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THE TOPOLOGY OF SCALAR FIELDS IN 2D AND 3D TURBULENCE

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1 Introduction

I am honoured to present the Opening Lecture at this Symposium on the *Geometry and Statistics of Turbulence.* I vividly recall a previous IUTAM Symposium in Tokyo in 1983 on the subject *Turbulence and Chaotic Phenomena in Fluids*; at that Symposium, I contributed a lecture with the title "Simple topological aspects of turbulent vorticity dynamics". During the last two decades, increasing attention has been paid to characteristic structures detected in both experimental work and in direct numerical simulation (DNS); indeed, it is these developments that have provided the main motivation for the present Symposium, and that will be described in many of the lectures on the programme.

In this introductory lecture, it may be appropriate to resume the theme of my 1983 lecture, but at a more basic level: I propose to consider the generic structure of scalar fields in both 2D and 3D turbulence, the topological description of these fields, and the manner in which the topology may change with time. I shall also seek to describe how, at the simplest level, the geometry of these fields may be related to the most basic statistical property – the field spectrum.

While the scalar field presents a tractable problem which it is sensible to consider as a starting point, the problem of providing a similar description of solenoidal vector fields such as velocity $\mathbf{u}$ or vorticity $\boldsymbol{\omega}$ in 3D is very much more difficult. Such fields generically exhibit chaos (as for example in the ABC-flow studied by Dombre et al 1986, or the quadratic STF-flow studied by Bajer & Moffatt 1990); and the problem of classifying such chaotic flows appears prohibitively difficult at present. This will continue to present a major challenge well into the new millennium.

T. Kambe et al. (eds.), IUTAM Symposium on Geometry and Statistics of Turbulence, 13–22.

2 Streamline topology in 2D turbulence

Consider first the problem of 2D turbulence in a periodic domain (topologically a torus). Let $\psi(x, y, t)$ be the streamfunction for the flow. The critical points at any instant of ψ are the points where $\nabla\psi = 0$, i.e. they are the (instantaneous) stagnation points of the flow. Taking origin O at one such critical point, the streamfunction has local Taylor expansion

$$\psi = \psi_0 + c_{ij}x_i x_j + O(|\mathbf{x}|^3)\,, \tag{1}$$

where $(x_1, x_2) \equiv (x, y)$ and where

$$c_{ij} = \tfrac{1}{2}\left(\partial^2\psi/\partial x_i \partial x_j\right)_{\mathbf{x}=0} = c_{ji}\,. \tag{2}$$

Let λ_1, λ_2 be the eigenvalues of c_{ij}. If we choose the axes Ox, Oy to be along the corresponding eigenvectors, then (1.1) takes the form

$$\psi = \psi_0 + \lambda_1 x^2 + \lambda_2 y^2 + O(|\mathbf{x}|^3)\,. \tag{3}$$

Non-degeneracy of the critical point means that $\lambda_1\lambda_2 \neq 0$. The *index* i of the critical point is defined as the number of negative eigenvalues, i.e. $i = 0$, 1 or 2 in this case. If $i = 0$, both λ_1 and λ_2 are positive and ψ is clearly minimal at O; if $i = 2$, λ_1 and λ_2 are negative and ψ is maximal at O. In either case, the streamlines $\psi =$ cst. are elliptic in the neighbourhood of O, the flow being clockwise if $i = 0$, anticlockwise if $i = 2$. If $i = 1$, then one of (λ_1, λ_2) is positive, the other negative, and O is a saddle point of ψ, the streamlines being locally hyperbolic with asymptotes

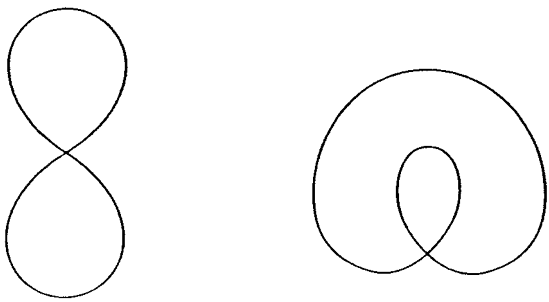
$$x/y = \pm(-\lambda_2/\lambda_1)^{1/2}\,. \tag{4}$$

Regarding the periodic domain as (topologically) a torus, Euler's index theorem implies that

$$\sum_{i=0}^{2}(-1)^i n_i = n_0 - n_1 + n_2 = 0\,, \tag{5}$$

i.e. the number of elliptic points $n_0 + n_2$ equals the number of hyperbolic points n_1. A uniform flow for which $n_0 + n_2 = n_1 = 0$ provides a trivial example.

