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COURSE 6

# THE TOPOLOGY OF TURBULENCE

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# THE TOPOLOGY OF TURBULENCE

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## 1 Introduction

Topological considerations enter the study of turbulence through quantities, often expressed as integrals over the fluid domain, which in idealized (non-dissipative) circumstances, are constant in time, *i.e.* invariants of the flow. The simplest example occurs for two-dimensional incompressible inviscid flow for which the vorticity field $\omega(x, y, t)$ satisfies the equation

$$\frac{D\omega}{Dt} \equiv \frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0. \tag{1.1}$$

Here $\mathbf{u} = (u(x,y,t), v(x,y,t), 0)$ is the velocity field with $\omega = \partial v/\partial y - \partial u/\partial x$. Equation (1.1) tells us that the vorticity associated with any fluid element is constant. The "isovorticity lines" $\omega =$ cst. are "frozen" in the fluid, so that the area $A(\omega)$ inside any closed isovorticity line $\omega =$ cst. is conserved. This function, described (Moffatt 1986) as the "signature" of the vorticity field, may be thought of as a topological invariant under the continuous deformation of the vorticity field by its self-induced velocity field $\mathbf{u}$.

If $F(\omega)$ is an arbitrary function of $\omega$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{S_L} F(\omega) \mathrm{d}x \mathrm{d}y = \int_{S_L} F'(\omega) \frac{D\omega}{Dt} \mathrm{d}x \mathrm{d}y = 0, \tag{1.2}$$

where $S_L$ is a "patch" of fluid bounded by a closed curve $C_L$ that moves with the fluid. We use the suffix $L$ here for "Lagrangian". This gives the family of invariants

$$I\{F\} = \int_{S_L} F(\omega) \mathrm{d}x \mathrm{d}y \tag{1.3}$$

(of which the best known, with $F(\omega) = \omega^2$ and $S_L$ the whole fluid domain, is the enstrophy of the flow).

Two important questions arise from these elementary considerations: (i) how are these families of invariants to be generalized to three-dimensional flows? (ii) How are these "invariants" modified, and what role do they play, when weak dissipative (*e.g.* viscous) effects are taken into consideration?

In these lectures, we first provide some tentative approaches to answer these questions. Then we discuss some particular situations in which topological considerations play a central role.

## 2 The family of helicity invariants

Let $\mathbf{v}(\mathbf{x},t)$ be an arbitrary three-dimensional incompressible flow and let $\mathbf{B}(\mathbf{x},t)$ be an arbitrary solenoidal field satisfying the evolution equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{v} \wedge \mathbf{B}), \ \nabla \cdot \mathbf{B} = 0. \tag{2.1}$$

It is well-known that this equation describes "frozen-field" transport of $\mathbf{B}$, the flux of $\mathbf{B}$ across any material surface element being conserved. If $\mathbf{B}$ is interpreted as the magnetic field in a perfectly conducting fluid, then this is none other than Alfven's theorem in magnetohydrodynamics.

Now let $\mathbf{A}$ be a vector potential for $\mathbf{B}$ satisfying $\nabla \cdot \mathbf{A} = 0$; then "uncurling" (2.1) gives

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \wedge \mathbf{B} - \nabla\varphi \tag{2.2}$$

for some scalar field $\varphi$. Equations (2.1) and (2.2) can be written in equivalent Lagrangian form

$$\begin{aligned} \frac{DB_j}{Dt} &= B_j \frac{\partial v_i}{\partial x_j} \\ \frac{DA_i}{Dt} &= v_j \frac{\partial}{\partial x_i} A_j - \frac{\partial \varphi}{\partial x_j} \end{aligned} \tag{2.3}$$

where $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$. We define the helicity $\mathcal{H}$ of the field $\mathbf{B}$ within any (Lagrangian) magnetic surface $S_L$ on which $\mathbf{n} \cdot \mathbf{B} = 0$ by

$$\mathcal{H} = \int_{V_L} \mathbf{A} \cdot \mathbf{B} \mathrm{d}V. \tag{2.4}$$

It is easily shown that this integral is gauge invariant (*i.e.* invariant under replacement of $\mathbf{A}$ by $\mathbf{A} + \nabla\psi$). It is further easily shown that

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_{V_L} \frac{D}{Dt}(\mathbf{A} \cdot \mathbf{B})\mathrm{d}V = \int_{S_L} (\mathbf{n} \cdot \mathbf{B})(-\varphi + \mathbf{v} \cdot \mathbf{A})\mathrm{d}S \tag{2.5}$$

and this vanishes since $\mathbf{n} \cdot \mathbf{B} = 0$ on $S_L$. Hence $\mathcal{H}$ is invariant (Woltjer 1958).

The invariant has topological interpretation (Moffatt 1969; Arnol'd 1974); thus for example if the field $\mathbf{B}$ is identically zero except in two flux tubes carrying fluxes $\Phi_1$, $\Phi_2$, and linked with (Gauss) linking number $n$, then

$$\mathcal{H} = \pm 2n\Phi_1\Phi_2 \tag{2.6}$$

(where $V_L$ is taken to be a volume containing both tubes). This statement requires immediate qualification: it is true provided the $\mathbf{B}$-lines within either tube are closed curves with no linkage between any pair; in this case, each tube on its own makes no contribution to the helicity. It is just the linkage of the tubes that contributes, *via* the formula (2.6).

Suppose instead that we have a single flux tube with flux $\Phi$ and axis in the form of a circle, the $\mathbf{B}$-lines in the tube being all "parallel" circles. Then there is no linkage and the helicity is zero. Imagine now that we cut the tube, twist one cut end through an angle $2\pi$ and rejoin the ends. Now each pair of $\mathbf{B}$-lines is linked, with linking number 1; if we imagine this twisted tube as built up through the addition of incremental fluxes $\mathrm{d}\varphi$, the total helicity is given by

$$\mathcal{H} = \pm 2 \int_0^{\Phi} \varphi \mathrm{d}\varphi = \pm \Phi^2 \tag{2.7}$$

the $+$ or $-$ being chosen according as the twist is right- or left-handed.

If the angle of twist is $2\pi h$, instead of $2\pi$, then clearly the helicity is linear in $h$ and is therefore given by

$$\mathcal{H} = \pm h\Phi^2. \tag{2.8}$$

Returning now to the two linked flux tubes, we may allow for "twists" $h_1$ and $h_2$ in the two tubes, and the total helicity is then

$$\mathcal{H} = \pm h_1\Phi_1^2 \pm h_2\Phi_2^2 \pm 2n\Phi_1\Phi_2 \tag{2.9}$$

where, for each term, the sign is chosen according as the twist (or linkage) is right- or left-handed.

