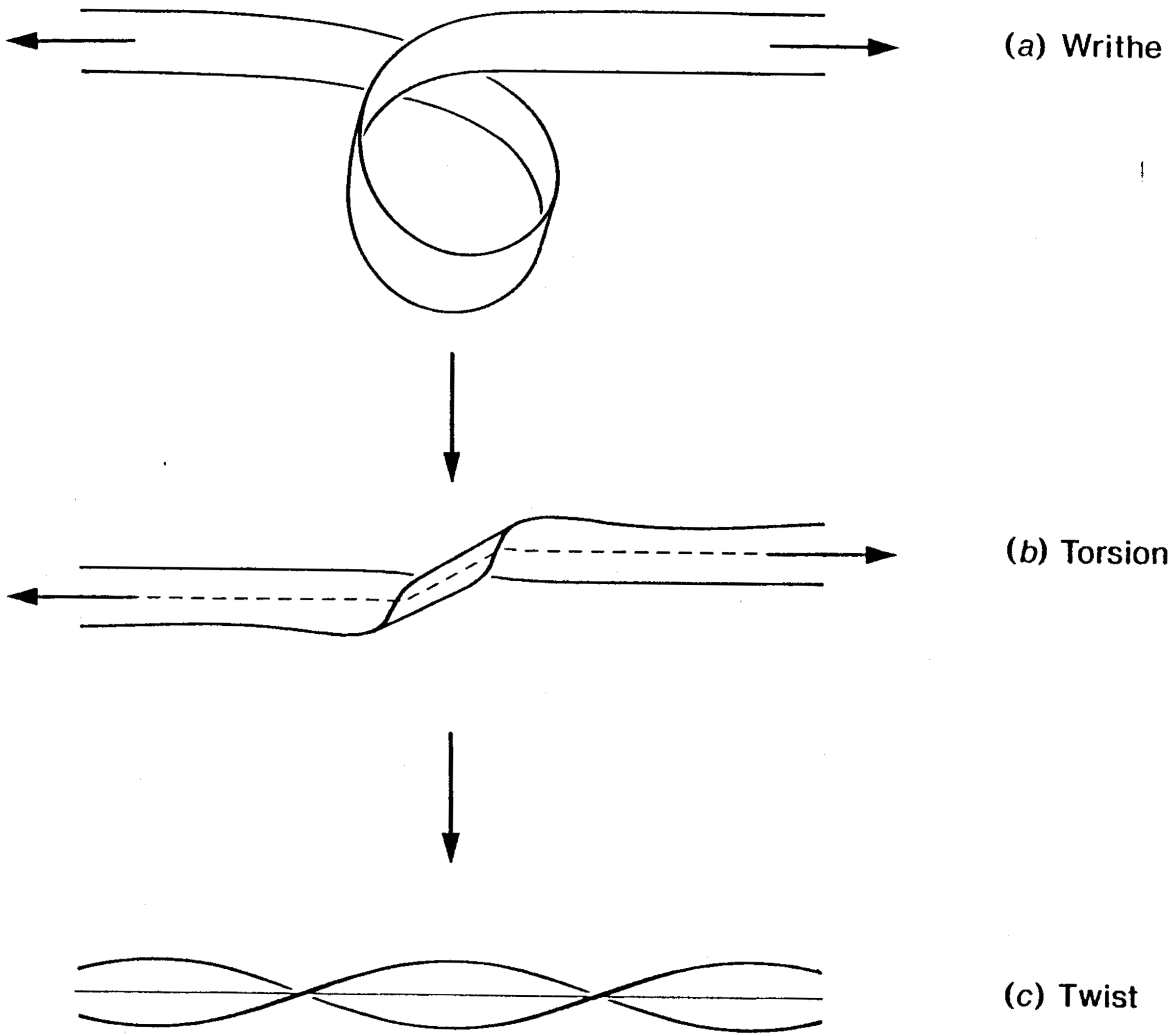


Figure 1

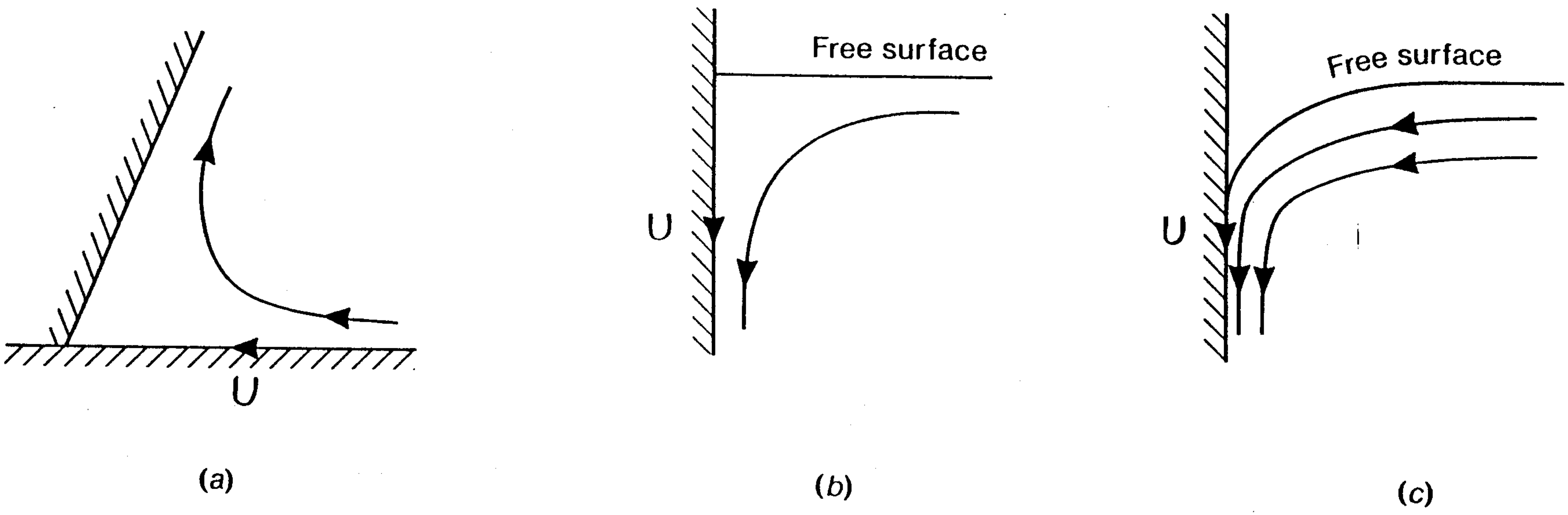
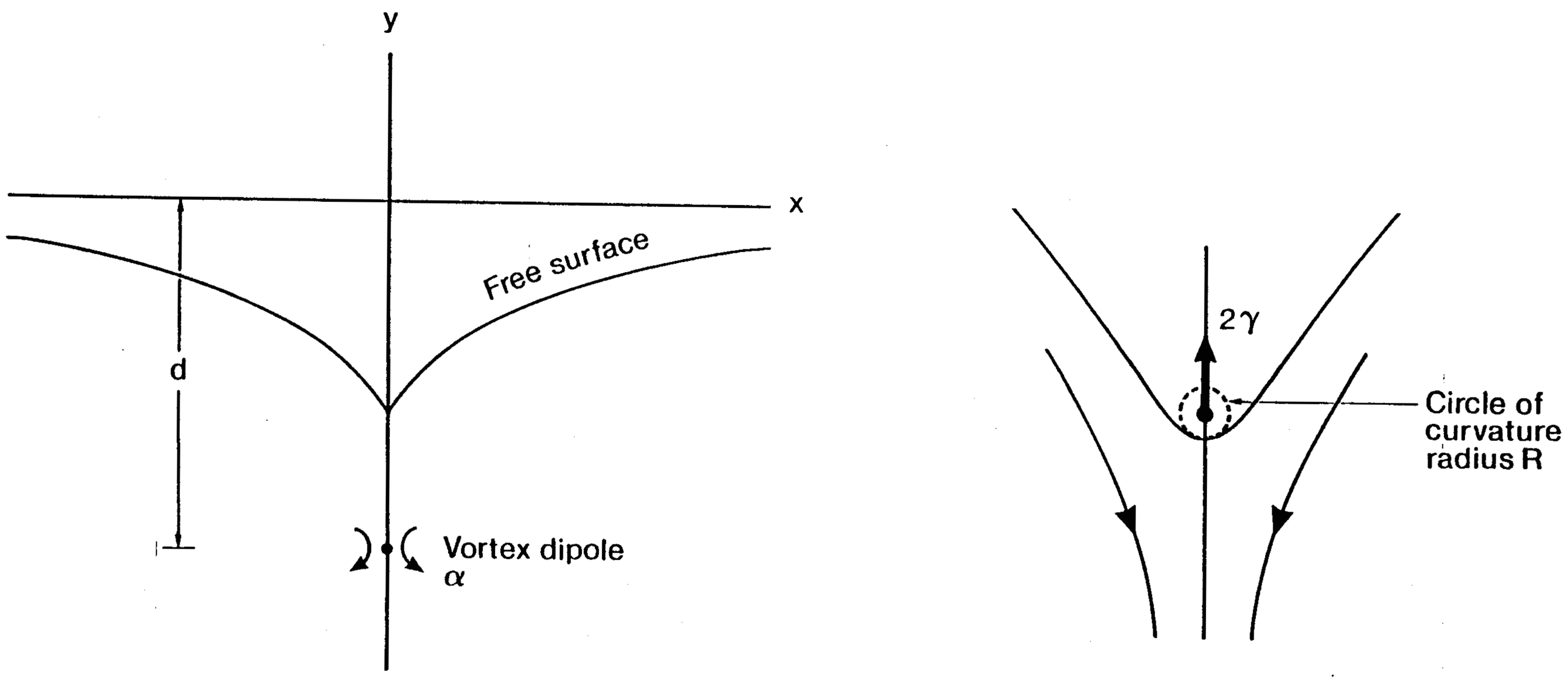


Figure 2


y
x
Free surface
d
Vortex dipole
α
2γ
Circle of
curvature
radius R

(a)

(b)

Figure 3

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