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VORTEX- AND MAGNETO-DYNAMICS – A TOPOLOGICAL PERSPECTIVE

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1 Introduction

The subject of vortex dynamics, within the broader field of fluid dynamics, was initiated by the pioneering studies of Helmholtz (1858) and Thomson (Lord Kelvin) (1869) on the laws of vortex motion. These laws are encapsulated in Kelvin's circulation theorem which applies to the motion of an ideal (i.e. inviscid) fluid in which pressure p and density ρ are functionally related, i.e. $p = p(\rho)$, and any body forces acting on the fluid are irrotational. Under these conditions, Kelvin showed that the circulation round any closed circuit C moving with the fluid is conserved:

$$\kappa = \oint_C \mathbf{u} \cdot d\mathbf{x} = \text{ cst.} \tag{1}$$

Here, $\mathbf{u}(\mathbf{x}, t)$ represents the velocity field. The circulation κ may equally be expressed as the flux of vorticity $\boldsymbol{\omega} = \text{curl}\, \mathbf{u}$ across any orientable surface S spanning C:

$$\kappa = \int_S \boldsymbol{\omega} \cdot \mathbf{n} dS\,, \tag{2}$$

and, since this applies to every material circuit C, including the boundaries of infinitesimal surface elements, it is readily deduced that "vortex lines are frozen in the fluid", i.e. the velocity field $\boldsymbol{\omega}(\mathbf{x}, t)$ is transported with the flow.

Kelvin immediately recognized that this result implied also the conservation of any linkage or any knottedness that might exist in the vorticity field at some reference instant $t = 0$. For example, if the vorticity field in a fluid is zero everywhere except in a closed tube which is knotted in the form of a knot K, then this topology of the vorticity field is conserved for all time; (it should be noted here and subsequently that this sort of result holds only for so long as the fluid can be regarded as truly inviscid; this is an idealisation that is never realised exactly in practice, except perhaps in liquid helium II in which quantum effects provide alternative complications). It was this insight that led Kelvin to propose his 'vortex theory of atoms' in which a correspondence is conjectured between atoms of different elements and knots of different knot types. This theory, although subsequently abandoned, provided a powerful stimulus for the major study of the classification of knots undertaken by Tait (1898, 1900) and in the subsequent development of topology as a distinct branch of mathematics.

2 Helicity and its topological interpretation

Remarkably, almost 100 years elapsed following Kelvin's great paper before the discovery of an invariant of the Euler equations of fluid motion which is truly topological in character and which indeed provides a natural bridge between fluid dynamics and topology. This invariant (J.-J. Moreau 1961, Moffatt 1969) is the *helicity* of a flow, defined as follows. Let S be any closed orientable surface moving with the fluid on which $\boldsymbol{\omega} \cdot \mathbf{n} = 0$, i.e. the vorticity field is tangential to S; S may be described as a 'vorticity surface', a condition that clearly persists if it holds for $t = 0$. The helicity of the flow in the volume V inside S is then defined by

$$\mathcal{H} = \int_V \mathbf{u} \cdot \boldsymbol{\omega} dV \,, \tag{3}$$

and this quantity is conserved under evolution governed by the Euler equations. To show this, it is best to express the Euler equations in Lagrangian form, viz.

$$\frac{D\mathbf{u}}{Dt} \left(= \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla h \tag{4}$$

where $h = \int \rho^{-1} dp$ (the need for the 'barotropic' condition $p = p(\rho)$ may be seen here). The curl of (4) coupled with the equation of mass conservation in the form

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} \tag{5}$$

leads to the vorticity equation in the form

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\right) \cdot \nabla \mathbf{u} \,. \tag{6}$$

Now from (3), it is readily shown that

$$\frac{d\mathcal{H}}{dt} = \int_V \frac{D\mathbf{u}}{Dt} \cdot \boldsymbol{\omega} dV + \int_V \mathbf{u} \cdot \frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) \rho dV \tag{7}$$

and, on using (4) and (6), this reduces to

$$\frac{d\mathcal{H}}{dt} = \int_S (\mathbf{n} \cdot \boldsymbol{\omega})(-h + \tfrac{1}{2}\mathbf{u}^2) dS = 0 \,, \tag{8}$$

on using the essential condition $\mathbf{n} \cdot \boldsymbol{\omega} = 0$ on S. Hence, $\mathcal{H}$ is indeed constant. Note that this result does not require that the flow be incompressible, although it does hold also in this special case (with $\rho =$ cst.). In general, it holds under precisely the same conditions that govern Kelvin's circulation theorem: inviscid fluid, barotropic flow, and irrotational body forces. It should be evident therefore that $\mathcal{H}$ must admit a topological interpretation.

