

Topological Dynamics of Fluids

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Abstract

Certain fluid dynamical problems having a peculiarly topological flavour are reviewed. First, some properties of the helicity invariant of the Euler equations are described. Then the technique of magnetic relaxation for the indirect determination of steady Euler flows is summarised. Applications to the relaxation of two-dimensional fields characterised by an invariant signature function, of knotted magnetic flux tubes to minimum energy states, and of chaotic three-dimensional fields are briefly discussed.

1 Introduction

Let us consider an incompressible fluid moving with velocity field $\mathbf{u}(\mathbf{x}, t)$ where $\nabla \cdot \mathbf{u} = 0$. Let ρ be the density of the fluid (assumed uniform and constant), $p(\mathbf{x}, t)$ the pressure field, $V(\mathbf{x}, t)$ the potential of any conservative forces (including gravity), and $\mathbf{f}(\mathbf{x}, t)$ any other non-conservative forces applied to the fluid. Then the evolution of the flow is described by the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \wedge \boldsymbol{\omega} - \nabla h + \nu \nabla^2 \mathbf{u} + \mathbf{f} \tag{1.1}$$

where $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$ is the vorticity field, $h = p/\rho + \frac{1}{2} \mathbf{u}^2 + V$, and ν is the kinematic viscosity of the fluid. This famous equation describes both laminar and turbulent flows. Only a handful of exact solutions are known, and these usually correspond to highly idealised boundary and/or initial conditions. In practical contexts, and particularly for turbulent flow fields, recourse must generally be had to numerical techniques, which are highly developed. These are however viable only for rather modest values of the Reynolds number Re , defined as $Re = UL/\nu$ where U is a typical velocity and L a typical length-scale of the flow field. In many practical contexts, $Re \geq 10^6$, or even (in meteorological contexts) $Re \geq 10^8$, and numerical techniques have to be coupled with semi-empirical assumptions of dubious validity. Asymptotic theories (for $Re \gg 1$) are still at a rudimentary stage of development, but the hope for the future must lie in a development of these asymptotic theories, and incorporation of the results in improved numerical techniques.

In the limit $Re \rightarrow \infty$ (or equivalently $\nu \rightarrow 0$) and assuming $\mathbf{f} = 0$, (1.1) reduces to the Euler equation of classical hydrodynamics

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \wedge \boldsymbol{\omega} - \nabla h \quad . \quad (1.2)$$

As is well-known, this is a singular limit which fails to describe boundary layers at the fluid boundaries, or the structure of vortex sheets, or other singularities, in the fluid interior. Excluding such singular regions, however, eqn. (1.2) may be expected to provide a reasonable description of the evolution of the flow.

The curl of (1.2) provides the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \wedge (\mathbf{u} \wedge \boldsymbol{\omega}) \quad , \quad (1.3)$$

which embodies the results of Helmholtz (1858) and Kelvin (1869), specifically that vortex lines move as if they were composed of fluid particles – in modern parlance, the vortex lines are ‘frozen in the fluid’. As Kelvin recognized, this means that all topological structures associated with the vorticity field are invariant: two closed vortex tubes remain linked for all time if they are linked at time $t = 0$; a single vortex tube remains knotted in the form of a knot K if it is so knotted at time $t = 0$; and so on. Equation (1.3) may be regarded as the ‘master equation’ that guarantees invariance of the complete topology of the vorticity field $\boldsymbol{\omega}(\mathbf{x}, t)$. This is a powerful statement, because, for example, all knot invariants may be regarded (although some may consider it eccentric to do so) as invariants of the nonlinear evolution equation (1.3).

When we adopt a topological (as opposed to an analytical or differential) approach to fluid dynamics, we seek to exploit the topological invariance embodied in (1.3). This forces us to focus on the global (as opposed to local) properties of the vorticity field and to identify specific invariants of (1.3) which have a topological character.

