

CHAPTER 12

KNOTS AND FLUID DYNAMICS

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1 Introduction

Fluid mechanics provides a very natural setting for the consideration of knotted or linked structures, and the manner in which these may 'relax' to an equilibrium configuration of minimal energy. For we may imbed within a fluid medium a solenoidal vector field $\mathbf{B}(\mathbf{x}, t)$ ($\nabla \cdot \mathbf{B} = 0$) which represents the knotted or linked structure in an appropriate way and which is 'frozen' in the fluid under any continuous fluid motion, so that its topological properties are conserved. If moreover the field $\mathbf{B}$ is assumed to impart a force to the fluid medium, then an associated energy W stored in the field $\mathbf{B}$ may be defined. Under the action of the force, this energy is converted to kinetic energy of motion and hence dissipated via the agency of viscosity. There is thus a natural mechanism for the decrease of W under the constraint of conservation of field topology; and we are then faced with the mathematical problem of determining the equilibrium field configuration, $\mathbf{B}^E(\mathbf{x})$ say, which minimises W subject to prescribed field topology. The fluid dynamical relaxation process provides a natural route towards this minimum energy equilibrium state.

In the following, we shall first review the general theory of relaxing fields, as developed by Moffatt (1985), and then particularise to fields defined within a tubular neighbourhood of a given knot K (Moffatt 1990). The internal twist of the field within this neighbourhood (related to the 'framing of the knot') affects the relaxation process, and the minimum energy attained depends upon this twist. If the twist is zero, then the minimum energy configuration is closely related to the 'ideal' configuration of Katritch *et al* (1996) in which the tube length L is minimised (and cross-section S maximised) for given volume $V = LS$; however the relaxation process does not constrain the cross-section to remain circular.

If the twist is large, then the behaviour is quite different: the dominant field component B_θ (where r, θ are polar coordinates on the tube cross-section) now tends to *decrease* S with consequential *increase* of L; at the same time, the tube is subject to kink instabilities which decrease the internal twist, but at the expense of increase in the *writhe* of the tube axis. What is conserved in this process is the *helicity* $\mathcal{H} = h\Phi^2$ of the tube field, where Φ is the (conserved) axial flux; and we shall show that the minimum energy scales like $|h|^{4/3}$ when $|h| \gg 1$, irrespective of knot type.

We shall adopt in what follows the language of magnetohydrodynamics (MHD), in which $\mathbf{B}(\mathbf{x}, t)$ is interpreted as the magnetic field in a perfectly conducting fluid medium. We need not, however, feel constrained by this interpretation; from an abstract point of view, $\mathbf{B}(\mathbf{x}, t)$ is to be regarded simply as a vector field, convected with the fluid motion in such a way that its flux across any *material* element of area (i.e. consisting always of the same fluid particles) is conserved.

2 The general theory of relaxation

The fundamental equation satisfied by any such 'frozen' field is well-known in the MHD context, namely

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{v} \wedge \mathbf{B}) \tag{1}$$

where $\mathbf{v}(\mathbf{x}, t)$ is the fluid velocity. We shall suppose that the flow is volume-preserving (incompressible) so that $\nabla \cdot \mathbf{v} = 0$. The flow determines a time-dependent mapping

$$\mathbf{x} \to \mathbf{X}(\mathbf{x}, t) \qquad (t > 0) \tag{2}$$

for each fluid particle starting at $\mathbf{x}$ at time $t = 0$; and the solution of (1) is given in terms of this mapping by

$$B_i(\mathbf{X}, t) = B_j(\mathbf{x}, 0) \frac{\partial X_i}{\partial x_j} \,. \tag{3}$$

The antisymmetric part of the deformation tensor $\partial X_i / \partial x_j$ is associated with rotation of a fluid volume element initially centred on $\mathbf{x}$, while the symmetric part represents irrotational deformation (whereby a small sphere deforms to an ellipsoid).