### *2.1 Chaotic fields*

In general, a field $\mathbf{B}$ in $\mathbb{R}^3$ may be expected to have chaotic field lines, *i.e.* field lines which do not lie on surfaces except in some sub-regions of "regularity" of the field. Examples of such fields may be found in Dombre *et al.* (1986) (the ABC-field) and in Bajer and Moffatt (1990) (the STF-field). Let $V_L$ be a subdomain in which the **B**-lines are chaotic; then for such a subdomain, we have the single helicity invariant (2.4), and there appears to be no analogue of the infinite families of invariants $A(\omega)$ and $I\{F\}$ introduced in Section 1. It may be noted that it is for this situation that Arnol'd (1974) established the generalized interpretation of (2.5) as the "asymptotic Hopf invariant" (*i.e.* asymptotic linking number) of the field **B** in the chaotic region.

### 2.2 *Simply degenerate fields*

In many situations of interest however, the **B**-lines do lie on a family of surfaces. Suppose this family is the family $\psi(\mathbf{x}, t) = 0$; then evidently, since the **B**-lines move with the fluid, we must have

$$D\psi/Dt = 0 \qquad \text{and} \qquad \mathbf{B}\cdot\nabla\psi = 0. \tag{2.10}$$

We may clearly define an invariant "helicity function"

$$\hat{\mathcal{H}}(\psi) = \int_{V_L(\psi)} \mathbf{A}\cdot\mathbf{B}\mathrm{d}V \tag{2.11}$$

where $V_L(\psi)$ is the volume inside the Lagrangian surface $\psi =$ cst. Here we have an obvious analogue of the signature function $A(\omega)$ of Section 1. Equally however if we define

$$h(\psi) = \mathrm{d}\hat{\mathcal{H}}/\mathrm{d}\psi \tag{2.12}$$

so that $h(\psi)\mathrm{d}\psi$ represents the (invariant) helicity trapped between surfaces labeled $\psi$ and $\psi+\mathrm{d}\psi$, then we may construct a family of invariants analogous to (1.3), namely

$$I\{F\} = \int_{V_L(\psi)} F(h(\psi))\mathrm{d}\psi \tag{2.13}$$

where $F$ is an arbitrary function of $h$; for

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \int F'(h)h'(\psi)\frac{D\psi}{Dt}\mathrm{d}\psi = 0 \tag{2.14}$$

by virtue of (2.11).

### 2.3 *Doubly degenerate fields*

In special circumstances, it may happen that every **B**-line is a closed curve; for example, the "twisted flux tube" described above, with angle of twist

$2\pi n$ ($n$ an integer) is a field for which each **B**-line is an unknotted closed curve, each pair of **B**-lines having linking number $n$. More generally, we may consider a situation in which the **B**-lines are the intersections of surfaces $\psi =$ cst. and $\chi =$ cst. (a doubly-infinite family), with

$$D\psi/Dt = 0, \; D\chi/Dt = 0, \; \mathbf{B} = \nabla\psi \wedge \nabla\chi. \tag{2.15}$$

Then (*cf.* (2.11)), we may define

$$\hat{\mathcal{H}}(\Delta\psi, \Delta\chi) = \int_{V_L(\Delta\psi,\Delta\chi)} \mathbf{A}\cdot\mathbf{B}\mathrm{d}V \tag{2.16}$$

where $\Delta\psi = \psi_1 - \psi_2$, $\Delta\chi = \chi_1 - \chi_2$, and $V_L$ is the tube-like volume bounded by the surfaces labeled $\psi_1, \psi_2, \chi_1, \chi_2$. Defining $h(\psi,\chi)$ by

$$h(\psi,\chi)\mathrm{d}\psi\mathrm{d}\chi = \hat{\mathcal{H}}(\mathrm{d}\psi, \mathrm{d}\chi) \tag{2.17}$$

for increments $\mathrm{d}\psi$, $\mathrm{d}\chi$, we now have a family of invariants (*cf.* (2.13)) of the form

$$I\{F\} = \int_{V_L(\Delta\psi,\Delta\chi)} F(h(\psi,\chi))\mathrm{d}\psi\mathrm{d}\chi. \tag{2.18}$$

Note that the greater the degree of degeneracy of the field, the richer is the family of topological invariants that it exhibits. These invariants are all constructed from (2.4), in which the Lagrangian volume $V_L$ is constrained only by the requirement that its surface $S_L$ be a surface on which $\mathbf{n}\cdot\mathbf{B} = 0$; the greater the degeneracy of the field, the greater is the freedom in the choice of such volumes.

## 3 The special case of Euler dynamics

If, in the above theory, we choose $\mathbf{v} = \mathbf{u}(\mathbf{x}, t)$, a velocity field evolving under the incompressible Euler equations, and $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$, the corresponding vorticity field, then (2.1) is clearly satisfied since

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \wedge (\mathbf{u} \wedge \boldsymbol{\omega}), \; \nabla \cdot \boldsymbol{\omega} = 0. \tag{3.1}$$

The special feature here is that the field $\boldsymbol{\omega}$ is transported by its own "self-induced" velocity field $\mathbf{u}$. This coupling of $\mathbf{u}$ to $\boldsymbol{\omega}$ makes (3.1) nonlinear; but this does not invalidate the manipulation leading to (2.5) which, with

$$\mathcal{H} = \int_{V_L} \mathbf{u}\cdot\boldsymbol{\omega}\mathrm{d}V, \tag{3.2}$$

now becomes

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \int_{S_L} (\mathbf{n}\cdot\boldsymbol{\omega}) \left(-p + \frac{1}{2}\mathbf{u}^2\right) \mathrm{d}V. \qquad (3.3)$$

Here $p$ is the fluid pressure, and for simplicity of notation we have taken the fluid density to be $\rho = 1$. Thus again, $\mathcal{H} =$ cst. provided $\mathbf{n}\cdot\boldsymbol{\omega} = 0$ on $S_L$ (a condition that persists under Euler evolution).