That it does so is best seen through consideration of the simplest possible 'prototypical' linkage of vortex lines: consider the situation in which $\boldsymbol{\omega}$ is zero except inside two unknotted but linked vortex tubes of circulations κ_1 and κ_2 and of small cross-sections; and suppose that the vortex lines within each such tube are themselves unlinked closed curves. Then $\mathcal{H}$ may be evaluated by first integrating across the cross-section of each tube, then along their axes C_1 and C_2. The result is

$$\mathcal{H} = \pm 2n\kappa_1\kappa_2 \,, \tag{9}$$

where n is the number of times that C_1 winds round C_2 before closing on itself (the Gauss linking number of C_1 and C_2), and the + or − is chosen according as this linkage (which is oriented by the direction of vorticity within each tube) is right-handed or left-handed. The velocity field $\mathbf{u}$ can be expressed in terms of vorticity $\boldsymbol{\omega}$ by the Biot-Savart law, and this leads to the well-known expression for n as an integral:

$$n = \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{(d\mathbf{x}_1 \wedge d\mathbf{x}_2) \cdot (\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3} . \tag{10}$$

This is the fundamental topological invariant of the two closed curves C_1 and C_2, and the bridge between topology and fluid dynamics is therefore established by the simple result (9).

The situation is not so simple when knotted, as opposed to linked, vortex tubes are considered (Moffatt & Ricca 1992). Suppose now that $\boldsymbol{\omega}$ is zero except in a single closed vortex tube whose axis C is knotted in the form of a knot K. Suppose that each vortex line in the tube is a closed curve 'nearly parallel' to C, by which we mean simply that if C' is one such curve, then C and C' form the edges of a closed ribbon of small width. Let $\mathbf{N}(s)$ be a unit spanwise normal directed from C to C' on this ribbon,where s is arclength on C; then the *twist* of the ribbon (see, for example, Fuller 1971) is defined by

$$Tw = \frac{1}{2\pi} \oint_C \left(\mathbf{N}(s) \wedge \mathbf{N}'(s)\right) \cdot d\mathbf{x} . \tag{11}$$

This twist can be decomposed in two parts:

$$Tw = \frac{1}{2\pi} \oint_C \tau(s) ds + N \tag{12}$$

where $\tau(s)$ is the torsion on C and N, an integer, is the number of rotations of $\mathbf{N}(s)$ relative to the Frenet triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ of unit vectors on C. If such a ribbon is cut and one of the cut ends twisted through 2π and then rejoined, then N changes by ± 1, depending on the sense of twist. N may be described as the 'intrinsic twist' of the ribbon.

Now suppose that the vortex tube is 'uniformly twisted' in the sense that every pair of vortex lines C', C'' has the same value of intrinsic twist N. Then the result analogous to (8) is the following:

$$\mathcal{H} = h\kappa^2 , \tag{13}$$

where

$$h = Wr + Tw , \tag{14}$$

Tw is given by (11), and the *writhe* Wr is given by

$$Wr = \frac{1}{4\pi} \oint_C \oint_C \frac{(d\mathbf{x} \wedge d\mathbf{x}') \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} , \tag{15}$$

i.e. by the Gauss formula but with the integral taken twice round the same curve. Under continuous deformation of C, both Wr and Tw vary continuously, but their

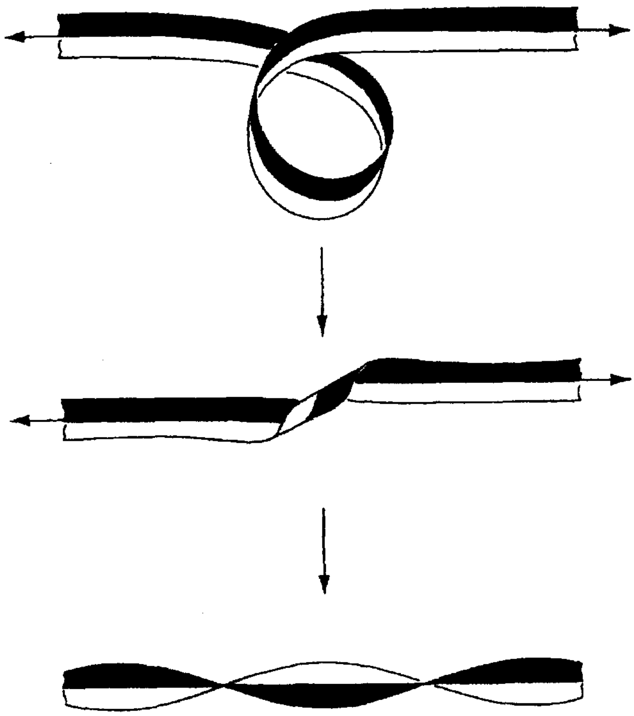

Figure 1. Conversion of writhe to twist through continuous distortion of a ribbon: (a) $Wr = 1, Tw = 0$; (b) $Wr + Tw = 1$; (c) $Wr = 0, Tw = 1$ [From Moffatt and Ricca (1992)].

sum is invariant (Călugăreanu 1961, White 1969). Again therefore it is conservation of helicity that actually underlies (via (14) and (15)) the essentially topological invariance of writhe plus twist.