Let us define the energy density of the field $\mathbf{B}$ as $\frac{1}{2}\mathbf{B}^2$; the rate of change of this energy density is, from (1),

$$\frac{\partial}{\partial t} \tfrac{1}{2}\mathbf{B}^2 = \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{B} \cdot \nabla \wedge (\mathbf{v} \wedge \mathbf{B}) \,. \tag{4}$$

By standard vector manipulation, this may be written

$$\frac{\partial}{\partial t} \tfrac{1}{2}\mathbf{B}^2 = -\mathbf{v} \cdot (\mathbf{j} \wedge \mathbf{B}) - \nabla \cdot \Big[(\mathbf{v} \wedge \mathbf{B}) \wedge \mathbf{B}\Big] \tag{5}$$

where $\mathbf{j} = \nabla \wedge \mathbf{B}$ is the *current* associated with $\mathbf{B}$. If, as we may suppose, $\mathbf{B}$ is a localised field, the divergence term vanishes when integrated over the fluid domain, and we have

$$\frac{dM}{dt} = -\int \mathbf{v} \cdot \mathbf{F} dV \tag{6}$$

where $M = \frac{1}{2}\int \mathbf{B}^2 dV$, and

$$\mathbf{F} = \mathbf{j} \wedge \mathbf{B} \tag{7}$$

is the effective force per unit volume exerted by the field $\mathbf{B}$ on the fluid. In electrodynamics, this is recognized as the Lorentz force, and is associated with the Maxwell stress tensor

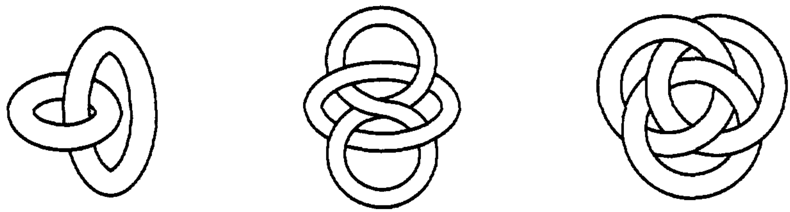
$$T_{ij} = B_i B_j - \tfrac{1}{2} B^2 \delta_{ij}$$

representing both tension in the lines of force of the $\mathbf{B}$-field, and pressure between adjacent lines of force. The field energy decreases through contraction of lines of force (thus relieving tension), but this can proceed only for so long as topological constraints present in the initial field $\mathbf{B}_0(\mathbf{x}) = \mathbf{B}(\mathbf{x}, 0)$ permit. If every $\mathbf{B}$-line is an unknotted closed loop which can be shrunk to a point in the fluid domain without cutting any other $\mathbf{B}$-lines, then there is no topological impediment to decrease of the field energy towards zero. This case (of *trivial topology*) is however exceptional; in general even simple fields for which all field lines are closed curves exhibit linkages that present topological barriers to this decrease of energy. Three examples are shown in figure 1: a flux linkage, a Whitehead link (in which the total flux trapped is zero), and a Borromean link (in which the total flux trapped is again zero and it is the relative position of different strands of flux that provides the topological barrier).

In each of these cases, if a closed field line C is shrunk to a point via a volume-preserving fluid motion, then the energy of the trapped field necessarily increases without limit. Thus for example, if C is taken to be the circle $x^2 + y^2 = a^2$, it may be shrunk to a point via the incompressible strain flow

$$\mathbf{v} = (-\alpha x\, , -\alpha y\, , 2\alpha z) \qquad (\alpha > 0) \tag{8}$$

under whose action the radius of the convected circle C is $r = ae^{-\alpha t}$ at time t. This flow involves persistent stretching in the z-direction under which the z-component of any trapped flux *increases* exponentially, with consequent increase of field energy. In a freely evolving situation, the contraction of C will presumably be arrested when an equipartition of the field energy is established between the trapping component (in a flux tube centred on C) and the trapped component normal to C.



a) b) c)

Figure 1: a) Flux linkage with helicity $\mathcal{H} = 2\Phi_1\Phi_2$ (see §3); b) the Whitehead link, with zero helicity; c) the Borromean link for which each pair of flux tubes is unlinked.