Thus the helicity $\mathcal{H}$ is an invariant of the nonlinear system (3.1). However, it should be interpreted within the broader canvass of Section 2, involving the system (2.1), which (for any given $\mathbf{v}$) is evidently linear in $\mathbf{B}$. From this point of view, it is evident that, under "artificial Euler dynamics" for which (3.1) is replaced by

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\wedge(\mathbf{v}\wedge\boldsymbol{\omega}),\ \ \nabla\cdot\boldsymbol{\omega} = 0 \qquad (3.4)$$

with $\boldsymbol{\omega} = \nabla\wedge\mathbf{u}$ and $\mathbf{v}$ an arbitrary field satisfying $\nabla\cdot\mathbf{v} = 0$, the helicity (3.2) is still invariant; this is because $\boldsymbol{\omega}$ is now transported by the $\mathbf{v}$-field and its topology is still conserved. In particular, if $\mathbf{v}$ is some functional of $\mathbf{u}$ (as for example in the artificial Euler dynamics of Vallis *et al.* 1989), helicity (and $\boldsymbol{\omega}$-topology) are still conserved. The more recent "averaged Euler equations" of Marsden *et al.* (2000) appear to have similar conservation properties.

## 4 Scalar field structure in 2D flows

The structure of a scalar field $s(\mathbf{x})$ may be described in terms of the critical points of the field where $\nabla s = 0$ and the structure of the iso-surfaces $s =$ cst. near these points (for details, see Moffatt 2001). Consider first the situation in a 2D periodic domain (as frequently adopted in numerical experiments on 2D turbulence). Here, the field $s$ may be the streamfunction $\psi$ of the flow, or the vorticity field $\omega = -\nabla^2\psi$, or it may be some other scalar field of physical significance, such as the pressure field $p$. The critical points are either elliptic (extrema) or hyperbolic (saddle points). In a periodic domain, the number of extrema is equal to the number of saddle points (a consequence of Euler's index theorem). In an evolving situation, pairs of critical points (always one saddle and one extremum) can appear or disappear (through "saddle-node" bifurcation). If $s$ is the vorticity field $\omega$, then it is obvious that such bifurcation (with consequent change of topology of the $\omega$-field) can occur only through the agency of viscosity.

In 2D turbulence, one may distinguish between the "eddies" that may be observed in instantaneous streamline plots $\psi =$ cst., and "vortices" that may be observed in instantaneous isovorticity plots $\omega =$ cst. Indeed, if we *define* an eddy as the region of closed streamlines circulating round an extremum of $\psi$, then the number of eddies $N_\psi$ in the domain is precisely the number of extrema of $\psi$. We may define a vortex similarly in terms of the $\omega$-field, so that the number of vortices $N_\omega$ is the number of extrema of $\omega$ (some positive, some negative). In a typical field of 2D turbulence at high Reynolds numbers, the vorticity field shows much more structure than the velocity field; this visual property suggests that $N_\omega$ is much greater than $N_\psi$. In fact, these numbers depend on the energy spectrum $E(k)$ of the turbulence. If there is an inertial range in which

$$E(k) \sim k^{-\lambda} \qquad (k_1 \ll k \ll k_2) \tag{4.1}$$

with $1 < \lambda < 5$, then $N_\psi$ and $N_\omega$ scale as

$$N_\psi \sim k_1^2 A, \; N_\omega \sim k_2^2 A \tag{4.2}$$

(Moffatt 2001) where $A$ is the area of the domain of periodicity of the flow, and the behavior $N_\omega \gg N_\psi$ is evident. The condition $1 < \lambda < 5$ is satisfied for all reasonable models of 2D turbulence (*e.g.* the enstrophy-cascade model of Kraichnan 1967 and Batchelor 1969 with $\lambda = 3$; or the spiral wind-up model of Gilbert 1988 for which $3 < \lambda < 4$).

## 5 Scalar field structure in 3D flows

The critical points of a scalar field $s(\mathbf{x})$ in 3D are again elliptic or hyperbolic, the elliptic points being either maxima or minima of the field. The hyperbolic (saddle)points are of two distinct types depending on whether $s$ decreases in one principal direction away from the saddle (and increases in the other two directions), or *vice versa.* Bifurcations of such a field can create (or destroy) critical points in pairs: a minimum and a saddle point of the first kind; a maximum and a saddle point of the second kind; or a pair of saddle points, one of each kind. Each such topological transition is compatible with Euler's index theorem, which for the case of a space periodic field, takes the form

$$n_0 - n_1 + n_2 - n_3 = 0; \tag{5.1}$$

here $n_0$ is the number of minima, $n_3$ the number of maxima and $n_1, n_2$ the number of saddle points of first and second kinds respectively. ($n_\alpha$ is the number of critical points, always assumed non-degenerate, with index $\alpha$; $\alpha$ is the number of negative eigenvalues of the matrix $\partial^2 s/\partial x_i \partial x_j$ at the critical point.) It is worth noting that, since saddle points may be created in pairs, whereas extrema cannot be created without creating an equal number of saddle points, the number of saddle points will in general be greater (and possibly much greater) than the number of extrema.

A simple measure of the topological complexity of a scalar field $s(\mathbf{x})$ in 3D is given by the number $N$ of critical points of the field per unit volume (irrespective of type). If $s(\mathbf{x})$ is a passive scalar field subjected to convection with negligible diffusion, then $Ds/Dt = 0$, and the number $N$ is conserved (since topological transitions involving either increase or decrease of $N$) cannot occur.

In the presence of weak molecular diffusivity $\kappa$, the field $s$ is governed by the advection-diffusion equation

$$Ds/Dt \equiv \partial s/\partial t + \mathbf{u} \cdot \nabla s = \kappa \nabla^2 s. \tag{5.2}$$

Following Batchelor (1959) and Batchelor *et al.* (1959), we may suppose that gradients of $s$ are maintained at a large length-scale, and that fluctuations of $s$, relative to its mean, cascade to very small "diffusive" scales, at which they are eliminated. In these circumstances, the number $N$ may be expected to attain a large mean value determined by the detailed process of molecular diffusion (which both creates and destroys critical point pairs in equal measure). When $\kappa \geq \nu$ (kinematic viscosity), the wave-number beyond which molecular diffusion becomes important is $k_c \sim (\epsilon/\kappa^3)^{1/4}$ (the "conduction cut-off") and the number $N$ of critical points in a periodicity volume $V$ may be estimated as

$$N \sim k_c^3 V = (\epsilon/\kappa^3)^{3/4} V. \tag{5.3}$$

The situation when $\kappa \ll \nu$ is physically less clear; all one can say in this case is that $N/V$ must be at least of order $k_v^3$, where $k_v = (\epsilon/\nu^3)^{1/4}$ is the conventional "Kolmogorov cut-off".

