

# Vortices subjected to non-axisymmetric strain – unsteady asymptotic evolution

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## 1. Introduction

Much interest has been recently re-focused on the problem of concentrated vortices and their interactions. Much of this interest has been stimulated by the discovery of such structures in direct numerical simulations of turbulence (see, for example, Jimenez et al 1993, and references therein). The relevance of stretched vortices to the problem of turbulence was first pointed out by Burgers (1948), and has recently been advocated by Moffatt, Kida & Ohkitani 1994 (hereafter MKO'94) who describe the situation in the following terms:

“Vortices have often been described as the ‘sinews’ of fluid motion. Just as sinews serve to connect a muscle with a bone or other structure, so the concentrated vortices of turbulence serve to connect large eddies of much weaker vorticity; and just as sinews can take the stress and strain of muscular effort, so the concentrated vortices can accommodate the stress associated with the low pressure in their cores and the strain imposed by relative motion of the eddies into which they must merge at their ends.”

The particular problem addressed by MKO'94 was that of a vortex subjected to non-axisymmetric uniform strain

$$\mathbf{U} = (\alpha x, \beta y, \gamma z) \quad (1.1)$$

with  $\alpha + \beta + \gamma = 0$ ,  $\alpha < \beta < 0 < \gamma$ , for which a steady high-Reynolds-number asymptotic solution may be obtained in which the leading order vorticity distribution is simply

$$\boldsymbol{\omega} = (0, 0, \omega_0(r)) \quad (1.2)$$

with  $r^2 = x^2 + y^2$ . The form of  $\omega_0(r)$  is determined by a solvability condition at order  $R_\Gamma^{-1}$ , where

$$R_\Gamma = \Gamma/\nu \quad (1.3)$$

and  $\Gamma$  is the total circulation of the vortex; this solvability condition in effect selects from the infinite family of 'inviscid' solutions of the form (1.2) the particular (Burgers) solution

$$\omega_0(r) = \frac{\gamma\Gamma}{4\pi\nu} \exp\left[-\frac{\gamma r^2}{4\nu}\right]. \quad (1.4)$$

This solution is selected as a result of the persistent (long-term) effect of viscosity; the vortex experiences (at leading-order) the average strain field (averaged around the axis  $Oz$  of positive strain rate) which favours an axisymmetric solution. Departures from axisymmetry appear at order  $R_\Gamma^{-1}$ , the vorticity distribution then having the form

$$\omega(r, \theta) = \omega_0(r) + \lambda R_\Gamma^{-1} \Omega(r) \sin 2\theta, \quad (1.5)$$

where  $\lambda = (\alpha - \beta)/(\alpha + \beta)$ , length is now scaled on the Burgers scale

$$\delta = (\nu/\gamma)^{\frac{1}{2}} \quad (1.6)$$

and vorticity on  $\gamma\Gamma/\nu$ ;  $\Omega(r)$  is then a universal function determined by numerical solution of a second-order ordinary differential equation with boundary conditions that  $\Omega(r)/r^2$  is finite as  $r \rightarrow 0$ , and  $\Omega(r)$  is exponentially small as  $r \rightarrow \infty$ .

The asymptotic solution (1.5) may be extended to higher order in the small parameter

$$\epsilon = R_\Gamma^{-1} = \nu/\Gamma, \quad (1.7)$$

the solution at each level  $O(\epsilon^n)$  being completed by imposition of a solvability condition at  $O(\epsilon^{n+1})$ . At the level  $n = 2$ , a uniformly valid approximation to the field of dissipation of energy

$$\Phi = 2\nu s_{ij} s_{ij}, \quad s_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (1.8)$$

may be obtained. The contours of constant  $\Phi$  obtained by Moffatt et al (1994) agree well with previously computed contours, using a finite-difference numerical scheme for solution of the governing nonlinear partial differential equations (obtained from the exact Navier-Stokes equations), and provide evidence that the asymptotic solution is accurate for  $\lambda\epsilon \lesssim 0.005$ , and becomes increasingly more accurate with increasing Reynolds number.

Since a steady-state solution may take a long time to establish (starting from an arbitrary initial condition), and since moreover a steady solution may be subject to modification due to perturbation vorticity advected inwards 'from infinity' by the strain field (1.1), or due to interaction with other vortices (by the processes of merger and/or cancellation considered in a numerical study by Buntine & Pullin 1989), it is natural to investigate whether the asymptotic procedure described above may be extended to deal with unsteady evolution. A clue is provided by the work of Neu (1984) who showed that such an approach is indeed possible for two-dimensional strain fields ( $\beta = 0$ ,  $\alpha = -\gamma$  in the notation of the present paper). Here, we adapt this approach to accommodate the general strain field (1.1) and discuss the interesting perspectives opened up by this type of analysis.