This picture has been formalised by Freedman (1988) who has proved that in any topologically nontrivial field of closed **B**-lines, the field energy is indeed bounded away from zero under any volume-preserving diffeomorphism. This means that, if we can construct a process by which the energy of such a field decreases monotonically, then that energy must tend to a positive limit.

Such a process may indeed be easily constructed. We simply suppose that the velocity field $\mathbf{v}$ is driven in the fluid by the force $\mathbf{F} = \mathbf{j} \wedge \mathbf{B}$ starting from rest. If the fluid has (uniform) density ρ and viscosity μ, it then flows according to the Navier-Stokes equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \mathbf{j}\wedge\mathbf{B} + \mu\nabla^2\mathbf{v}\,. \tag{9}$$

If the fluid is assumed to fill $\mathbb{R}^3$, the localised force $\mathbf{j}\wedge\mathbf{B}$ generates a localised *vorticity* field $\boldsymbol{\omega} = \nabla\wedge\mathbf{v}$; the corresponding velocity field has a quadrupole character (the total impulse imparted by $\mathbf{j}\wedge\mathbf{B}$ being zero) and is therefore $O(r^{-4})$ as $r = |\mathbf{x}| \to \infty$. The kinetic energy of motion

$$K = \tfrac{1}{2}\int \rho\mathbf{v}^2 dV \tag{10}$$

is therefore certainly finite. Its rate of change is

$$\frac{dK}{dt} = \int \mathbf{v}\cdot(\mathbf{j}\wedge\mathbf{B})dV - \mu\int(\nabla\wedge\mathbf{v})^2 dV \tag{11}$$

the second term on the right representing viscous dissipation of energy (into heat). Coupling this with (6) gives

$$\frac{d}{dt}(K+M) = -\mu \int \omega^2 dV \tag{12}$$

so that the total energy certainly decreases monotonically. Since M is bounded below and K is positive, this means that $K+M$ must tend to a limit, and the enstrophy of the flow must tend to zero:

$$\int \omega^2 dV \to 0\,. \tag{13}$$

In this limit, if we ignore the (unphysical) possibility of the appearance and persistence of singularities in vorticity in a viscous fluid, the flow is irrotational; again excluding singularities, the only possibility compatible with the boundary condition $\mathbf{v} \to 0$ as $r \to \infty$ is that $\mathbf{v} = 0$ everywhere. Thus, as $t \to \infty$, the field structure does relax to an equilibrium minimum energy structure $\mathbf{B}^E(\mathbf{x})$, in which the force field $\mathbf{j}^E \wedge \mathbf{B}^E$ is in equilibrium with the pressure gradient ∇p^E, i.e.

$$\mathbf{j}^E \wedge \mathbf{B}^E = \nabla p^E\,, \tag{14}$$

and the fluid is at rest.

The dynamical model adopted above (eqn. 9) is not the only possibility, if the only requirements of the relaxation process are that (i) the topology of the **B**-field be conserved; and (ii) energy be dissipated. The requirement (i) is guaranteed by equation (1). The requirement (ii) may be satisfied by the simpler 'porous medium' model in which $\mathbf{v}$ is directly related to $\mathbf{j} \wedge \mathbf{B}$ via the equation

$$k\mathbf{v} = -\nabla p + \mathbf{j} \wedge \mathbf{B}\,. \tag{15}$$

The pressure term is still needed in order to ensure that $\mathbf{v}$ remains solenoidal (i.e. $\nabla \cdot \mathbf{v} = 0$). It is easy to show that the energy equation associated with (15) is now

$$\frac{dM}{dt} = -k \int \mathbf{v}^2 dV\,. \tag{16}$$

Kinetic energy does not appear since, in effect, in the dynamical model (15) inertia forces are neglected. Equation (16) implies monotonic decrease of field energy M until $\mathbf{v} \equiv 0$ (again neglecting the possibility of the appearance of unphysical singularities of $\mathbf{v}$). The model (15) has been adopted by Linardatos (1993) and by Chui & Moffatt (1996) in the determination of a variety of two-dimensional nonlinear magnetostatic equilibrium states.