It proves fruitful to widen the investigation by considering an arbitrary ‘frozen-in’ solenoidal field $\mathbf{B}(\mathbf{x}, t)$ ($\nabla \cdot \mathbf{B} = 0$) satisfying the evolution equation (cf. (1.3))

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) \quad . \quad (1.4)$$

This is the equation satisfied by the magnetic field $\mathbf{B}(\mathbf{x}, t)$ in a perfectly conducting fluid moving with velocity $\mathbf{u}(\mathbf{x}, t)$; it also serves as a model for any other situation in which a vector field \mathbf{B} is transported by the fluid, the flux of \mathbf{B} through any material surface element being conserved. The field \mathbf{u} may be independent of \mathbf{B} (in which case \mathbf{B} is a ‘passive’ vector field and eqn. (1.4) is linear); or \mathbf{u} may be influenced by \mathbf{B} through dynamic effects (e.g. through the Lorentz force in the magnetohydrodynamic context), and then (1.4) is in effect a nonlinear evolution equation, like (1.3).

2 The Helicity Invariant

Consider a closed material surface S_ω within the fluid on which $\mathbf{n} \cdot \boldsymbol{\omega} = 0$, a condition that persists for all t when $\boldsymbol{\omega}$ satisfies (1.3). Let V be the volume inside S_ω ; we may then define the helicity \mathcal{H} of the flow within V as

$$\mathcal{H} = \int_V \mathbf{u} \cdot \boldsymbol{\omega} dV \quad , \quad (2.1)$$

and there is a family of such integrals corresponding to any family of vorticity surfaces S_ω . It is known (Moreau 1961, Moffatt 1969) that, under the Euler evolution (1.3), \mathcal{H} is invariant:

$$\mathcal{H} = \text{cst.}, \quad (2.2)$$

and that this invariance is a consequence of the conserved topology of the vorticity field. Thus, for example, if the vorticity field consists of two (possibly linked) vortex tubes in each of which the vortex lines are untwisted, then

$$\mathcal{H} = \pm 2n\kappa_1\kappa_2 \quad , \quad (2.3)$$

where κ_1, κ_2 are the (conserved) circulations of the tubes, n is the Gauss linking number of their axes, and the + or - is chosen according as the linkage is positive or negative. Eqn. (2.3) provides a clear bridge between an invariant of classical fluid dynamics and the fundamental topological invariant n of two linked curves.

A further bridge is provided through consideration of a single knotted vortex tube of circulation κ , the axis C of the tube being knotted in the form of a knot K (Moffatt & Ricca 1991). Here we cannot avoid consideration of the twist of the $\boldsymbol{\omega}$ -lines within the tube. Suppose that this twist is uniform in the sense that each pair of $\boldsymbol{\omega}$ -lines in \mathcal{T} has the same linking number n . Then the helicity of the tube may be calculated explicitly in the form

$$\mathcal{H} = \pm n\kappa^2 \quad , \quad (2.4)$$

where again the + or - is chosen according as the linkage is right-handed or left-handed. In this situation, the number n may be related to geometric properties of the ribbon whose boundaries are the axis C (itself a vortex line) and any other vortex line in the tube. Let $\mathbf{N}(s)$ be a unit transverse vector on this ribbon, where s is arclength on C . Then, following Fuller (1971) we may define the twist of the ribbon by

$$Tw = \frac{1}{2\pi} \oint_C (\mathbf{N} \wedge \mathbf{N}') \cdot d\mathbf{x} \quad , \quad (2.5)$$

where $\mathbf{N}' = d\mathbf{N}/ds$. We may also define the writhe Wr of C by

$$Wr = \frac{1}{4\pi} \oint_C \oint_C \frac{d\mathbf{x} \wedge d\mathbf{x}' \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \quad . \quad (2.6)$$