There is a further subtlety in relation to the decomposition of twist (12) in which the first term depends only on C, while the second depends on the mutual configuration of C and C'. Under continuous deformation of the ribbon, it may at discrete instants pass through 'inflexional configurations', i.e. configurations for which C contains an inflexion point at which the curvature $c(s)$ is zero, and the torsion is undefined. As the ribbon passes through such a configuration, the integral of the torsion $\tau(s)$ jumps by $\pm 2\pi$, but there is a corresponding jump ∓ 1 in the integer N, so that the sum in (12) varies continuously through the transition. The delicate interplay of writhe, torsion and intrinsic twist can be visualised in the process of stretching a twisted ribbon (see figure 1).

3 Magnetic relaxation

For simplicity, let us focus now on the case of incompressible (or volume-preserving) flow with $\rho =$ cst. , for which $\nabla \cdot \mathbf{u} = 0$, and equation (6) may be written in the equivalent form

$$\partial\boldsymbol{\omega}/\partial t = \nabla\wedge(\mathbf{u}\wedge\boldsymbol{\omega}). \tag{16}$$

This equation is of course nonlinear (through the dependence of $\boldsymbol{\omega}$ on $\mathbf{u}$). It proves fruitful to consider an associated linear equation having a very similar structure, namely

$$\partial\mathbf{B}/\partial t = \nabla\wedge(\mathbf{v}\wedge\mathbf{B}) \tag{17}$$

where $\nabla\cdot\mathbf{B} = 0$, $\nabla\cdot\mathbf{v} = 0$ and $\mathbf{B}$ and $\mathbf{v}$ are otherwise independent fields. Equation (17) means that the field $\mathbf{B}(\mathbf{x},t)$ is transported by the 'velocity' field $\mathbf{v}(\mathbf{x},t)$, the flux of $\mathbf{B}$ through every material circuit being conserved. Equation (17) is in fact the equation satisfied by a magnetic field $\mathbf{B}$ in a perfectly conducting fluid moving with velocity $\mathbf{v}$. This interpretation may be helpful, but is by no means essential to the argument; we shall however use the terminology of magnetohydrodynamics (MHD) in what follows. The important property of (17) is that, no matter what the field $\mathbf{v}$ may be, it conserves the topological structure of $\mathbf{B}$, at least for all finite time; the question of what may happen as $t\to\infty$ is of particular interest, and will be discussed below.

Let us define the 'energy' M of the field $\mathbf{B}$ in the obvious way, i.e.

$$M = \tfrac{1}{2}\int \mathbf{B}^2 dV \tag{18}$$

where the integral is taken over the domain $\mathcal{D}$ of fluid, and it is supposed that

$$\mathbf{n}\cdot\mathbf{B} = 0\,,\ \mathbf{n}\cdot\mathbf{v} = 0 \ \text{ on } \ \partial\mathcal{D}. \tag{19}$$

We pose the question: can we choose $\mathbf{v}(\mathbf{x},t)$ in such a way that the energy M decreases to a minimum compatible with the conserved topology of $\mathbf{B}$? In fact, there are various possible ways of choosing $\mathbf{v}$ to achieve this end. The simplest choice (irrelevant constant factors being set equal to unity) is

$$\mathbf{v} = \mathbf{j}\wedge\mathbf{B} - \nabla p \tag{20}$$

where $\mathbf{j} = \nabla\wedge\mathbf{B}$ (the current distribution, if $\mathbf{B}$ is indeed a magnetic field), and p (the 'pressure field') is chosen so that $\nabla\cdot\mathbf{v} = 0$ and $\mathbf{n}\cdot\mathbf{v} = 0$ on $\partial\mathcal{D}$. We may write (20) in the more compact form

$$\mathbf{v} = (\mathbf{j}\wedge\mathbf{B})_S \tag{21}$$

where the notation $(\ldots)_S$ is used to denote the 'solenoidal projection' of a vector field. With this choice of $\mathbf{v}$, equation (17) becomes

$$\partial\mathbf{B}/\partial t = \nabla\wedge((\mathbf{j}\wedge\mathbf{B})_S\wedge\mathbf{B})\,, \tag{22}$$

an evolution equation with cubic nonlinearity. It follows moreover that

$$\begin{aligned}\frac{dM}{dt} &= \int \mathbf{B}\cdot\nabla\wedge(\mathbf{v}\wedge\mathbf{B})dV \\ &= \int(\nabla\wedge\mathbf{B})\cdot(\mathbf{v}\wedge\mathbf{B})dV \\ &= -\int \mathbf{v}^2 dV \end{aligned} \tag{23}$$