3 Conservation of field helicity

There is a global invariant associated with the frozen field equation (1), which plays an important part in the subsequent application to knotted structures. This is the field *helicity* $\mathcal{H}$. To be specific, let us assume that the field $\mathbf{B}$ has compact support in $\mathbb{R}^3$, and let $\mathbf{A}$ be a vector potential satisfying

$$\mathbf{B} = \nabla \wedge \mathbf{A}\,. \tag{17}$$

Then the helicity of $\mathbf{B}$ is defined by

$$\mathcal{H} = \int \mathbf{A} \cdot \mathbf{B}\, dV\,, \tag{18}$$

the integral being over the support of $\mathbf{B}$ (or equivalently, over all space). As shown by Woltjer (1958), this helicity is invariant, a result interpreted by Moffatt (1969) in terms of the conserved linkages of magnetic field lines in a frozen-field situation. Note that $\mathcal{H}$ does not depend upon the choice of gauge for the field $\mathbf{A}$.

The interpretation of $\mathcal{H}$ for the case that is of particular interest here, when $\mathbf{B}$ is confined to a tubular neighbourhood $\mathcal{T}$ of a given knot K, has been established by Moffatt & Ricca (1992). Suppose that the knot K is itself a $\mathbf{B}$-line of the field, and that every $\mathbf{B}$-line in $\mathcal{T}$ is a closed curve which cuts any section of $\mathcal{T}$ normal to K only once and which has winding number h around K. Let Φ be the flux of $\mathbf{B}$ across each section of $\mathcal{T}$ normal to K. Then

$$\mathcal{H} = h\Phi^2\,. \tag{19}$$

Moreover, as shown by Călugăreanu (1959, 1961), h may be expressed as the sum of 'writhe' and 'twist' components:

$$h = Wr + Tw\,. \tag{20}$$

Here, Wr is defined as a double integral round K:

$$Wr = \frac{1}{4\pi} \oint_K \oint_K \frac{(d\mathbf{x} \wedge d\mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\,, \tag{21}$$

and is a number, determined solely by the geometry of K, and which varies continuously under continuous distortion of K.

If K is deformed to be nearly in a plane, then Wr tends to an integer equal to the number of positive crossings minus the number of negative crossings in the projected knot. More generally, Wr is the average over all projections of this difference between positive and negative crossings.

The twist Tw depends not only on the geometry of K, but also on the *relative geometry* of any two lines of force within the tube $\mathcal{T}$. Let s be arc length on K and let $\mathbf{N}(s)$ be a unit vector normal to K in the direction from K to a given $\mathbf{B}$-line in $\mathcal{T}$. Then, by definition,

$$Tw = \frac{1}{2\pi} \oint_K \left(\mathbf{N}' \wedge \mathbf{N}\right) \cdot \mathbf{t} ds \tag{22}$$

when $\mathbf{t}$ is a unit tangent vector on K. This can be thought of as the twist of a ribbon whose boundaries are K and the given $\mathbf{B}$-line, this twist being assumed the same for all $\mathbf{B}$-lines. The twist Tw can be further decomposed in the form

$$Tw = \mathcal{T} + n$$

where

$$\mathcal{T} = \frac{1}{2\pi} \oint_K \tau(s) ds \tag{23}$$

with $\tau(s)$ the *torsion* of K, and n is the number of rotations of the vector $\mathbf{N}(s)$ relative to the Frenet triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ on K in one passage around K. $\mathcal{T}$ and n are well-defined only if K has no inflexion points (i.e. points of zero curvature). If K is subjected to continuous distortion that takes it through an 'inflexional configuration' (i.e. a configuration having an inflexion point) then both $\mathcal{T}$ and n jump by an integer, the sum however varying continuously. This behaviour was recognized by Călugăreanu (1961); it was shown to be generic by Moffatt & Ricca (1992). Details of the mechanism by which twist may be converted to writhe have been recently investigated by Longcope & Klapper (1997).