Then, as shown by Călugăreanu (1961) and White (1969),

$$n = Wr + Tw \quad , \quad (2.7)$$

so that the sum of writhe and twist is a topological invariant of the ribbon. The twist may be further decomposed in the form

$$Tw = \mathcal{T} + N_0 \quad , \quad (2.8)$$

where

$$\mathcal{T} = \oint \tau(s) ds \quad , \quad (2.9)$$

$\tau(s)$ is the torsion on C , and N_0 is the number of rotations of $\mathbf{N}(s)$ relative to the Frenet triad on C in one passage round C . It is interesting to note here that if C is deformed through an inflexion point, then both \mathcal{T} and N_0 are discontinuous (by ± 1 and ∓ 1 respectively) but $\mathcal{T} + N_0$ is continuous. The self-linking number SL introduced by Pohl (1968) (half the algebraic sum of the cross-tangents on C) may be identified with $Wr + \mathcal{T}$, and is likewise discontinuous in deformation of C through any inflexional configuration.

3 Relaxation With Conservation of Topology

Regarding (1.2) as a dynamical system in the phase-space of solenoidal velocity fields, attention first focuses on the fixed points of this dynamical system, i.e. on the steady Euler flows satisfying

$$\mathbf{u} \wedge \boldsymbol{\omega} = \nabla h \quad . \quad (3.1)$$

These flows are characterised by the property that their energy

$$E = \frac{1}{2} \int \mathbf{u}^2 dV \quad (3.2)$$

is stationary under 'isovortical' displacements, i.e. perturbation under which the vorticity field is frozen. Moreover, if E is extremal (either maximum or minimum) under such perturbations, then the flow is stable (Arnol'd 1966). It makes sense therefore to attempt to construct relaxation processes which drive the system towards a state of stationary energy, and then to consider the nature of the stationary point, whether saddle or extremum. One such process has been proposed by Moffatt (1985), following an earlier suggestion of Arnol'd (1974). This first involves consideration of the relaxation of a magnetic field \mathbf{B} in a perfectly conducting, but viscous, fluid, a process governed by the equations

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{v} \wedge \mathbf{B}) \quad (3.3)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \mathbf{j} \wedge \mathbf{B} + \nu \nabla^2 \mathbf{v} \quad (3.4)$$

where \mathbf{v} is the velocity field (the change of notation here is deliberate), and $\mathbf{j} = \nabla \wedge \mathbf{B}$ is the current distribution (the usual factor μ_0^{-1} is here irrelevant and is omitted). Viscosity is included in order to provide a means of dissipating energy which now has both kinetic (K) and magnetic (M) contributions. Eqn.

(3.3) however guarantees that the topology of the field \mathbf{B} is conserved. If this topology is nontrivial, then as shown by Freedman (1988), the magnetic energy

$$M = \frac{1}{2} \int_V \mathbf{B}^2 dV \quad (3.5)$$

is bounded away from zero. Here, V now represents the whole fluid domain, and we may assume that M is finite. Since $K + M$ is monotonic decreasing and bounded below, it must tend to a constant. Thus in the absence of any boundary forcing and assuming non-pathological behaviour, the velocity \mathbf{v} must tend to zero everywhere, since otherwise viscous dissipation would continue to reduce $K + M$. In this limiting situation, (3.4) reduces to

$$\mathbf{j} \wedge \mathbf{B} = \nabla p \quad , \quad (3.5)$$

i.e. we reach a state of magnetostatic equilibrium in which the field $\mathbf{B}(\mathbf{x})$ has, with one qualification, the same topology as that of the 'initial field' for the relaxation problem. The qualification is that tangential discontinuities of \mathbf{B} may appear during the process of relaxation (and consideration of the relaxation of fields with linked flux tubes indicates that such discontinuities do appear generically – Moffatt 1985).