## 2. Long-term quasi-axisymmetric evolution

We consider a vorticity field

$$\boldsymbol{\omega} = (0, 0, \omega(x, y, t)) \quad (2.1)$$

subjected to the action of the uniform irrotational strain (1.1). We may suppose that  $\omega$  is localised in the sense that  $|\omega|$  is exponentially small for large  $r = (x^2 + y^2)^{1/2}$ , and that the total circulation

$$\Gamma = \iint \omega(x, y, t) dx dy \quad (2.2)$$

is finite and non-zero. We assume that the Reynolds number  $R_\Gamma = \Gamma/\nu$  is large. The evolution of  $\omega$  is described by the nonlinear vorticity equation. In polar coordinates  $(r, \theta)$  with  $x = r \cos \theta$ ,  $y = r \sin \theta$ , this equation takes the form

$$\begin{aligned} \frac{\partial \omega}{\partial t} - \frac{1}{r} \frac{\partial(\psi, \omega)}{\partial(r, \theta)} = & \gamma \left( \omega + \frac{1}{2} r \frac{\partial \omega}{\partial r} \right) + \nu \nabla^2 \omega \\ & + \frac{1}{2} (\beta - \alpha) \left( \cos 2\theta r \frac{\partial}{\partial r} - \sin 2\theta \frac{\partial}{\partial \theta} \right) \omega \quad , \end{aligned} \quad (2.3)$$

where  $\psi$  is the streamfunction of the rotational ingredient of the flow, i.e.

$$\nabla^2 \psi = -\omega \quad . \quad (2.4)$$

Note that when  $\beta = \alpha$  (the axisymmetric case), equation (2.3) admits the steady solution (1.4).

We now non-dimensionalise (2.3), (2.4) with the substitutions

$$r \rightarrow (\nu/\gamma)^{1/2} r, \quad t \rightarrow \gamma^{-1} t, \quad \psi \rightarrow \Gamma \psi \quad . \quad (2.5)$$

Equation (2.3) then becomes

$$-\frac{1}{r} \frac{\partial(\psi, \omega)}{\partial(r, \theta)} = -\epsilon L_0 \omega - \epsilon \lambda L_1 \omega \quad , \quad (2.6)$$

where  $\epsilon = R_\Gamma^{-1}$ ,  $\lambda = (\beta - \alpha)/\gamma$ , and

$$L_0 = 1 + \frac{1}{2} r \frac{\partial}{\partial r} + \nabla^2 - \partial/\partial t \quad , \quad (2.7)$$

$$L_1 = \frac{1}{2} \left( \cos 2\theta r \frac{\partial}{\partial r} - \sin 2\theta \frac{\partial}{\partial \theta} \right) \quad (2.8)$$

Equation (2.6) is in the form obtained by MKO'94, but the operator  $L_0$  is modified by the inclusion of the term  $-\partial/\partial t$ . Note that by scaling time on  $\gamma^{-1}$ , we focus on evolution on a long slow time-scale (relative to the fast time-scale  $(\nu/\Gamma)\gamma^{-1}$ ). Equation (2.6) cannot therefore describe the details of vortex merger, for example, which occurs on a time scale short compared with  $\gamma^{-1}$  (see §4 below); but it *can* describe the slow evolution consequent upon such merger.

We now seek an asymptotic solution of (2.6) in the form

$$\psi(r, \theta, t) = \psi_0(r, \theta, t) + \epsilon \psi_1(r, \theta, t) + \epsilon^2 \psi_2(r, \theta, t) + \dots \quad (2.9)$$

with a corresponding expansion for  $\omega$ . At order  $\epsilon^0$ , we obtain the steady Euler equation

$$\frac{\partial(\psi_0, \omega_0)}{\partial(r, \theta)} = 0 \quad (2.10)$$

Here we focus on solutions of the form

$$\psi_0 = \psi_0(r, t) \quad , \quad \omega_0 = -\nabla^2 \psi_0 = \omega_0(r, t) \quad (2.11)$$

Thus the streamlines at leading order are circles  $r = \text{cst.}$ , and  $t$  appears merely as a parameter. The dependence of  $\psi_0$  on  $r$  and  $t$  can only be determined through application of a solvability condition at order  $\epsilon^1$ .