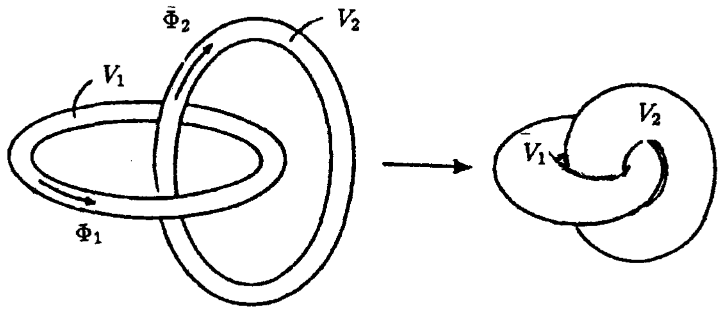

Figure 2. Relaxation of two linked flux tubes which lose energy through contraction.

on using the boundary condition (19) to eliminate the pressure term. Thus the energy of the field $\mathbf{B}$ does decrease monotonically for so long as $\mathbf{v} \not\equiv 0$.

However the conserved topology of $\mathbf{B}$ implies that (if this topology is nontrivial) there is a positive lower bound for M (Freedman 1988). Again, the prototypical configuration of two linked flux tubes makes this clear (figure 2): the field 'relaxes' as a result of contraction of the $\mathbf{B}$-lines (due to the Maxwell tension associated with the Lorentz force); both the fluxes Φ_1 and Φ_2 and the volumes V_1, V_2 are conserved during this process, which must evidently be arrested when the two tubes make contact with each other. 'Nontriviality' of the topology means simply that there exist field lines which cannot be continuously contracted to a point without 'trapping' other field lines in the process.

Thus, as $t \to \infty$, we must conclude that, for any nontrivial field topology,

$$\int \mathbf{v}^2 dV \to 0 \text{ and } M \to M^E , \tag{24}$$

where $M^E (> 0)$ is the asymptotic (relaxed) energy. Unless singularities of $\mathbf{v}$ appear during this relaxation process (a possibility that appears extremely unlikely, but has not as yet been rigorously eliminated), it follows further that

$$\mathbf{v} \to 0 \text{ and } \mathbf{B}(\mathbf{x}, t) \to \mathbf{B}^E(x)$$

as $t \to \infty$. From (20), the field $\mathbf{B}^E(\mathbf{x})$ satisfies

$$\mathbf{j}^E \wedge \mathbf{B}^E = \nabla p^E , \tag{25}$$

i.e. it is a magnetostatic equilibrium with pressure $p^E(\mathbf{x})$. The linked tube example suggests rather strongly that in general, the relaxed field $\mathbf{B}^E(\mathbf{x})$ may contain tangential discontinuities (as where the two tubes ultimately make contact); these tangential discontinuities of $\mathbf{B}^E$ are current sheets, and we should here emphasise that it is the assumption of perfect conductivity that permits the appearance of such current sheets. If the least resistivity is permitted in the fluid, then the current sheets will diffuse to finite thickness, and may be subject to 'resistive instabilities'; here we deliberately exclude such resistive effects.

4 Magnetic knots

The above relaxation process is particularly intriguing and illuminating when we consider the case of an initial field $\mathbf{B}_0(\mathbf{x})$ with 'knotted tube' topology. If the field lines within such a tube are 'uniformly twisted' so that the helicity (cf (13)) is given by $\mathcal{H} = h\Phi^2$, where Φ is the axial flux of $\mathbf{B}_0$, then the key parameters that remain constant during the relaxation process are h, Φ and the volume V of the tube (the fluid being still supposed incompressible). Hence the asymptotic energy M^E is determined by these three parameters, no others being available. There is only one dimensional possibility:

$$M^E = m(h)\Phi^2 V^{-1/3}, \tag{26}$$

a result first obtained by Moffatt (1990). Here $m(h)$ is a function of the dimensionless parameter h, and this function is determined (in principle) solely by the topology of the tube knot K. It may of course happen that there are multiple equilibrium states, which may be ordered so that

