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SOME REMARKS ON TOPOLOGICAL FLUID MECHANICS

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Abstract. Some fluid dynamical problems having a topological flavour are briefly reviewed, and some further problems having at least a topological starting point are posed.

1. Introduction

Topological Fluid Mechanics is primarily concerned with structures within a flow field which retain some coherence over a significant period of time. Under circumstances that may be described as 'ideal' relative to the type of structure considered, this 'significant period of time' is infinite; but insofar as circumstances are never ideal in reality, we must be equally concerned with the manner in which structural (or topological) properties of a flow may change with time (generally under the influence of some diffusive process).

These statements suffer from a degree of imprecision that can be removed only through consideration of particular problems. The purpose of this brief paper is to set out a number of such problems, all of which have at least a starting point that can be described as topological, and most of which are unsolved. There is no shortage of challenging problems of this type for which a combination of analytical, computational and experimental (ACE) techniques will be required if real progress is to be made.

2. Particle Paths

For any given flow field $\mathbf{u}(\mathbf{x}, t)$, whether in a finite or infinite domain, the path $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$ of the particle initially at position $\mathbf{x}$ is determined by the

R.L. Ricca (ed.), An Introduction to the Geometry and Topology of Fluid Flows, 3–10.

dynamical system

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X}, t), \quad \mathbf{X}(\mathbf{x}, 0) = \mathbf{x}. \tag{1}$$

We shall restrict attention to incompressible flows for which $\nabla \cdot \mathbf{u} = 0$. Even with this restriction it is only in the simplest circumstances that the system (1) is integrable; for example, for steady two-dimensional flow with streamfunction $\psi(x, y)$, the particle paths coincide with the streamlines $\psi =$ cst. , and the distance travelled by a fluid particle along a streamline is obtained by integrating along the streamline.

As shown by Aref [1], the introduction of a time-periodic ingredient in a two-dimensional flow field, is sufficient to induce chaos in the particle paths, this chaos appearing near homoclinic or heteroclinic trajectories of the time-averaged velocity field $\bar{\mathbf{u}}(\mathbf{x})$. The particular case studied was that of the 'blinking vortices' for which this time-averaged flow has a single homoclinic (saddle) point; but there seems little doubt that the associated appearance of chaos is indeed a generic phenomenon. A passive scalar field $\theta(\mathbf{x}, t)$ subjected to advection in such a flow field suffers a very rapid stretching of its iso-scalar curves $\theta =$ cst. and an associated rapid decrease of scale to the level at which molecular diffusion becomes operative in smoothing out field variation [17].

In three dimensions, even steady flows $\mathbf{u}(\mathbf{x})$ have associated particle paths (coinciding with streamlines) that exhibit chaos. This behaviour was first identified for space-periodic flows, specifically the 'ABC-flow'

$$\mathbf{u} = (C \cos kz + B \sin ky\,,\ A \cos kx + C \sin kz\,,\ B \cos ky + A \sin kx)\,, \tag{2}$$

by Arnold [2] and Hénon [9], the analysis of this flow being developed later by Dombre *et al.* [7].

Chaos can likewise occur for three-dimensional steady flows in a bounded domain. The case of quadratic flows of the form

$$u_i(\mathbf{x}) = a_i + b_{ij}x_j + c_{ijk}x_j x_k \tag{3}$$

satisfying $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{n} \cdot \mathbf{u} = 0$ on $|\mathbf{x}| = 1$ was treated by Bajer & Moffatt [3]. The vorticity field $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ in such flows is linear in $\mathbf{x}$ and so satisfies $\nabla^2 \boldsymbol{\omega} = 0$. Thus, the flows are Stokes flows in the sphere $|\mathbf{x}| < 1$, driven by a prescribed tangential velocity on the surface $|\mathbf{x}| = 1$. These flows in general exhibit regions in which the particle paths are chaotic. This behaviour may be detected visually through computation of Poincaré sections; more formally, the chaotic regions may be defined as the set of initial positions $\mathbf{x}$ for which the Liapunov exponent