4 Relaxation of knotted fields

Suppose now that the initial field $\mathbf{B}_0(\mathbf{x})$ of §2 is taken to be a 'tubular field', each pair of field lines in the tube having winding number h as described above. Under the relaxation process of §2, each field line 'wishes' to contract in length. During this process, the helicity $\mathcal{H} = h\Phi^2 = (Wr + Tw)\Phi^2$ is constant, and there may have to be some kind of trade-off between writhe and twist contributions. The volume V of the tube is also constant (under the assumption of incompressibility). Under these conditions, the minimum (or equilibrium) energy M^E is determined by the flux Φ, the volume V and the helicity $\mathcal{H}$, and by no other parameters. On dimensional grounds, it must take the form

$$M^E = m(h)\Phi^2 V^{-1/3} \tag{24}$$

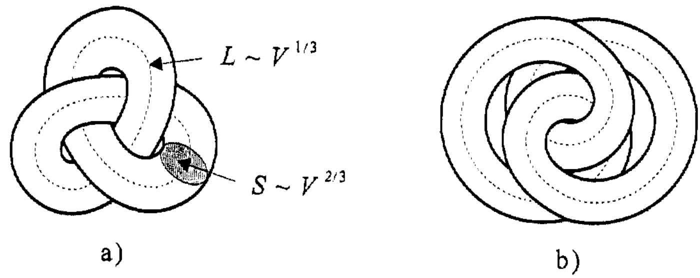


Figure 2: Two distinct minimum energy states that may be expected when K is the trefoil knot.

where $m(h)$ is a (dimensionless) function of the dimensionless parameter h (Moffatt 1990). We have until now restricted h to be an integer; however we may now allow h to be any real number; if h is rational, then the **B**-lines in $\mathcal{T}$ are closed curves, while if it is irrotational they do not close, but lie on nested toroidal surfaces with axis K. Note that, by Dehn surgery (cut, twist and reconnect the tube) the value of h may be set to any prescribed value for the initial field. This prescription of h is equivalent to prescribing a 'framing' of the knot. The particular choice $h = 0$ constitutes 'zero-framing'. This for example is the choice adopted by Berger & Field (1984).

Suppose we start with this natural choice $h = 0$. As noted in §4, relaxation proceeds through contraction of **B**-lines; for an untwisted tube, this can be best accomplished by contraction of the axis of the tube and corresponding increase of cross-section (to conserve volume). This process is arrested when the tube gets so 'fat' that it makes contact with itself in such a way that no further increase of cross-section is achievable. In this sense, the attainment of a minimum has clear parallels with the concept of 'ideal' knot configuration, developed by Katritch *et al* (1996). Of course, in the present context, the tube cross-section is not constrained to remain circular, and some flattening of the cross-section in regions where transverse contacts are made is to be expected.

A final caveat is needed: even when $h = 0$, there is no guarantee of uniqueness for the ultimate minimum energy state attained under a relaxation process. There may be more than one minimum (each being a 'local' minimum in the function space of accessible configurations), and in this case we should speak of the 'energy spectrum' of the knot, this being the whole set of energy levels of minimal states. Figure 2 shows two distinct minimum energy a)

b)

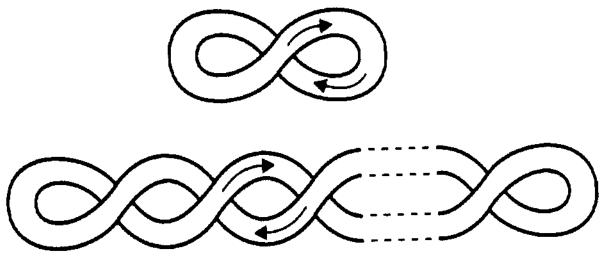


Figure 3: Preferred configurations of an unknotted but twisted tube, a) for $h = 1$, b) for $h = n > 1$.

states expected when K is the trefoil knot. This type of configuration has been studied in detail by Chui & Moffatt (1995).