$$0 < m_0(h) \leq m_1(h) \leq m_2(h) \leq \dots . \tag{27}$$

We may then talk of the 'energy spectrum' of the knot,

$$\mathbf{m}(h) = (m_0(h), m_1(h), m_2(h), \dots) . \tag{28}$$

This type of argument may now be carried somewhat further. Suppose we consider the relaxed state of lowest energy $m_0(h)\Phi^2 V^{-1/3}$. The corresponding magnetostatic equilibrium $\mathbf{B}^E(\mathbf{x})$ can exist in an incompressible fluid at rest. Let us suppose that that fluid is ideal (i.e. inviscid as well as perfectly conducting). The equilibrium is stable (being one of minimum magnetic energy). We may ask, in the spirit of Kelvin, what are the normal modes of vibration about such an equilibrium? If we linearise the equations of ideal MHD,

$$\partial\mathbf{B}/\partial t = \nabla\wedge(\mathbf{u}\wedge\mathbf{B}) \tag{29}$$

$$\partial\mathbf{u}/\partial t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{j}\wedge\mathbf{B} \tag{30}$$

about the equilibrium state $\mathbf{u} = 0$, $\mathbf{B} = \mathbf{B}^E(\mathbf{x})$, then, writing

$$\mathbf{u} = \mathbf{u}_1(\mathbf{x})e^{i\omega t}\,,\ \mathbf{B} = \mathbf{B}^E(\mathbf{x}) + \mathbf{B}_1(\mathbf{x})e^{i\omega t}\,, \tag{31}$$

we obtain

$$i\omega\mathbf{B}_1 = \nabla\wedge(\mathbf{u}_1\wedge\mathbf{B}^E)\,, \tag{32}$$

$$i\omega\mathbf{u}_1 = -\nabla p_1 + \mathbf{j}_1\wedge\mathbf{B}^E + \mathbf{j}^E\wedge\mathbf{B}_1\,, \tag{33}$$

an eigenvalue problem (when coupled with appropriate boundary conditions) for the pair of fields $\{\mathbf{u}_1(\mathbf{x}), \mathbf{B}_1(\mathbf{x})\}$. Note that in the notation used here, the field $\mathbf{B}$ has been scaled so that its units are those of velocity (actually, $\mathbf{B}$ *is* the local Alfven velocity).

Here again there is presumably a spectrum of frequencies $0 < \omega_0 \leq \omega_1 \leq \omega_2 \leq \dots$, the fundamental frequency ω_0 being of greatest interest. Since this is determined in principle by the field $\mathbf{B}^E(\mathbf{x})$ which is in turn determined uniquely by h, Φ and V, we may conclude on dimensional grounds that

$$\omega_0 = \Omega_0(h)\Phi V^{-1} \tag{34}$$

where again $\Omega_0(h)$ is a dimensionless function of h determined solely by the topology of the knot K. Of course there will presumably be a matrix $\Omega_{ij}(h)$ of such functions, where j labels the member of the spectrum of relaxed fields, and i labels the normal mode of vibration of this member. It is of course one thing to assert the existence of such functions; it is an altogether more difficult matter, as yet beyond analytical or computational capabilities, to determine and evaluate them.

5 The relaxation of chaotic fields

The situation considered in §4 in which all magnetic field lines are closed curves is very exceptional. The field lines of an arbitrary field $\mathbf{B}(\mathbf{x})$ are the trajectories of the system

$$\frac{dx}{B_x(x,y,z)} = \frac{dy}{B_y(x,y,z)} = \frac{dz}{B_z(x,y,z)}, \tag{35}$$

and if the components B_x, B_y, B_z are nonlinear functions of (x,y,z), then in general these trajectories are chaotic within the domain $\mathcal{D}$ of definition of the field. An example of such a chaotic field within a sphere $|\mathbf{x}| < 1$ has been studied by Bajer & Moffatt (1990); the field is quadratic in the space variables, i.e.

$$B_i = c_{ijk} x_j x_k\,, \tag{36}$$

the tensor c_{ijk} being such that $\nabla \cdot \mathbf{B} = 0$ and $\mathbf{n} \cdot \mathbf{B} = 0$ on $|\mathbf{x}| = 1$. Even for this simple form of nonlinearity, the lines of the force of $\mathbf{B}$ are chaotic: they are not closed curves, neither do they lie on a family of surfaces. The system (35) is technically 'non-integrable'. Figure 3 shows a Poincaré section of the field for a particular choice of c_{ijk} – it shows the points in which a single field line of $\mathbf{B}$ intersects the plane of section; the widespread scatter of these points is a familiar symptom of chaotic behaviour; at the same time, one should note the existence of a certain order within this chaos, an order that can be analysed and understood by means of 'adiabatic' techniques.

Suppose now that we adopt such a field as the initial field $\mathbf{B}_0(\mathbf{x})$ in the relaxation process described in §3. During relaxation, the chaotic character of the field clearly persists – there is no obvious mechanism by which a field line which is initially chaotic could, under transport by a continuous velocity field, rearrange itself to lie upon a surface. The inference is that the relaxed field $\mathbf{B}^E(\mathbf{x})$ must also therefore exhibit the above symptom of chaos.