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln |\boldsymbol{\xi}| \tag{4}$$

is positive; here $\boldsymbol{\xi}$ is an infinitesimal material line element originally at $\mathbf{x}$; λ represents the asymptotic rate of stretching of this material line element. Note that if $\mathbf{x}_1$, $\mathbf{x}_2$ lie on the same streamline, then $\lambda(\mathbf{x}_1) = \lambda(\mathbf{x}_2)$. In regions of 'regularity' of the flow, $|\boldsymbol{\xi}|$ grows merely linearly with time, and $\lambda(\mathbf{x}) = 0$.

A quantity of key importance in relation to the stirring efficiency of a flow is the volume fraction μ of the fluid domain in which $\lambda(\mathbf{x}) > 0$. This volume fraction is a fundamental structural parameter; it is topological in the sense that it is invariant under continuous volume-preserving deformations of the flow field (which deform the regions of chaos without change of volume). We may thus pose a first problem of topological character:

XXI Century Problem 1 *For a given bounded domain $\mathcal{D}$ and a given continuous flow field $\mathbf{u}(\mathbf{x})$ $(\mathbf{x} \in \mathcal{D})$, satisfying $\nabla \cdot \mathbf{u} = 0$ in $\mathcal{D}$ and $\mathbf{n} \cdot \mathbf{u} = 0$ on $\partial\mathcal{D}$, to determine the volume fraction μ of $\mathcal{D}$ for which $\lambda(\mathbf{x}) > 0$, i.e. for which the flow has chaotic particle paths.*

This problem has obvious generalisations for space-periodic flows and for time-periodic two-dimensional flows.

3. Scalar Field Problems

Let $\theta(\mathbf{x}, t)$ be a scalar field that is convected by a continuous velocity field $\mathbf{u}$, and suppose for the moment that molecular diffusion is negligible. Then θ is constant for each fluid particle, i.e.

$$\frac{D\theta}{Dt} \equiv \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla\theta = 0\,. \tag{5}$$

To be specific, suppose that the flow domain $\mathcal{D}$ is in $\mathbb{R}^3$, and that $\mathbf{u} \cdot \mathbf{n} = 0$, $\theta =$ cst. on $\partial\mathcal{D}$. Equation (5) of course implies that the surfaces $\theta =$ cst. are transported with the flow. Their topology is therefore conserved. How is this topology to be described?

A start has been made [14] through consideration of the saddle points S_i of the field θ and the homoclinic iso-scalar surfaces Σ_i through these S_i. Together with $\partial\mathcal{D}$, the Σ_i divide $\mathcal{D}$ into a number of subdomains $\mathcal{D}_\alpha$ $(\alpha = 0, 1, 2, \ldots)$ where $\mathcal{D}_0$ is the subdomain that is bounded externally by $\partial\mathcal{D}$, and each $\mathcal{D}_\alpha$ $(\alpha = 1, 2, \ldots)$ is bounded by parts of one or (at most) two of the homoclinic surfaces. The volume of each $\mathcal{D}_\alpha$ is conserved under (5) as is the topology of the surfaces Σ_i.

Knowledge of the relative configurations of the Σ_i and of the volumes of the $\mathcal{D}_\alpha$ is an important first step in classifying the possible topologies of the θ-field. This is not all however. Within each $\mathcal{D}_\alpha$ are a family of surfaces $\theta =$ cst. and we may define a *signature function* $V_\alpha(\theta)$ in $\mathcal{D}_\alpha$ with the property that $(dV_\alpha/d\theta)\delta\theta$ is the volume between surfaces labelled $\theta, \theta + \delta\theta$; the function $V_\alpha(\theta)$ is then defined up to a constant C_α which may be chosen so that the signature function varies continuously in moving from one subdomain to another. The set of signature functions $\{V_\alpha(\theta)\}$ thus defined is clearly invariant under the evolution (5) and is therefore a topological property of the field θ.