Numerical simulations of the passive scalar field problem (with artificial "prescribed" $\mathbf{u}$-field having a Kolmogorov spectrum) can be carried out, and it should be a straightforward matter to locate and count the critical points of the scalar field, with a view to testing the above conclusions.

## 6 Vector field structure in 3D flows

If we pass from the passive scalar field problem to the passive vector field problem for which a field $\mathbf{B}(\mathbf{x}, t)$ (conventionally magnetic field) evolves according to the equation

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \ \nabla \cdot \mathbf{B} = 0, \tag{6.1}$$

then we are faced with the extremely difficult problem of classifying possible topological structures for the field $\mathbf{B}$. We have already noted the possibility of a "degenerate" field whose $\mathbf{B}$-lines lie on surfaces $\psi =$ cst. However, for general $\mathbf{u}$, such a condition will persist only if $\eta = 0$. If $\eta \neq 0$, then the

**B**-lines cannot be expected to remain on surfaces (it would be interesting to investigate for what special class (if any) of flows **u**, the **B**-lines *do* remain on surfaces!). Generically, the **B**-lines must be expected to have a chaotic character. Two such fields (already mentioned in Sect. 2) have been studied in some detail. The first is the "ABC-field",

$$\mathbf{B} = (C\sin kz + B\cos ky,\ A\sin kx + C\cos kz,\ B\sin ky + A\cos kx) \quad (6.2)$$

which, for arbitrary constants $A, B, C$, satisfies the Beltrami condition $\nabla\wedge\mathbf{B} = k\mathbf{B}$. The chaotic character of the **B**-lines of this field (when $ABC \neq 0$) was conjectured by Arnold (1965), explored numerically by Hénon (1966), and analyzed further in detail by Dombre *et al.* (1986). The second is the "STF-field"

$$\mathbf{B} = (\alpha z - 8xy,\ 11x^2 + 3y^2 + z^2 + xy - 3,\ -\alpha x + 2yz - xy) \quad (6.3)$$

(Bajer and Moffatt 1990), which, as may be easily verified, satisfies $\nabla \cdot \mathbf{B} = 0$ and $\mathbf{n} \cdot \mathbf{B} = 0$ on $|\mathbf{x}| = 1$. (The field also satisfies the "Stokes" condition' $\nabla^2(\nabla\wedge\mathbf{B}) = 0$.) When the parameter $\alpha$ equals zero, the field **B** is doubly degenerate, each **B**-line being a closed curve; this imposes a certain character on the chaotic wandering of **B**-lines for $\alpha \neq 0$. Note that the STF-field was originally constructed as a velocity field in a spherical ball $|\mathbf{x}| < 1$ having a combination of stretch, twist and fold ingredients – hence the STF-label.

Apart from these two fields, whose properties are now reasonably well documented, very little is known concerning the generic structure of solenoidal vector fields in 3D. The beginnings of a general structural theory may perhaps be seen in the work of Ghrist (1997) and Kuperberg (1999); but these are only the beginnings, and the general classification problem for such 3D fields is still wide open.

## 7 Helicity and the turbulent dynamo

The evolution of a magnetic field in a conducting fluid moving with velocity $\mathbf{u}(\mathbf{x}, t)$ is governed by equation (6.1). It has long been recognized that field intensification will result from the stretching of **B**-lines associated with the exponential separation of initially neighboring particles in turbulent flow. However, this intensification is achieved at the expense of a reduction of scale in the **B**-field, and a consequential enhancement of the effect of diffusion represented by the term $\eta\nabla^2\mathbf{B}$ in (6.1); this ultimately becomes important no matter how small the diffusivity parameter $\eta$ may be. The question then arises whether sustained intensification of magnetic field (or "dynamo" action) can arise, and if so through what mechanism that somehow bypasses the "enhanced diffusivity" effect.

As pointed out earlier, if the velocity field $\mathbf{u}$ can be specified independently of $\mathbf{B}$, then (6.1) is linear in $\mathbf{B}$, which behaves as a "passive vector field". The only constraint that need be imposed on $\mathbf{u}$ is the kinematic constraint of incompressibility $\nabla \cdot \mathbf{u} = 0$; and the corresponding phase of field evolution is covered by the term "kinematic dynamo problem". If the field grows exponentially, then this phase cannot last forever, because the Lorentz force $\mathbf{j} \wedge \mathbf{B}$ (with $\mathbf{j} = \mu_0^{-1} \nabla \wedge \mathbf{B}$) obviously reacts back upon the dynamics of the flow, *i.e.* $\mathbf{u}$ can no longer be specified independently of $\mathbf{B}$; we then enter upon the phase of the fully "magnetohydrodynamic dynamo problem", when it becomes necessary to specify also the nature of the forces agitating the fluid (or equivalently the nature of the source of kinetic energy of the turbulence).

### *7.1 The kinematic phase*

A double-length-scale approach is usually adopted as a starting point. It is supposed that the dominant scale of the turbulence (*i.e.* the scale of the "energy-containing eddies") is $l_0$, and one focuses on the evolution of the field $\bar{\mathbf{B}}(\mathbf{x}, t)$ on a scale $L$ much greater than $l_0$. This is the "mean-field" approach pioneered by Steenbeck *et al.* (1966) (see Moffatt 1978; Krause and Rädler 1980). If the turbulence is homogeneous with zero mean, then a mean electromotive force $\mathcal{E}(\mathbf{x}, t)$ is generated on the scale $L$, given by an expansion of the form

$$\mathcal{E}_i = \langle \mathbf{u} \wedge \mathbf{b} \rangle_i = \alpha_{ij} \bar{B}_j + \beta_{ijk} \partial \bar{B}_j / \partial x_k + \dots , \tag{7.1}$$

where $\alpha_{ij}$, $\beta_{ijk}$, ... are pseudo-tensors determined (in principle) by the statistical properties of the turbulence and the parameter $\eta$. The field $\mathbf{b}$ is the small-scale magnetic perturbation induced by the flow $\mathbf{u}$ across $\bar{\mathbf{B}}$. The important feature of (7.1) is the linearity between the fields $\mathcal{E}$ and $\bar{\mathbf{B}}$. This linearity results from the linearity of (6.1); no assumption need be made concerning the relative magnitudes of $|\mathbf{b}|$ and $|\bar{\mathbf{B}}|$ in order to deduce the structure (7.1). Conventional estimates suggest that

$$\mathbf{b} = O(R_m) \bar{\mathbf{B}}, \tag{7.2}$$

where

$$R_m = (\bar{\mathbf{u}}^2)^{1/2} l_0 / \eta, \tag{7.3}$$

the magnetic Reynolds number of the turbulence. In situations of interest in planetary physics and astrophysics, $R_m$ is a least of order unity, and generally much greater than unity; in these circumstances, the fluctuating ingredient of the field $\mathbf{b}$ may be much greater than the mean $\bar{\mathbf{B}}$.