Note now the structural similarity of (3.1) and (3.5), which are indeed isomorphic under the identifications

$$\mathbf{u} \leftrightarrow \mathbf{B} \quad , \quad \boldsymbol{\omega} \leftrightarrow \mathbf{j} \quad , \quad h \leftrightarrow p_0 - p \quad (3.6)$$

for arbitrary constant p_0 . This means that to every magnetostatic equilibrium obtained by the process of magnetic relaxation, there corresponds a steady Euler flow obtained by replacing the magnetostatic $\mathbf{B}(\mathbf{x})$ by $\mathbf{u}(\mathbf{x})$. Tangential discontinuities of \mathbf{B} become vortex sheets in this analogue Euler flow; and the topology of \mathbf{u} is the same as the topology of \mathbf{B} , which may be arbitrarily prescribed via the initial condition for the relaxation problem.

This procedure for obtaining solutions of the steady Euler equations is one of considerable subtlety, involving as it does an interplay between two distinct analogies – first that between frozen fields $\mathbf{B}(\mathbf{x}, t)$ and $\boldsymbol{\omega}(\mathbf{x}, t)$ in unsteady evolution, and second, that between $\mathbf{B}(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$ in equilibrium states. The latter analogy does *not* extend to unsteady evolution, and so the stability criteria for magnetostatic equilibria and for steady Euler flows are quite distinct; the first involves consideration of isomagnetic disturbances, while the second involves consideration of isovortical disturbances (Moffatt 1986, 1990a). A magnetostatic equilibrium obtained by the process of magnetic relaxation is a minimum energy state with respect to isomagnetic perturbations, and is therefore stable in an ideal fluid. The analogous Euler flow may however be unstable, and indeed is presumably unstable (to Kelvin-Helmholtz instabilities) if vortex sheets are present.

4 Two-Dimensional Relaxation

The process described above has been implemented numerically (with a simplified dissipation mechanism) by Bajer (1989) and by Linardatos (1993) for two-dimensional configurations. If the initial field $\mathbf{B}(\mathbf{x}, 0)$ has any saddle points, then it is found that in general the X -type structure near a saddle point collapses with formation of a current sheet of finite extent as $t \rightarrow \infty$. The technique thus leads to numerical determination of steady Euler flows with imbedded vortex sheets of finite extent – which may be expected to be unstable for the reason indicated above.

If the initial field $\mathbf{B}(\mathbf{x}, 0)$ has only one neutral point of O -type, and no saddle-points, then the field lines have a natural tendency to relax to circular form, but this tendency is impeded by the boundary constraint $\mathbf{B} \cdot \mathbf{n} = 0$ if the boundary is noncircular. The asymptotic equilibrium represents a compromise between these two conflicting demands. The field lines may be represented by a family of curves $\chi = \text{cst.}$ which move with the fluid during the relaxation process. The area $A(\chi)$ inside each such curve is constant (the flow being incompressible). For this simple field structure, the field topology is totally described by this invariant function $A(\chi)$, the *signature* of the field (Moffatt 1990a); under the assumed conditions, $A(\chi)$ is monotonic.

A result of great interest has recently been proved by Davidson (1994): this is that Euler flows obtained via two-dimensional magnetic relaxation are stable (to two-dimensional disturbances) provided the initial field has the ‘simplest’ topology as described in the preceding paragraph. Thus for arbitrary two-dimensional boundary and arbitrary (monotonic) signature function $A(\chi)$, we have a means of obtaining a corresponding stable Euler flow. The inference is that the instabilities to which more general Euler flows may be subject are necessarily localised in the neighbourhood of vortex sheets – but this requires further investigation.

Davidson’s result applies equally to axisymmetric configurations in which the fields (magnetic or velocity) have only meridional components – such fields yielding stable Euler flows. The argument fails however if zonal field components are included: axisymmetric Euler flows *with swirl* obtained via the magnetic relaxation technique may be unstable.