But here we are driven to a curious conclusion. The relaxed field $\mathbf{B}^E$ satisfies the magnetostatic condition (25), so that

$$\mathbf{B}^E \cdot \nabla p^E = 0\,, \tag{37}$$

i.e. the field lines of $\mathbf{B}^E$ lie on surfaces $p^E =$ cst. If, by the above argument, they do *not* lie on such surfaces, then ∇p^E must be identically zero in the region of

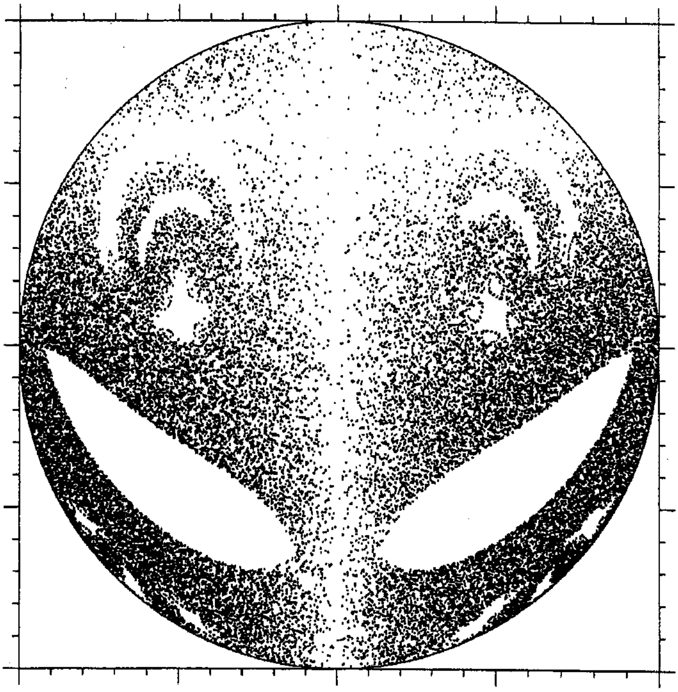

Figure 3. Poincaré section showing the intersections of a single chaotic field line of a quadratic field of the form (36) with an equatorial plane of section in the sphere $|x| < 1$ (from Bajer & Moffatt 1990).

chaos, and so $\mathbf{j}^E \wedge \mathbf{B}^E \equiv 0$. It then follows that

$$\mathbf{j}^E = \alpha \mathbf{B}^E \quad \text{where} \quad \mathbf{B}^E \cdot \nabla \alpha = 0 \tag{38}$$

and again, $\mathbf{B}^E$-lines lie on surfaces $\alpha =$ cst. , unless $\alpha \equiv 0$. It therefore appears that $\mathbf{B}^E$-lines can be chaotic in some subdomain $\mathcal{D}'$ of $\mathcal{D}$ only if

$$\nabla \wedge \mathbf{B}^E = \alpha \mathbf{B}^E \quad \text{in} \quad \mathcal{D}' \tag{39}$$

with α constant in $\mathcal{D}'$. Equation (39) expresses the fact that $\mathbf{B}^E$ is a *Beltrami field* in $\mathcal{D}'$, a very special type of field.

As first pointed out by Arnol'd (1974, 1986), there cannot possibly be enough generality in the solutions of (39) (if applied to $\mathcal{D}$ rather than $\mathcal{D}'$) to accommodate the arbitrary topology that may be assured for the initial chaotic field $\mathbf{B}_0(\mathbf{x})$. How can we escape this paradox?

The explanation that has been suggested (Moffatt 1985) is that, within any chaotic field, there are always 'islands of regularity' (large islands can be clearly seen in the example of figure 3, but there are many smaller islands also, below the level of visual detection). Under relaxation, the boundaries of these islands may become considerably distorted, and the subdomain $\mathcal{D}'$ of chaos acquires a correspondingly complex geometry. On this picture, the complexity of the initial field $\mathbf{B}_0$ translates to complexity of the geometry of the domain $\mathcal{D}'$ in which the relaxed field is chaotic. Whether this is the correct explanation must await direct numerical simulation of the relaxation process, a computational experiment that has not as yet been accomplished in three dimensions.

6 Two-dimensional relaxation

Numerical relaxation to minimum energy states has however been carried out for two-dimensional fields. These have the advantage that the topology can be completely prescribed (Moffatt 1999) in terms of the homoclinic field lines (or 'separatrices') through all the hyperbolic neutral points (i.e. 'saddle points') of the field. If placed on the sphere S^2, each such separatrix is a figure-of-eight and the generic separatrix structure consists of two families of nested figure-of-eights. Under relaxation, the incompressibility condition implies that the area $A(\chi)$ inside any field line $\chi =$ cst. remains constant; the function $A(\chi)$ (or set of such functions for different regions within separatrix loops) is called the signature of the field (Moffatt 1986a) and is invariant during relaxation; it is in effect a topological property of the field.