If the turbulence is isotropic as well as homogeneous, then the pseudo-tensors $\alpha_{ij}$, $\beta_{ijk}$, ... must also be isotropic, *i.e.*

$$\alpha_{ij} = \alpha\delta_{ij}, \ \beta_{ijk} = \beta\epsilon_{ijk}, \ \dots \tag{7.4}$$

where $\alpha$ is a pseudo-scalar and $\beta$ a pure scalar (the "pseudo" property of $\beta_{ijk}$ being taken up by the pseudo-tensor $\epsilon_{ijk}$). The expansion (7.1) then takes the simpler form

$$\mathcal{E} = \alpha\bar{\mathbf{B}} - \beta\nabla\wedge\bar{\mathbf{B}} + \dots, \tag{7.5}$$

so that (with $\alpha, \beta$ constants by virtue of homogeneity),

$$\nabla\wedge\mathcal{E} = \alpha\nabla\wedge\bar{\mathbf{B}} + \beta\nabla^2\bar{\mathbf{B}}. \tag{7.6}$$

The "mean-field" equation for $\bar{\mathbf{B}}$ then becomes

$$\partial\bar{\mathbf{B}}/\partial t = \alpha\nabla\wedge\bar{\mathbf{B}} + \eta_T\nabla^2\bar{\mathbf{B}}, \tag{7.7}$$

where $\eta_T = \eta+\beta$, a diffusivity augmented by the turbulent contribution of $\beta$; one would clearly expect $\beta$ to be positive, although there is no guarantee of this, at this level of argument.

Equation (7.5) clearly admits exponentially growing modes of "force-free" structure satisfying

$$\nabla\wedge\bar{\mathbf{B}} = K\bar{\mathbf{B}}, \tag{7.8}$$

where $K$ is a constant; for such modes satisfy

$$\partial\bar{\mathbf{B}}/\partial t = \alpha K\bar{\mathbf{B}} - \eta_T K^2\bar{\mathbf{B}}, \tag{7.9}$$

and hence $\bar{\mathbf{B}} \sim e^{pt}$ where

$$p = \alpha K - \eta_T K^2, \tag{7.10}$$

showing exponential growth provided

$$|\alpha/K| > \eta_T. \tag{7.11}$$

This condition is satisfied if $K$ has the same sign as $\alpha$ and the scale $L \sim |K^{-1}|$ of $\bar{\mathbf{B}}$ is sufficiently large (consistent with the "two-scale" assumption $L \gg l_0$).

This dynamo mechanism is due to the "generation term" $\alpha\nabla\wedge\bar{\mathbf{B}}$ in (7.5), or equivalently to the term $\alpha\bar{\mathbf{B}}$ in (7.3). This is the famous "$\alpha$-effect"; a growing understanding of this generic phenomenon has been one of the most dramatic developments of turbulence theory of the last half-century.

The fact that $\alpha$ is a pseudo-scalar implies that the effect can occur only in turbulence that "lacks reflexional symmetry". The mean helicity $\mathcal{H} = \langle \mathbf{u} \cdot \boldsymbol{\omega} \rangle$ of the turbulence (also a pseudo-scalar) is in general non-zero in such circumstances, and a link between $\alpha$ and $\mathcal{H}$ is to be expected.

To illustrate the essential mechanism by which an $\alpha$-effect is produced, let us calculate $\alpha$ for the case of an "ABC"-type velocity field (with $A = B = C = u_0$)

$$\mathbf{u} = u_0(\sin(kz - \omega t) + \cos(ky - \omega t),\ \sin(kx - \omega t) + \cos(kz - \omega t),\\ \sin(ky - \omega t) + \cos(kx - \omega t)), \tag{7.12}$$

for which $\boldsymbol{\omega} = k\mathbf{u}$, and so

$$\mathcal{H} = \langle \mathbf{u} \cdot \boldsymbol{\omega} \rangle k\bar{\mathbf{u}}^2 = 3ku_0^2. \tag{7.13}$$

The helical character of (7.9) is evident in that the motion consists of a superposition of three circularly polarized waves propagating parallel to the axes $Ox, Oy, Oz$.

To calculate $\alpha$, it is legitimate to assume a uniform mean field $\bar{\mathbf{B}}$, and the equation for the fluctuating field $\mathbf{b}$ (from 6.1) is then

$$\partial \mathbf{b}/\partial t = (\bar{\mathbf{B}} \cdot \nabla)\mathbf{u} + \nabla \wedge \mathbf{G} + \eta \nabla^2 \mathbf{b}, \tag{7.14}$$

where $\mathbf{G} = \mathbf{u} \wedge \mathbf{b} - \langle \mathbf{u} \wedge \mathbf{b} \rangle$. If the wave amplitude $u_0$ is sufficiently weak, then the nonlinear term $\nabla \wedge \mathbf{G}$ may be neglected (the "first-order smoothing" approximation); the field $\mathbf{b}$ may then be found by elementary methods, and $\mathcal{E} = \langle \mathbf{u} \wedge \mathbf{b} \rangle$ constructed. The result is indeed $\mathcal{E} = \alpha \bar{\mathbf{B}}$, where

$$\alpha = -\frac{1}{3}\left(\frac{\eta k^2}{\omega^2 + \eta^2 k^4}\right)\mathcal{H} \tag{7.15}$$

showing the expected dependence on mean helicity. Note also (i) the negative sign in (7.15), *i.e.* positive helicity generates a negative $\alpha$-effect, and (ii) the fact that (for $\omega \neq 0$), $\alpha \to 0$ as $\eta \to 0$, *i.e.* an $\alpha$-effect (at least in this first-order smoothing approximation) requires a non-zero molecular diffusivity $\eta$.