At order  $\epsilon^1$ , (2.6) gives

$$\frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial \omega_0}{\partial r} \psi_1 - \frac{\partial \psi_0}{\partial r} \omega_1 \right) = L_0 \omega_0 + \lambda L_1 \omega_0 \quad (2.12)$$

The solvability condition is obtained by integrating over  $\theta$ , and is evidently

$$L_0 \omega_0 \equiv \left( 1 + \frac{1}{2} r \frac{\partial}{\partial r} + \nabla^2 - \frac{\partial}{\partial t} \right) \omega_0 = 0 \quad (2.13)$$

This equation determines a unique evolution of  $\omega_0(r, t)$  from arbitrary initial condition

$$\omega_0(r, 0) = \hat{\omega}_0(r) \quad (2.14)$$

towards the asymptotic steady state

$$\omega_0(r, t) \sim \frac{1}{4\pi} e^{-r^2/4} \quad (t \rightarrow \infty) \quad , \quad (2.15)$$

i.e. the Burger's vortex. Equation (2.13) describes a 'quasi-axisymmetric' evolution, the rapid rotation within the vortex being such as to maintain axisymmetry at each instant  $t$ , despite the perturbing effect of the non-axisymmetric strain.

A particular solution of (2.13) which indicates the manner in which the asymptotic solution (2.15) is approached is given by

$$\omega_0(r, t) = \frac{1}{4\pi\delta^2} e^{-r^2/4\delta^2} \quad , \quad (2.16)$$

where

$$\delta^2 = 1 + (\delta_0^2 - 1) e^{-t} \quad (2.17)$$

A solution of precisely this kind was obtained by Neu (1984) for the case of a vortex subjected to plane strain ( $\beta = 0$  in the present notation).

For  $t \gg 1$ , (2.16) may be expanded in the form

$$\omega_0(r, t) \sim \frac{1}{4\pi} \left( 1 + \left( \frac{r^2}{4} - 1 \right) (\delta_0^2 - 1) e^{-t} + O(e^{-2t}) \right) e^{-\frac{1}{4}r^2} \quad (2.18)$$

Here the leading term carries the total circulation (normalised to unity) and the subsequent terms of the series have zero total circulation. The general solution of (2.13)

(depending only on  $r$  and  $t$ ) with unit total circulation in fact has the form

$$\omega_0(r, t) = \frac{1}{4\pi} \left( 1 + A \left( \frac{r^2}{4} - 1 \right) e^{-t} + O(\epsilon^{-2t}) \right) e^{-\frac{1}{4}r^2} , \quad (2.19)$$

where the constant  $A$  depends on the initial conditions.

### 3. The perturbation from axisymmetry

Having satisfied the solvability condition (2.13), (2.12) may be integrated with respect to  $\theta$  to give

$$\frac{1}{r} \left( \frac{\partial \omega_0}{\partial r} \psi_1 - \frac{\partial \psi_0}{\partial r} \omega_1 \right) = \frac{1}{4} \lambda \sin 2\theta r \frac{\partial \omega_0}{\partial r} - q(r, t) . \quad (3.1)$$

The solution for  $\psi_1(r, \theta, t)$  has the form

$$\psi_1 = \lambda f(r, t) \sin 2\theta + g(r, t) , \quad (3.2)$$

where

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{4}{r^2} f = \left( f - \frac{1}{4} r^2 \right) \eta(r, t) , \quad (3.3)$$

where

$$\eta(r, t) = - \frac{\partial \omega_0 / \partial r}{\partial \psi_0 / \partial r} . \quad (3.4)$$

If we use the similarity solution (2.16), this gives

$$\eta(r, t) = - \frac{r^2}{4\delta^4(e^{r^2/4\delta^2} - 1)} . \quad (3.5)$$

By the re-scaling

$$\hat{r} = r/\delta , \quad \hat{f} = f/\delta^2 \quad (3.6)$$

equation (3.3) becomes

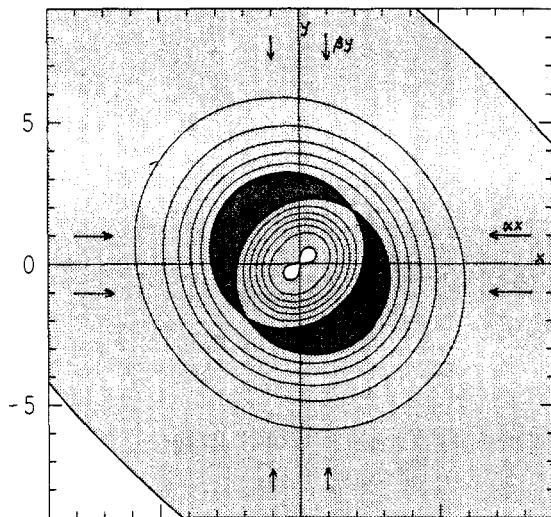
$$\frac{\partial^2 \hat{f}}{\partial \hat{r}^2} + \frac{1}{\hat{r}} \frac{\partial \hat{f}}{\partial \hat{r}} - \frac{4}{\hat{r}^2} \hat{f} = \left( \hat{f} - \frac{1}{4} \hat{r}^2 \right) \hat{\eta}(\hat{r}) , \quad (3.7)$$