It has been pointed out in §3 that tangential discontinuities of $\mathbf{B}$ may appear during relaxation as $t \to \infty$. This can occur also in the two-dimensional case: such behaviour is located near the saddle points, and results from the collapse of the separatrices (to zero angle) near the saddle points (Linardatos 1993), a behaviour that has been subsequently re-examined and confirmed by Vainshtein *et al* (1999).

It may be conjectured that saddle points of a field $\mathbf{B}$ will play an equally significant role in three-dimensional relaxation; but it is by no means essential that saddle points be present to initiate such discontinuities (see Parker 1994 for an extended discussion of the spontaneous formation of such discontinuities in the important context of the solar coronal magnetic field).

7 Analogous Euler flows

What, it may be asked, does magnetic relaxation, in a perfectly conducting viscous fluid, have to do with the problem that we started with, namely the flow of an inviscid non-conducting fluid in the absence of any magnetic effects? The answer is it provides a powerful, albeit indirect, method for establishing the existence of steady Euler flows (i.e. steady solutions of the Euler equations of an incompressible fluid) having arbitrary streamline (NB *not* vortex line) topology. For the equation for such steady flow may be written in the form

$$\mathbf{u} \wedge \boldsymbol{\omega} = \nabla H \tag{40}$$

where $H = p/\rho + \frac{1}{2}\mathbf{u}^2$ is the Bernoulli function, and $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$. There is an obvious analogy between equations (25) and (40) through the identifications

$$\mathbf{B}^E \to \mathbf{u}\,,\; \mathbf{j}^E \to \boldsymbol{\omega}\,,\; p_0 - p^E \to H \tag{41}$$

where p_0 is an arbitrary constant. Thus the magnetic relaxation mechanism, which establishes the existence of fields $\mathbf{B}^E$ satisfying (25) (and recall that $\mathbf{j}^E = \nabla \wedge \mathbf{B}^E$), simultaneously determines an analogous Euler flow $\mathbf{u}$ via the analogy (41). Of course, care must be taken to ensure that the boundary conditions on the flow are compatible with the analogy (see Moffatt 1985).

Thus, for example, since the arguments of §§3 and 4 establish the existence of 'knotted magnetic flux tube equilibria' for any knot class K, it follows via the above analogy that steady Euler flows having similarly knotted streamtubes also exist! It is not quite as visualised by Kelvin who considered knotted vortex tubes; there *may* exist steady knotted vortex tube configurations, but no technique has as yet been found to prove the existence of such configurations.

Note that the tangential discontinuities of $\mathbf{B}^E$ (i.e. current sheets) that may appear during the relaxation process translate via the analogy (41) to tangential discontinuities of $\mathbf{u}$, i.e. vortex sheets, imbedded within the Euler flows thus determined. Now it is well-known that vortex sheets are prone to instability (the Kelvin-Helmholtz instability) and one may infer that the steady Euler flows may be unstable within the context of the Euler equations despite the fact that the analogous magnetostatic equilibria are, by their construction, stable within the context of the magnetohydrodynamic equations in a viscous, perfectly conducting, fluid.

This may appear surprising, but it should be recognised that the analogy (41) applies only to the steady states, but not to the (different) problems of the stability of these steady states. The differences between the two types of stability problem has been discussed by Moffatt (1986b); and it has in fact been shown by Rouchon (1991) that the sufficient condition for stability of an Euler flow obtained by Arnol'd (1966) is never satisfied for flows that are fully 3-dimensional and lack any obvious symmetry.

Thus, although the magnetic relaxation technique yields a rich harvest of information about the existence of steady solutions of the Euler equations, the downside is that any such solution of nontrivial topology is almost certainly unstable.

8 Relaxation to steady solutions of the MHD equations

Let us consider the full MHD equations for an ideal (i.e. inviscid, perfectly conducting) fluid in the form

$$\partial \mathbf{u}/\partial t = \mathbf{u} \wedge \boldsymbol{\omega} + \mathbf{j} \wedge \mathbf{B} - \nabla H \qquad (42)$$

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) \qquad (43)$$

and let us now, following Vladimirov, Moffatt & Ilin (1999), construct a relaxation process that yields topologically interesting steady solutions of these equations.