The helicity $\langle \mathbf{b} \cdot \nabla \wedge \mathbf{b} \rangle$ of the fluctuation field may be calculated under first-order smoothing (Moffatt 1978, Sect. 11.2). Not surprisingly, it has the same sign as the "driving" kinetic helicity $\langle \mathbf{u} \cdot \boldsymbol{\omega} \rangle$. Now look again at the mean-field equation (7.5) (with now slowly varying $\bar{\mathbf{B}}$); if (say) $\langle \mathbf{u} \cdot \boldsymbol{\omega} \rangle$ is positive, then $\alpha$ is negative and so negative helicity $\bar{\mathbf{B}} \cdot \nabla \wedge \bar{\mathbf{B}}$ is generated in the large-scale field. It seems that the positive magnetic helicity that is generated (and dissipated) on scales of order $l_0$ is at least partly compensated by negative magnetic helicity generated on the large scales of order $L \gg l_0$. We know that if $\eta = 0$, then total magnetic helicity is conserved. It would appear that dissipation of small-scale helicity is essential for dynamo action precisely because this allows the sustained increase of magnetic helicity (of opposite sign) in the large-scale field.

This manifestation of the effect of helicity in the turbulence was recognized in the seminal paper of Pouquet *et al.* (1976), who used the "eddy-damped quasi-normal Markovian" (EDQNM) closure scheme to analyze spectral evolution of fully magnetohydrodynamic turbulence. More recent direct numerical simulation of MHD turbulence (*e.g.* Brandenberg 1992), although limited to rather modest Reynolds and magnetic Reynolds numbers, shows similar trends.

### *7.2 The dynamic phase*

Let us now consider briefly what happens when dynamo action occurs as a result of the $\alpha$-effect, and a large-scale magnetic field grows exponentially until the stage at which the back-reaction of the Lorentz force on the flow becomes important. As previously indicated, it is necessary at this stage to specify the nature of the source of energy for the motion. Let us suppose for the sake of argument that there is a random body force $\mathbf{f}(\mathbf{x}, t)$ on the scale $l_0$, and that this force is independent of both $\mathbf{u}$ and $\mathbf{B}$.

The large-scale field $\bar{\mathbf{B}}$ evolves relatively slowly, and may, for the purpose of this analysis, be treated as constant. When $\bar{\mathbf{B}}$ is sufficiently strong, its effect is to severely control the amplitude of motion on scales of order $l_0$. It is reasonable then to linearize the equations for the velocity $\mathbf{u}$ and fluctuating field $\mathbf{b}$. These equations (in units such that $\mu_0\rho = 1$) become

$$\left.\begin{aligned} \partial\mathbf{u}/\partial t &= -\nabla P + \bar{\mathbf{B}}\cdot\nabla\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{f} \\ \partial\mathbf{b}/\partial t &= \bar{\mathbf{B}}\cdot\nabla\mathbf{u} + \eta\nabla^2\mathbf{b} \end{aligned}\right\} \tag{7.16}$$

where $P$ is the sum of fluid and magnetic pressure. These are just the equations for forced Alfven waves traveling on the field $\bar{\mathbf{B}}$; indeed if we consider a single Fourier component $\hat{f}\exp i(\mathbf{k}\cdot\mathbf{x} - \omega t)$ of $\mathbf{f}$, the corresponding solution of (716) is

$$\hat{\mathbf{u}} = D^{-1}(-i\omega + \eta k^2)\hat{f},\ \hat{\mathbf{b}} = D^{-1}i(\bar{\mathbf{B}}\cdot k)\hat{f}, \tag{7.17}$$

where

$$D = \omega^2 + i\omega(\nu + \eta)k^2 - \nu\eta k^4 - (\bar{\mathbf{B}}\cdot\mathbf{k})^2. \tag{7.18}$$

Where $\nu$ and $\eta$ are both small, there is a sharp resonance near $\omega = \pm\bar{\mathbf{B}}\cdot\mathbf{k}$, the frequency of freely propagating non-dissipative Alfven waves. When we construct $\langle \mathbf{u} \wedge \mathbf{b} \rangle$ for a spectrum of forcing, the result is dominated by the amplitude and width of these resonances in $(\omega, \mathbf{k})$ space.

The details, which are quite subtle, have been worked out with the additional complication of Coriolis effects in a rotating body of fluid (Moffatt 1972). The combined effects of Lorentz and Coriolis forces induce anisotropy in the turbulence; the "effective $\alpha$" (*e.g.* $\alpha = \frac{1}{3}\alpha_{ii}$) can still however be calculated; this is now a decreasing function of mean field strength $|\bar{\mathbf{B}}|$, on account of the decreasing width of the resonant layers in $(\omega, \mathbf{k})$ space as $|\bar{\mathbf{B}}|$ increases.

This effect in which $\alpha(\bar{\mathbf{B}})$ is a decreasing function of $|\bar{\mathbf{B}}|$ (tending to zero as $|\bar{\mathbf{B}}| \to \infty$) has since been described as "$\alpha$-quenching". Clearly, it leads to saturation in the growth of the mean field. For the simple model of Section 7.1, this saturation will occur for scale $L \sim K^{-1}$ when

$$\alpha(\bar{\mathbf{B}}) = \eta_T K. \tag{7.19}$$

Note however that a field saturating at this level can still apparently grow on scales much larger than $K^{-1}$. Thus, under sustained forcing on some scale $l_0$ which, either through the intrinsic character of the forcing, or through a Coriolis effect, generates turbulence with non-zero helicity, we may expect the dynamo-generated magnetic field to saturate at successively larger length scales, the spectral shape of the resulting field being given, in qualitative terms, by solving (7.19) for $\bar{\mathbf{B}}$:

$$\bar{\mathbf{B}} = \alpha^{-1}(\eta_T K) \tag{7.20}$$

and by interpreting this as providing the spectrum (in $\mathbf{K}$-space) of $\bar{\mathbf{B}}$.

Of course the parameter $\beta$ (contributing to $\eta_T$) will also be subject to a measure of quenching, which could also be included in the above type of analysis.

## 8 Magnetic relaxation

A situation of great interest arises when we consider a different type of initial value problem: suppose that at time $t = 0$, a random magnetic field $\mathbf{B}_0(\mathbf{x})$ exists in a fluid which is at rest. The associated current is $\mathbf{j}_0 = \nabla \wedge \mathbf{B}_0$ and the Lorentz force $\mathbf{F}_0 = \mathbf{j}_0 \wedge \mathbf{B}_0$ is in general rotational (*i.e.* $\nabla \wedge \mathbf{F}_0 \neq 0$). This force cannot therefore be balanced by pressure gradients, and the fluid will move; energy is then dissipated by viscosity. At the same time, the field is transported by the flow, and the Lorentz force distribution $\mathbf{F}(\mathbf{x}, t)$ evolves in time.