5 The Minimum Energy of Knotted Flux Tubes

The magnetic relaxation technique may be applied to a magnetic field confined to a single flux tube \mathcal{T} knotted in the form of a knot K . Let Φ be the flux of \mathbf{B} in the tube, conserved under frozen-field evolution. If the twist h of the field lines in \mathcal{T} is uniform, then the magnetic helicity (also conserved) is given by

$$\mathcal{H}_M = h\Phi^2 \quad (5.1)$$

If h is an integer n , then this is the linking number of any pair of \mathbf{B} -lines in \mathcal{T} as discussed in §2 above. However h need not be an integer (the \mathbf{B} -lines being then ergodic on nested toroidal surfaces within \mathcal{T}).

If the flux tube is allowed to relax to a minimum energy state in an incompressible medium, then its volume V is conserved (like Φ and h). On dimensional grounds, the minimum energy M_{\min} is proportional to $\Phi^2 V^{-\frac{1}{3}}$; thus (Moffatt 1990b)

$$M_{\min} = m(h)\Phi^2 V^{-\frac{1}{3}}, \quad (5.2)$$

where $m(h)$ is a dimensionless function of the dimensionless parameter h , this function being determined solely by the topology of the knot K .

The function $m(h)$ (described by Moffatt 1990b as the 'ground-state' energy of the 'framed' knot) may be determined, in principle, by numerical implementation of the magnetic relaxation procedure. The need to ensure strict conservation of field topology however makes for practical difficulties that have not yet been overcome. A variational technique has recently been developed (Chui & Moffatt 1995) using a local non-orthogonal coordinate system attached to the flux tube, to obtain the minimum energy function $\bar{m}(h)$ under certain additional mild constraints, this providing an upper bound on, and a reasonable approximation to, $m(h)$ when h is not too large. The function $\bar{m}(h)$ has been computed for various torus knots; for each knot considered it is a convex curve with minimum at $h = h_m \neq 0$; and the evidence available so far indicates that when $h = O(1)$, $\bar{m}(h)$ increases with increasing knot complexity, as is to be expected.

The concept of a relaxed minimum energy state for a knotted tube or ribbon is appealing in many physical and biological contexts; the procedure outlined above provides a viable approach for the actual determination of the minimum energy and the corresponding geometrical configuration. The concept may also be extended in a natural way to links.

6 Relaxation of Chaotic Fields

An arbitrary solenoidal vector field \mathbf{B} defined within a finite domain \mathcal{D} and with $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial\mathcal{D}$ in general exhibits the phenomenon of chaotic field line wandering, i.e. the \mathbf{B} -lines do not lie on surfaces nested within \mathcal{D} , but are more nearly space-filling in character. Poincaré sections of such a field exhibit the familiar scatter of points (for a single \mathbf{B} -line) characteristic of chaotic behaviour. The components of \mathbf{B} need not be particularly complicated functions of the coordinates (x, y, z) to give this sort of behaviour; thus, for example, if each component is a quadratic function of (x, y, z) , then the field lines in general exhibit chaos (Bajer & Moffatt 1990), although frequently 'islands of regularity' are apparent within the 'chaotic sea'.

If such a field is allowed to relax, its chaotic topology being conserved, then as we have seen the asymptotic field satisfies the magnetostatic equation (3.5)

with the consequence $\mathbf{B} \cdot \nabla p = 0$. This means that the \mathbf{B} -lines lie on surfaces $p = \text{cst.}$; but by definition within any chaotic region in \mathcal{D} , the \mathbf{B} -lines do not lie on surfaces. The only escape from this apparent contradiction is to conclude that $\nabla p = 0$ within any such region, i.e. from (3.5), $\mathbf{j} = \alpha(\mathbf{x})\mathbf{B}$ for some scalar field $\alpha(\mathbf{x})$. Now since $\nabla \cdot \mathbf{j} = 0$ and $\nabla \cdot \mathbf{B} = 0$ we have immediately $\mathbf{B} \cdot \nabla \alpha = 0$, so that by repetition of the argument, $\nabla \alpha = 0$ also in the chaotic region. Thus in this region we have