There are now two interesting directions that merit investigation. We indicate these in the form of problems:

XXI Century Problem 2 *Suppose now that molecular diffusivity κ is included, so that (5) is replaced by*

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \kappa\nabla^2\theta\,. \tag{6}$$

What transitions in the topology of the set of homoclinic surfaces $\{\Sigma_i\}$ are possible as a result of this diffusion; and how does the set of signature functions $\{V_\alpha(\theta)\}$ evolve, particularly during such a change of topology?

XXI Century Problem 3 *Suppose that the velocity field* **u** *is itself 'driven' by inhomogeneity of the θ-field, according to some well-defined dynamical prescription (e.g. θ could represent temperature variation in a gravity field, the flow being driven by the buoyancy force in the Boussinesq approximation (see, for example, [4]). The problem is to examine the evolution of the θ-field in the neighbourhood of its saddle-points, to determine whether singularities of $\nabla\theta$ can develop, and to examine the influence of weak molecular diffusivity κ in controlling the approach to such singularities.*

4. Vector Field Problems

Each of the above scalar field problems has a counterpart in the context of a transported vector field, such as the magnetic field $\mathbf{B}(\mathbf{x}, t)$ in a conducting fluid. This field is divergence-free, i.e. $\nabla\cdot\mathbf{B} = 0$, and satisfies the induction equation

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}\,, \tag{7}$$

the counterpart of (6). In the diffusionless (perfectly conducting) limit $\eta = 0$, (7) implies that **B**-lines are 'frozen' in the fluid, the flux of **B** through any closed material curve C being conserved. This has the important consequence that any linkage between **B**-lines is conserved – clearly a result with topological content. The simplest measure of net linkage of the field inside any Lagrangian (i.e. material) surface S on which $\mathbf{n} \cdot \mathbf{B} = 0$ is the magnetic helicity

$$\mathcal{H}_M = \int_V \mathbf{A} \cdot \mathbf{B} \, dV \tag{8}$$

where V is the volume inside S, and **A** is a vector potential for **B**, i.e. $\mathbf{B} = \nabla \times \mathbf{A}$. The integral (8) is gauge-independent, but it is usual to choose the gauge of **A** so that $\nabla \cdot \mathbf{A} = 0$.

XXI Century Problem 4 *Consider two linked unknotted flux tubes, each carrying flux Φ, the field within each tube being untwisted, so that the helicity $\mathcal{H}_M$ is $2\Phi^2$ [11]; we assume here that the linkage is right-handed. Suppose that the fluid motion brings the tubes into close proximity and that weak diffusion ($\eta > 0$) causes reconnexion of* **B***-lines in such a way that the two tubes become a single tube carrying flux Φ. In this process, the helicity (or at least some proportion of it) may survive through the appearance of internal twist in the resultant tube [15]. The problem is to determine precisely what is the total field helicity after reconnexion, the whole process being governed by equation (7).*

In the above problem, if the field **B** is sufficiently weak, then presumably it may be treated as dynamically passive, the velocity **u** being then independently prescribable. More realistically, however, the Lorentz force $\mathbf{j} \times \mathbf{B}$ where $\mathbf{j} = \nabla \times \mathbf{B}$, plays an important part in the reconnexion process. This is particularly the case when η is very weak since then very strong stretching of field lines occurs in conjunction with the flow that brings sections of the two initial tubes into close proximity. This leads to

XXI Century Problem 5 *Consider two flux tubes oblique to each other, carrying the same flux Φ, and driven towards each other by an imposed strain field; it is required to analyse the process of reconnexion, its dependence on Φ (via the Lorentz force), and again to determine how much twist is created in the reconnected tubes.*

Just as for the scalar field problem, there are circumstances in which the velocity field **u** is entirely driven by the Lorentz force distribution. If the fluid is viscous but perfectly conducting, then the field energy converts to kinetic energy which is dissipated by viscosity, and during this process the field topology is conserved. There is however an outstanding problem in relation to this 'magnetic relaxation' scenario that remains open:

XXI Century Problem 6 *Consider a smooth localised magnetic field of finite and nonzero magnetic helicity in a viscous, perfectly conducting, incompressible fluid initially at rest. It is known [12] that the kinetic energy tends to zero as $t \to \infty$. It is required to prove that $|\mathbf{u}(\mathbf{x}, t)| \to 0$ at all points $\mathbf{x}$ as $t \to \infty$.*

This is almost certainly true, since otherwise the appearance of singularities (of implausible form in a viscous fluid) is implied; a proof should not be impossibly difficult.

If the initial field is confined to a single knotted flux tube of volume V and carrying flux Φ, and with internal twist such that the helicity is $\mathcal{H}_M = h\Phi^2$, then, on dimensional grounds the 'relaxed' state has minimal magnetic energy M^E given by

$$M^E = m(h)\Phi^2 V^{-1/3} \tag{9}$$

where $m(h)$ is a positive dimensionless function of the dimensionless twist parameter h, which depends only on the form of the knot. Particular interest attaches to value of h (for a given knot K) for which $m(h)$ is minimal, $m_{\min}$ say. It is to be expected that $m_{\min}$ increases with increasing knot complexity.

XXI Century Problem 7 *Determine $m_{\min}$ for knots of minimum crossing number 3, 4, 5, ...*

This problem presents a considerable computational challenge. A start has been made for torus knots by Chui & Moffatt [6].

5. The Finite-Time Singularity Problem

It is but a small step from the above 'magnetically active' problems to the Euler and Navier-Stokes problems that lie at the heart of fluid mechanics. We simply replace $\mathbf{B}$ in (7) by the vorticity field $\boldsymbol{\omega}$, and we take $\mathbf{u}$ to be the inverse curl of $\boldsymbol{\omega}$:

$$\mathbf{u} = (\text{curl})^{-1}\boldsymbol{\omega}, \quad \nabla \cdot \mathbf{u} = 0; \tag{10}$$

and of course we replace η by kinematic viscosity ν.

The simplicity of the functional relationship (10) between $\mathbf{u}$ and $\boldsymbol{\omega}$ might suggest that, for example, the problem of viscous vortex tube reconnexion should be no more difficult than the problem of magnetic flux tube reconnexion (with Lorentz forces included). This however is just wishful thinking! Vortex tube reconnexion has attracted much study, both analytical [5] and computational [10], [18] (see also this volume), and yet we are still in the dark as regards the details of the process. In particular, we do not know whether the vorticity field within a zone of reconnexion remains finite for all t, or conversely whether a singularity of vorticity may develop within a finite time. The computational work cited above provides quite strong evidence for the appearance of finite-time singularities for Euler evolution ($\nu = 0$), whereas analytical studies based on the Leray similarity transformation [16], [19] point to the non-existence of finite-time singularities when $\nu > 0$. I have argued [13] that if finite-time singularities appear under Euler evolution, then the same type of singularity should appear when ν is positive but sufficiently small, i.e. that weak viscosity may not be able to prevent the formation of finite-time singularities. This suggests a problem that may provide a helpful stepping-stone towards the central ($1m) finite-time singularity problem as posed by the Clay Institute [8].

XXI Century Problem 8 *Suppose that there exists a smooth velocity field* $\mathbf{u}_0(\mathbf{x})$ *of finite energy in a bounded domain* $\mathcal{D}$ *such that, under Euler evolution starting from this initial condition, a singularity of* $\boldsymbol{\omega}(\mathbf{x},t)$ *appears at some finite time* t^*. *Prove that, for* $0 < \nu < \nu_c$ *where* ν_c *is small, and with the same initial condition,* $\boldsymbol{\omega}(\mathbf{x},t)$ *still becomes singular at finite time; or conversely, prove that* $\boldsymbol{\omega}(\mathbf{x},t)$ *remains smooth for all* t.

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