The situation is of particular interest if the fluid is a perfect conductor (*i.e.* $\eta = 0$), since then the field topology is conserved during this "relaxation" process. Energy is however still dissipated by viscosity. The magnetic helicity is conserved, and this acts as a "topological barrier" that prevents the magnetic energy from decaying to zero. We are then faced with a variational problem with an unusual twist: to find the minimum energy state of a field whose topology is prescribed as that of the initial field $\mathbf{B}_0(\mathbf{x})$.

The equations of magnetohydrodynamics (with $\eta = 0$, $\nu \neq 0$) provide the natural dynamics that drives the system towards such a minimum energy state (Moffatt 1985). In the minimum energy state, the velocity is again zero (since otherwise it would continue to dissipate energy); the corresponding field, $\mathbf{B}^E(\mathbf{x})$ say, is therefore magnetostatic, *i.e.*

$$\mathbf{j}^E \wedge \mathbf{B}^E = \nabla p^E \qquad (8.1)$$

for some scalar (pressure) field $p^E$, and $\mathbf{j}^E = \nabla \wedge \mathbf{B}^E$. The field $\mathbf{B}^E$ is "topologically accessible" from $\mathbf{B}_0(\mathbf{x})$, in the sense that it is obtained by continuous distortion by a velocity field $\mathbf{v}(\mathbf{x}, t)(0 < t < \infty)$ which dissipates a finite total amount of energy (the difference between the energies of the fields $\mathbf{B}_0$ and $\mathbf{B}^E$).

Although the above relaxation process is simple to describe, and seems transparently clear, it should be noted that point-wise convergence of the field $\mathbf{B}(\mathbf{x}, t)$ to an equilibrium field $\mathbf{B}^E(\mathbf{x})$ has not been proved, and remains an open problem.

### 8.1 The analogy with Euler flows

There is an exact analogy between (8.1) and the equation

$$\mathbf{u} \wedge \boldsymbol{\omega} = \nabla h \qquad (8.2)$$

describing steady Euler flows with $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$. The analogy is between the fields $\mathbf{B}^E$ and $\mathbf{u}$, (or equivalently between $\mathbf{j}^E$ and $\boldsymbol{\omega}$). Note that $h$ in (8.2) must be regarded as the analogue of $-p^E$. To any solution of (8.1), there corresponds *via* this analogy a corresponding solution of (8.2). It is thus apparent that (subject to pointwise convergence of the magnetic relaxation process) there exists a steady solution $\mathbf{u}(\mathbf{x})$ of the Euler equations having arbitrarily prescribed topology (*i.e.* that of the initial field $\mathbf{B}_0(\mathbf{x})$ in the magnetic relaxation problem).

Note here that it is the topology of the velocity field (rather than that of the vorticity field) that can be prescribed. It would be more interesting if a relaxation procedure conserving the topology of $\boldsymbol{\omega}$ (*i.e.* that of $\mathbf{j}$ in the magnetic relaxation problem!) could be devised, because that would be "more natural" for Euler dynamics. Only in 2D flows has this been found to be possible (Vallis *et al.* 1989). In this context, an upper bound can be placed on the energy of a flow of prescribed enstrophy, and a "relaxation" procedure that *increases* energy to a limiting value can be constructed.

In 3D, the steady Euler flows obtained *via* the magnetic relaxation technique, are all apparently unstable (Rouchon 1991). This is probably highly significant for turbulence! Nevertheless, it has been hypothesized (Moffatt 1990) that long-lived coherent vortices may be associated with "maximal helicity" regions where $\boldsymbol{\omega}$ is parallel to $\mathbf{u}$ (so that (8.2) is certainly satisfied). Recent analysis of 3D turbulent flows using wavelet transforms (Farge *et al.* 2001) lend some support to this hypothesis.

## 9 The blow-up problem

The question of smoothness of solutions of the Navier–Stokes and/or Euler equations for an incompressible fluid has remained open since first posed by Leray (1934). Numerical investigations (*e.g.* Kerr 1993; Pelz 1997) provide evidence for the blow-up of solutions of the Euler equations at finite time; but numerical codes have limited validity where singularities are concerned, and numerical results can be no more than suggestive in this context. On the analytical side, it is known (Beale *et al.* 1986) that if any breakdown of regularity of solutions of the Euler equations occurs at some finite time $t = t^*$, then the maximum value of the vorticity magnitude $|\boldsymbol{\omega}_{\max}(t)|$ must blow-up as $t \to t^*$ in such a way as to make the integral of this quantity (from $t$ to $t^*$) diverge. The simplest possibility is that

$$|\omega_{\max}(t)| \sim \frac{1}{t^* - t} \qquad \text{as} \qquad t \uparrow t^*, \tag{9.1}$$

and this is indeed the sort of behavior that has been inferred from numerical experiments (Pumir and Siggia 1988; Pelz 1997).

Why, it may be added, are we so interested in blow-up in the turbulence context? The reason is that if blow-up is a generic feature of any fully 3D time-dependent flow at very high Reynolds number, then the "spotty" or intermittent character of turbulent dissipation is immediately understandable in terms of the behavior near points where singularities of vorticity (and the related deformation tensor) occur.

Even more interesting is the question of how singularities (if they occur) must in practice be resolved. Singularities of the Euler equations may conceivably be resolved through inclusion of the effects of weak viscosity. But if it turns out that singular behavior can persist even for the Navier–Stokes equations (*i.e.* even when weak viscous effects are indicated), then we must look to other physical effects to resolve such behavior. The obvious effect that should then be considered is compressibility; for a singularity of vorticity will also imply a singularity of pressure (there being a large reduction of pressure in the core of an intense vortex). Compressibility in a gas results in the propagation of acoustic waves by the Lighthill mechanism

(Lighthill 1953), a mechanism that must surely prevent the formation of any pressure singularity in the fluid interior. In a liquid there is another mechanism: the liquid will cavitate wherever the pressure falls below the vapor pressure, and small bubbles will form, an effect that will be obviously dependent on the mean pressure applied to the system. This mechanism also mitigates against the formation of pressure singularities. the detailed mechanism of energy dissipation will be influenced by the presence of such cavitation bubbles; this deserves study!