The streamlines through the saddle points (or separatrices) play an important role in relation to the field topology. In general, the values of ψ at the critical points within the periodic box will be all different. This means that, in general, 'heteroclinic' separatrices connecting one critical point to another do not occur. All separatrices are homoclinic in the sense that each connects back to a single critical point. They may connect in two distinct ways as indicated in figure 1; the first is a figure-of-eight configuration, while the second may be described as an 'inverted figure-of-eight'.



(a) (b)

Figure 1: Homoclinic separatrices associated with a hyperbolic stagnation point. *a)* figure-of-eight structure; *b)* inverted figure-of-eight.

A flow may be topologically simplified through the coalescence of an elliptic point and a hyperbolic point (leaving (5) satisfied). This generic transition may be represented (locally in x, y and t) by the streamfunction

$$\psi_1 = -x^2 + 3yt + y^3\,, \tag{6}$$

which, for $t < 0$, has an elliptic critical point at $(0, \sqrt{-t})$ and a hyperbolic critical point at $(0, -\sqrt{-t})$. At $t = 0$, these points coalesce, and for $t > 0$, the flow has no critical points. It is interesting to note that at the instant of coalescence, the streamline $\psi = 0$ has a cusp at $x = y = 0$ (figure 2); at this instant, the critical point is degenerate.

Conversely, a flow may be topologically 'complexified' by time-reversal of this process: at any point within a flow, a local distortion may introduce a hyperbolic-elliptic pair (otherwise known as a saddle-node bifurcation), as illustrated in figure 3. This process indicates that the 'generic' instantaneous separatrix structure of a 2D flow consists of an array of nested figure-of-eights and inverted figure-of-eights, more complex flows containing higher-order nested structures.

The existence of the streamfunction (6) indicates that the transition indicated in figure 2 is kinematically possible; to show that it is dynamically possible, we need to consider the Navier-Stokes equation in dimensionless form

$$\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} = \frac{1}{Re}\nabla^4\psi\,. \tag{7}$$

With ψ_1 given by (6), we have

$$\frac{\partial}{\partial t}(\nabla^2\psi_1) = 0\,, \quad \frac{\partial(\psi_1, \nabla^2\psi_1)}{\partial(x,y)} = -12x \tag{8}$$

and (7) is not satisfied. However, if we redefine ψ_1 as

$$\psi_1 = -x^2 + y^3 + 3ty + (Re/10)x^5 \tag{9}$$

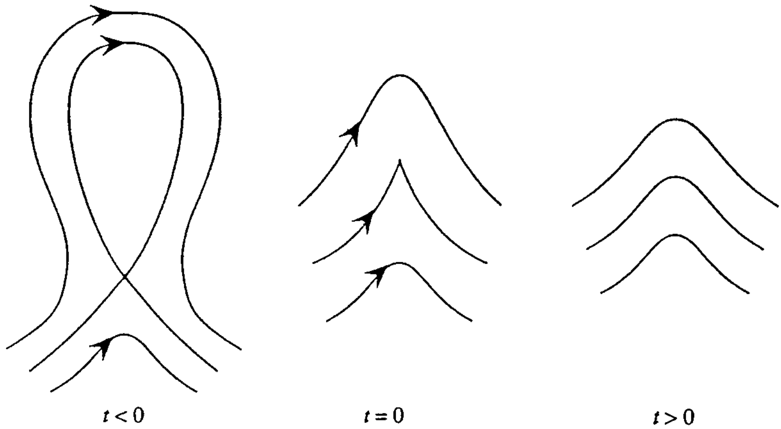


Figure 2: Coalescence of an elliptic point and a hyperbolic point represented by the streamfunction $\psi = -x^2 + yt + y^3$.

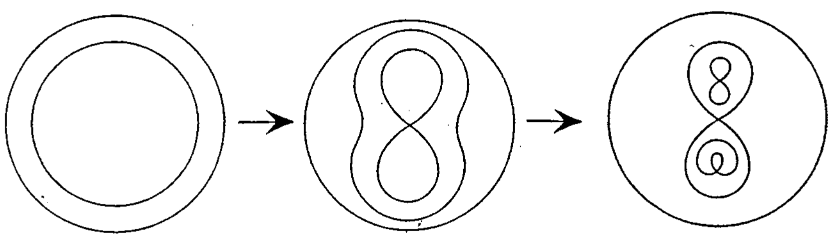


Figure 3: Flow complexification, through successive introduction of hyperbolic-elliptic pairs.