$$\nabla \wedge \mathbf{B} = \alpha \mathbf{B} \quad , \quad (6.1)$$

i.e. \mathbf{B} is a Beltrami field with constant α . And yet in this same region, \mathbf{B} is chaotic. It is a completely remarkable, and as yet ill-understood, phenomenon, that such a simple equation as (6.1) can apparently describe fields having an arbitrary degree of chaos. The islands of regularity may hold the key to this puzzling conclusion, a conjectural note on which it may be appropriate to terminate this lecture.

References

1. Arnold'd, V.I. 1966 Sur un principe variationnel pour les écoulements stationnaires des liquides parfaits et ses applications aux problèmes de stabilité non-linéaires. *J. Méc.* **5**, 29-43.
2. Arnold'd, V.I. 1974 The asymptotic Hopf invariant and its applications [English translation: *Sel. Math. Sov.* **5**, 327-345 (1986)].
3. Bajer, K. 1989 Flow kinematics and magnetic equilibria. PhD thesis. Cambridge University.
4. Bajer, K. & Moffatt, H.K. 1990 On a class of steady confined Stokes flows with chaotic streamlines. *J. Fluid Mech.* **212**, 337-363.
5. Călugăreanu, G. 1961 Sur les classes d'isotopie des noeuds tridimensionnels et leurs invariants. *Czechoslovak Math. J.* **11**, 588-625.
6. Chui, A. & Moffatt, H.K. 1995 Energy and helicity of knotted magnetic flux tubes. Preprint.
7. Davidson, P. 1994 Global stability of two dimensional and axisymmetric Euler flows. *J. Fluid Mech.* **246**, 273-305.
8. Freedman, M.H. 1988 A note on topology and magnetic energy in incompressible perfectly conducting fluids. *J. Fluid Mech.* **194**, 549-551.
9. Fuller, F.B. 1971 The writhing number of a space curve. *Proc. Natl. Acad. Sci. USA* **68**, 815-819.
10. Helmholtz, H. von 1858 *Crelle's Journal* **55**. [English translation: *Phil. Mag.* (4) **33**, 485-512 (1867)]
11. Kelvin, Lord (then W. Thomson) 1869 On vortex motion. *Trans. Roy. Soc. Edin.* **25**, 217-260.
12. Linardatos, D. 1993 Determination of two-dimensional magnetostatic equilibria and analogous Euler flows. *J. Fluid Mech.* **246**, 569-591.
13. Moffatt, H.K. 1969 The degree of knottedness of tangled vortex lines. *J. Fluid Mech.* **35**, 117-129.
14. Moffatt, H.K. 1985 Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology Part 1. Fundamentals. *J. Fluid Mech.* **159**, 359-378.

15. Moffatt, H.K. 1986 Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 2. Stability considerations. *J. Fluid Mech.* **166**, 359-378.
16. Moffatt, H.K. 1990a Structure and stability of solutions of the Euler equations: a Lagrangian approach. *Phil. Trans. R. Soc. London A* **333**, 321-342.
17. Moffatt, H.K. 1990b The energy spectrum of knots and links. *Nature Lond.* **347**, 367-369.
18. Moffatt, H.K. & Ricca, R. 1991 Helicity and the Călugăreanu invariant. *Proc. Roy. Soc. Lond. A* **439**, 411-429.
19. Moreau, J.J. 1961 Constants d'un ilot tourbillonnaire en fluide parfait barotrope. *C.r. hebd. Seanc. Acad. Sci. Paris* **252**, 2810-2812.
20. Pohl, W.F. 1968 The self-linking number of a closed space curve. *J. Math. Mech.* **17**, 975-985.
21. White, J.H. 1969 Self-linking and the Gauss integral in higher dimensions. *Am. J. Math.* **91**, 693-728.