### 9.1 Interaction of skewed vortices

All studies of the potential blow-up of vorticity have focussed on the behavior of skewed vortex tubes, when for one reason or another these are driven into close proximity with one another. The simplest scenario (Moffatt 2000) is that in which two skewed vortex pairs propagate on a collision course towards each other. These interact strongly when the separation between the pairs becomes of the same order as the separation of the vortices within each pair. We may define an inner "interaction" zone whose scale $a(t)$ is a decreasing function of time; and an outer zone, where the vortex pairs continue to propagate with negligible interaction.

If all characteristic length-scales (*e.g.* vortex separations, vortex radii of curvature, vortex core radii, ...) decrease in proportion to $a(t)$ (and this is a big IF), then the behavior in the inner zone is self-similar and can be described by the Leray (1934) scaling:

$$\mathbf{u}(\mathbf{x},t) = \left(\frac{\Gamma}{t^* - t}\right)^{1/2} \mathbf{U}(\mathbf{X}), \;\; \mathbf{X} = \frac{\mathbf{x}}{(\Gamma(t^* - t))^{1/2}} \cdot \tag{9.2}$$

where $\Gamma$ is a constant having the dimensions of a circulation. The corresponding vorticity is then

$$\boldsymbol{\omega}(\mathbf{x},t) = \frac{1}{t^* - t}\boldsymbol{\Omega}(\mathbf{X}), \;\; \boldsymbol{\Omega} = \nabla_{\mathbf{X}} \wedge \mathbf{U}, \tag{9.3}$$

and the vorticity equation transforms to

$$0 = \nabla \wedge \left[\left(\mathbf{U} + \frac{1}{2}\mathbf{X}\right) \wedge \boldsymbol{\Omega}\right] + \epsilon\nabla^2\boldsymbol{\Omega}, \tag{9.4}$$

where $\epsilon = \nu/\Gamma$. If any smooth solution of this equation, satisfying "acceptable" boundary conditions, can be found, then the corresponding solution of the native Navier–Stokes equation clearly has a singularity (with maximum vorticity behaving as in (9.1)) at $\mathbf{x} = 0$ as $t \to t^*$.

But what are the acceptable boundary conditions? Suppose that

$$|\boldsymbol{\Omega}(\mathbf{X})| \sim |\mathbf{X}|^{-\alpha} \quad \text{as} \quad |\mathbf{X}| \to \infty.$$

Then, for each fixed finite $\mathbf{x}$, from (9.2) and (9.3),

$$\omega(\mathbf{x},t) \sim \frac{1}{|\mathbf{x}|^*}\frac{\Gamma^{\alpha/2}}{(t^*-t)^{1-\alpha/2}}\,. \tag{9.5}$$

If $\alpha < 2$, this vorticity blows up for all finite $\mathbf{x}$ as $t \to t^*$, a behavior that is totally implausible. If $\alpha > 2$, then the vorticity goes identically to zero for all $\mathbf{x}$ as $t \to t^*$, a behavior that is equally implausible. The only realistic possibility therefore is that $\alpha = 2$, so that

$$|\mathbf{\Omega}(\mathbf{X})| \sim |\mathbf{X}|^{-2} \quad \text{as} \quad |\mathbf{X}| \to \infty \tag{9.6}$$

(and correspondingly $|\mathbf{U}| \sim |\mathbf{X}|^{-1}$) as would be realized, for example, by conical expansion of vortex tubes as they leave the interaction zone. Correspondingly we require that

$$|\omega(\mathbf{x},t)| \sim \Gamma/|\mathbf{x}|^2 \qquad \text{as} \quad |\mathbf{x}| \to 0 \tag{9.7}$$

as the inner "boundary condition" for the outer region. Equations (9.6) and (9.7) indicate only radial behavior; angular dependence is unconstrained. Again, there is evidence of this behavior in the numerical work of Pelz (1997) who studied, by vortex filament techniques, the implosion of 6 vortex pairs towards the origin, the whole configuration having cubic symmetry.

Against this scenario, two theorems have been proved by functional analytic techniques. Nečas *et al.* (1996) have shown that, for $\epsilon > 0$, (9.4) has no nontrivial solution $\mathbf{U}(\mathbf{X})$ in $L^3(\mathbb{R}^3)$ (*i.e.* for which $|\mathbf{U}(\mathbf{X})|^3$ is integrable over the whole $\mathbf{X}$-space). This immediately rules out solutions for which $|\mathbf{U}| \sim |\mathbf{X}|^{-q}$ (and $|\mathbf{\Omega}| \sim |\mathbf{X}|^{-(q+1)}$) for $q > 1$; it does not rule out the behavior (9.6), which, perhaps significantly, lies just outside this function space. However, more seriously, Tsai (1998) has shown (again for $\epsilon > 0$) that the theorem of Nečas *et al.* (1996) can be extended to cover the non-existence of nontrivial solutions $\mathbf{U}(\mathbf{x})$ of (9.4) in $L^q(\mathbb{R}^3)$ for all $q$ in the range $3 < q < \infty$; this certainly does exclude the behavior (9.6). The full implications of Tsai's theorem are as yet unclear (to this writer!), but it does imply that singularities of the Navier–Stokes equations with $\nu > 0$ cannot in fact be described in terms of the Leray scaling (9.2).

The above remarks do not apply to the Euler limit ($\epsilon = 0$ in (9.4)); in this limit, we may integrate (9.4) to give

$$\left(\mathbf{U} + \frac{1}{2}\mathbf{X}\right) \wedge \mathbf{\Omega} = \nabla H \tag{9.8}$$

for some scalar $H(\mathbf{X})$ (the scaled Bernoulli function). Since $\mathbf{\Omega} \cdot \nabla H = 0$, vortex lines lie on surfaces $H =$ cst., which in effect define the vortex tubes of the flow in the inner (Leray) region. The self-induced velocity $\mathbf{U}(\mathbf{X})$ must satisfy

$$\left(\mathbf{U} + \frac{1}{2}\mathbf{X}\right) \cdot \nabla H = 0, \tag{9.9}$$

*i.e.* there must be an inward flow across each vortex tube to compensate the outward transport represented by the term $\frac{1}{2}\mathbf{X} \cdot \nabla H$ (which can be traced to the space-scaling relating $\mathbf{X}$ and $\mathbf{x}$ in (9.2)). This inflow can in principle be compensated by outflow along the vortex tubes emanating from the interaction zone. But the $ million question is whether there is any vortex configuration which induces a velocity field which in turn keep the configuration steady *via* the condition (9.8). It seems likely that this question will continue to present a profound challenge over the next few years, if not decades!

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