Note first that there are two classes of topological invariants (or 'Casimirs') associated with (42), (43); these are first the magnetic helicity invariants

$$\mathcal{H}_M = \int_V \mathbf{A} \cdot \mathbf{B} dV \qquad (44)$$

where $\mathbf{B} = \nabla \wedge \mathbf{A}$ (cf (3)), and V is any material volume on whose surface $\mathbf{B} \cdot \mathbf{n} = 0$; and second the cross-helicity invariants

$$\mathcal{H}_C = \int_V \mathbf{u} \cdot \mathbf{B} dV \,. \qquad (45)$$

The cross-helicity is topological in that it provides a measure of 'mutual linkage' between vorticity and magnetic fields; this is conserved even although vortex lines are no longer frozen in the fluid, the Lorentz force in (42) being in general rotational (Moffatt 1969).

In addition to these topological invariants, the total energy

$$E = \tfrac{1}{2}\int(\mathbf{u}^2 + \mathbf{B}^2)dV \tag{46}$$

is also an invariant of (42), (43), the integral being over the whole fluid domain. The evolution of the system (42), (43) follows a trajectory on which $E =$ cst., this trajectory lying on an 'isomagnetovortical' folium in the function space of solenoidal fields $\{\mathbf{u}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ – i.e. a subspace in which the set (44), (45) of Casimirs take prescribed values.

To construct a relaxation process in which energy decreases, we must obviously modify the dynamics in some way, and we seek to do this in a special way that still conserves the Casimirs. This can be done by replacing (42), (43) by the modified equations

$$\partial\mathbf{u}/\partial t = \mathbf{v}\wedge\boldsymbol{\omega} + \mathbf{c}\wedge\mathbf{B} - \nabla H \tag{47}$$

$$\partial\mathbf{B}/\partial t = \nabla\wedge(\mathbf{v}\wedge\mathbf{B}) \tag{48}$$

where $\mathbf{v}$ and $\mathbf{c}$ are arbitrary fields satisfying $\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{c} = 0$ (and corresponding boundary conditions). It may be verified directly that the Casimirs (44), (45) do indeed survive this modification; however we now find that

$$\frac{dE}{dt} = -\int\Big\{\mathbf{v}\cdot(\mathbf{u}\wedge\boldsymbol{\omega} + \mathbf{j}\wedge\mathbf{B}) + \mathbf{c}\cdot(\mathbf{u}\wedge\mathbf{B})\Big\}dV \tag{49}$$

and this is, in general, non-zero. We can ensure that E decreases through choosing $\mathbf{v}$ and $\mathbf{c}$ in an obvious way (cf 21):

$$\mathbf{v} = (\mathbf{u}\wedge\boldsymbol{\omega} + \mathbf{j}\wedge\mathbf{B})_S\ ,\quad \mathbf{c} = (\mathbf{u}\wedge\mathbf{B})_S\,. \tag{50}$$

It then follows that

$$\frac{dE}{dt} = -\int(\mathbf{v}^2 + \mathbf{c}^2)dV\,, \tag{51}$$

so that E is monotonic decreasing for so long as $\mathbf{v}$ and/or $\mathbf{c}$ are nonzero. Moreover, the Cauchy-Schwarz inequality,

$$\tfrac{1}{2}\int(\mathbf{u}^2 + \mathbf{B}^2)dV \geq |\int\mathbf{u}\cdot\mathbf{B}dV| = |\mathcal{H}_C| \tag{52}$$

here places an obvious positive lower bound on E whenever the total cross-helicity is non-zero. (Actually a nonzero value of $\mathcal{H}_C$ for any subdomain V bounded by a magnetic surface is sufficient to provide a lower bound for E. Hence, E tends to a positive limit, and so, from (51), excluding the possibility of (point) singularities appearing in $\mathbf{v}$ or $\mathbf{c}$, we must have $\mathbf{v}\to 0$, $\mathbf{c}\to 0$. Hence, from (50), the limit fields $\{\mathbf{u}^E(\mathbf{x}), \mathbf{B}^E(\mathbf{x})\}$ satisfy precisely the equations of *steady* MHD.

It would of course be nice to go to the limit of zero magnetic field in the above argument, which would yield a relaxation procedure for the Euler equations. However, the lower bound (52) gives no useful information in this limit, and the energy can relax to zero in this situation.

So we have the curious result that a technique is available for treatment of the ideal MHD equations, but this technique fails for what, on the face of it, is a simpler system, namely the Euler equations for ideal fluids. The Euler equations, in a sense, emerge victorious, resistant as yet to the above type of general treatment that is available for more complex systems. The great, and enduring, difficulty of the Euler equations lies in their purity, within which the central intractable nonlinearity continues to defy progress at a fundamental level. It is this purity and associated intractability that lies at the heart of the still unsolved problem of turbulence – a problem that will continue to challenge and frustrate for many decades into the 21st century.

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