5 Relaxation of strongly twisted tubes

If $|h|$ is large, then a very different behaviour is to be expected. In this situation, the twist component of field B_θ in the tube $\mathcal{T}$ is strong compared to the axial component B_s (θ being an angular coordinate in the plane of cross-section of $\mathcal{T}$), and relaxation is achieved by shrinkage of the cross-section S (with then B_θ decreasing like $S^{1/2}$) and associated *increase* in length L of the tube axis (with $V = LS$ =cst., and $B_s \sim L \sim S^{-1}$). This process reaches equilibrium when B_θ and B_s are of the same order of magnitude, with $B_s = \Phi/S$, $B_\theta \sim |h|\Phi S^{1/2}/V$, giving $S \sim |h|^{-2/3}$, $L \sim |h|^{2/3}$. The associated magnetic energy is of order $B_s^2 V$, i.e.

$$M^E \sim |h|^{4/3}\Phi^2 V^{-1/3}\,, \tag{25}$$

and this result apparently holds irrespective of knot type K.

There is however a further effect that leads to reduction of magnetic energy. This reduction can be achieved by a 'kink instability', analogous to the instability of an elastic wire subject to twist. This instability occurs even for an unknotted configuration; for example if the tube $\mathcal{T}$ has circular axis, and it is subjected to unit twist ($h = 1$), then it will 'prefer' to deform to a figure-of-eight configuration for which the twist Tw is reduced to zero, and the writhe Wr is increased to 1 (figure 3a); the increase in writhe energy in this deformation is more than compensated by the complete loss of twist energy.

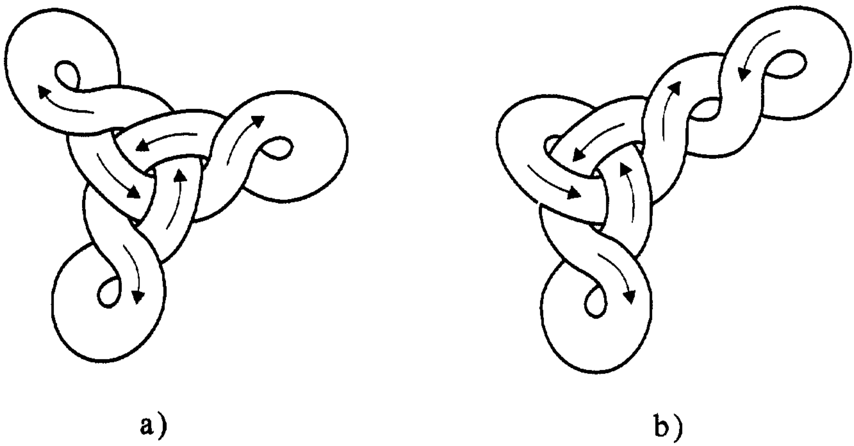


Figure 4: Conjectured minimum energy configurations for the trefoil knot when $h = 6$. a) symmetric state; b) non-symmetric state.

These energy changes have been calculated for the case of a twisted wire with standard elastic properties by Wadati *et al* (1989) (see also Calladine 1980), and the calculation for a magnetic flux tube is very similar.

Likewise, if h is any positive integer n, then again twist Tw may be reduced to zero via the deformation indicated in figure 3b. In this relaxed configuration, $Wr = n$ and the tube length is $L \sim 2nS^{1/2} = 2hS^{1/2}$; the magnetic energy $M = \frac{1}{2}B^2V$ with $B = \Phi/S$ and $V = LS$ is still given in order of magnitude by (25); the reduction in energy that can be achieved by the kink instability does not therefore change its order of magnitude. Its precise calculation involves determination of the dimensionless constant of proportionality in (25) for the two competing configurations (one with $Tw = n$, $Wr = 0$; the other with $Tw = 0$, $Wr = n$); this remains an open problem at present, although one may conjecture with confidence that the second configuration (depicted in figure 3b) wins. Similarly, for any knotted tube, twist can be converted to writhe through kink instabilities in disjoint sections. Figure 4a illustrates what may happen for the trefoil knot with $h = 6$. The tube can adjust itself so that $Tw = 0$, and $Wr = 3 + (3 \times 1)$ is accounted for by the writhe ($= 3$) of the trefoil knot itself and the additional writhe ($= 1$) of each of the three twisted arms in a symmetrical configuration. The possibilities for multiple (non-symmetric) minimum energy states become apparent here (figure 4b).

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