where

$$\hat{\eta}(\hat{r}) = - \frac{\hat{r}^2}{4(e^{\hat{r}^2} - 1)} . \quad (3.8)$$

The problem for  $\hat{f}$  is thus reduced to precisely the form studied for the steady situation ( $\delta = 1$ ) by MKO'94. All of the results concerning the elliptic distortion of the vortex may therefore be carried over directly to the unsteady situation considered here. The isovorticity contours and the dissipation contours are thus geometrically similar for the steady and unsteady problems, the only modification that is required being the introduction of the scaling factor  $\delta(t)$ .

We should note here however that this is not the only possibility. If a different zero-order solution of (2.13) is used, then  $\eta(r, t)$  and  $f(r, t)$  will be correspondingly modified, and more complex structures may be expected.



**Figure 1** [adapted from MKO'94]. Dissipation contours for a vortex subjected to non-axisymmetric strain. The maxima are in the dark shaded crescent-shaped regions and are indicated by \*.

In general terms however the structure of the perturbed dissipation field may be anticipated from the following argument. For the gaussian structure (2.16), the dissipation field  $\Phi$  is axisymmetric, with a minimum at  $r = 0$ , and a maximum on the circle  $r = 2.67\delta$  (where the radial shear is greatest). From a topological point of view, this distribution is degenerate; a non-axisymmetric perturbation proportional to  $\cos 2\theta$  (equations (2.8) and (2.12) will replace the circle of maxima by just two isolated maxima at  $r = 2.67\delta$ ,  $\theta = 3\pi/4, 7\pi/4$  (figure 1). These maxima are surrounded by two crescent-shaped regions determined by the separatrices of the field  $\Phi(r, \theta)$ . The minimum at  $r = 0$  may also suffer a topological transformation: the calculation of MKO'94 (to order  $\epsilon^2$ ) reveals that this minimum is split into two minima on the other diagonals  $\theta = \pi/4, 5\pi/4$ , with a saddle point at the origin, the separatrix through the origin being a small figure-of-eight (figure 1).

For the time-dependent situation described by (2.16), the above dissipation field is quasi-static, time-dependence entering only through the radial scale  $\delta(t)$ . For large  $t$ ,  $\delta(t) \rightarrow 1$ , and the behaviour settles down to the steady state treated by MKO'94.

#### 4. Merging of vortices

If two vortices of like circulations (of order  $\Gamma$ ) are driven towards each other by the strain field (1.1), then they will ultimately merge and form a single vortex in the manner described by Buntine & Pullin (1989). This process involves the engulfment

of each vortex by the other, a double spiral pattern of vorticity contours being formed during the initial stage of merger before viscosity eliminates strong radial variations of vorticity.

The time-scale of merger when the Reynolds number (based on either vortex) is large may be estimated as follows. The length  $\ell(t)$  of the spiral 'tongues' increases linearly with  $t$  and the radial 'striation' scale therefore decreases like  $t^{-1}$  (the area of the tongues being conserved during the nearly inviscid evolution. The local wave-number in the radial direction  $k(t)$  therefore increases as  $t$ , and the effect of viscous interdiffusion of tongues from the two vortices may be represented by a factor

$$\exp -\nu \int_0^t k^2 dt = \exp \left\{ -C\nu \left( \frac{\Gamma}{\delta^3} \right)^2 t^3 \right\} , \quad (4.1)$$

where  $C$  is a constant of order unity. Hence the time-scale for the merging process is

$$t_M \sim t_0 R_\Gamma^{\frac{1}{3}} \sim t_V R_\Gamma^{-\frac{2}{3}} , \quad (4.2)$$

where  $t_0 = \delta^2/\Gamma$ ,  $t_V = R_\Gamma t_0$ . This is the time-scale conjectured by MKO'94 (p. 246), and it is not incompatible with the trend indicated by the computations of Buntine & Pullin (1989) for axisymmetric strain at Reynolds numbers  $R_\Gamma/2\pi = 40, 160, 640$  and 1280.

We thus expect that when two like-signed vortices are swept together, the initial merging process occurs through the above spiral-tongue mechanism on a time-scale  $t_0 R_\Gamma^{\frac{1}{3}}$ ; after this initial stage, the vorticity field is approximately axisymmetric and subsequent relaxation to the ultimate steady form (2.15) takes place on the viscous time-scale  $t_V = t_0 R_\Gamma$ , by the process described in §2 above. The mechanism is similar to that described by Rhines & Young (1982), and in the closely related problem of magnetic flux expulsion by Moffatt & Kamkar (1983).

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