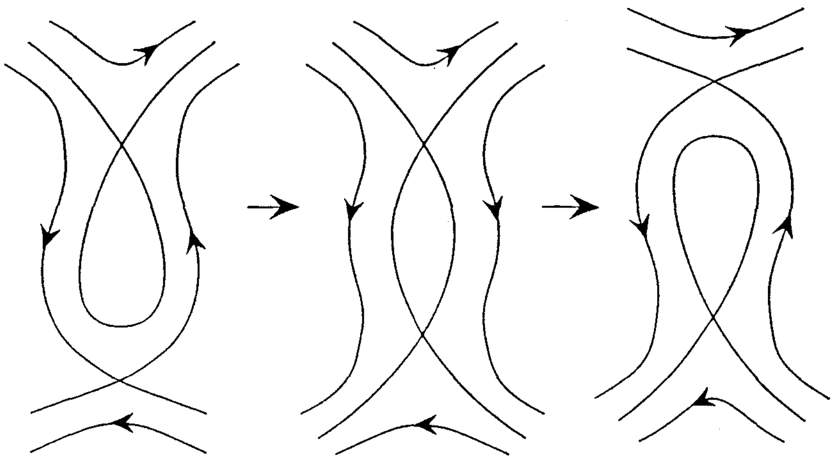


Figure 4: Change of separatrix topology through heteroclinic connexion

then (7) is satisfied at leading order, i.e. at the order of terms linear in x and y. Thus, the transition of figure 2 *is* dynamically possible, and since t may be replaced by $-t$, this transition is possible in either direction. (Alternative modifications of (6) are equally possible.)

As mentioned above, heteroclinic connexions are not generally present; however, they may occur instantaneously in an unsteady flow. This is illustrated by the streamfunction

$$\psi_2(x, y, t) = \tfrac{1}{2}(x^2 + y^2) - y^4/4\epsilon^2 - At(y - y_0)^2 \tag{10}$$

where ϵ is small. When $t = 0$, there are saddle points at $(0, \pm\epsilon)$ with heteroclinic connexion. Also, (7) is satisfied at leading order provided $A = 3(\epsilon^2 Re)^{-1}$. As t passes through zero, the topology of the flow changes as indicated in figure 4. The whole diagram lies within the region $|\mathbf{x}| = O(\epsilon)$. Thus again, this type of 'heteroclinic transition' appears to be both kinematically and dynamically possible. Note that

$$\frac{\partial(\psi_2, \nabla^2\psi_2)}{\partial(x, y)} = -3xy/\epsilon^2 = O(1) \tag{11}$$

but this term can be compensated by the introduction of higher-order terms in (9).

Similar considerations apply to the vorticity field $\omega = -\nabla^2\psi$ in 2D turbulence. The critical points of ω are now those points where $\nabla\omega = 0$, and these critical points are again elliptic or hyperbolic in equal numbers (for a periodic domain). The elliptic points (maxima or minima) may be identified with the centres of the 'concentrated vortices' that ultimately emerge in freely decaying turbulence at high Reynolds number. In the Euler limit ($Re = \infty$), the iso-vorticity contours $\omega =$ cst. move with the fluid, and so the topology of the ω-field is conserved. Thus topological transitions can now occur *only* through the agency of viscosity. Examples of such transitions can be easily constructed. For example, a transition eliminating an inverted figure-of-eight occurs whenever a strong positive vortex engulfs and eliminates through viscous action a neighbouring weak negative vortex.

3 Eddies and vortices in a 2D field of turbulence

Consider first a simple flow whose streamfunction is

$$\psi = A_1 \sin k_1 x \sin k_1 y + A_2 \sin k_2 x \sin k_2 y \tag{12}$$

where $k_2 \gg k_1$, $A_1 > 0$, $A_2 \geq 0$. The vorticity distribution $\omega = -\nabla^2 \psi$ is then

$$\omega = 2A_1 k_1^2 \sin k_1 x \sin k_1 y + 2A_2 k_2^2 \sin k_2 x \sin k_2 y\,. \tag{13}$$

If $A_2 = 0$, then $\omega = 2k_1^2 \psi$ and the isovorticity curves $\omega =$ cst. coincide with the streamlines $\psi =$ cst. . The critical points are at $(n\pi/k_1,\ m\pi/k_1)$, and the separatrices are all heteroclinic – a degenerate situation. The number of critical points in a square of side πL is $(k_1 L)^2$.

Suppose now that A_2/A_1 is slowly increased from zero. The x-component of velocity on $y = 0$ is

$$u = \partial\psi/\partial y|_{y=0} = A_1 k_1 \sin k_1 x + A_2 k_2 \sin k_2 x\,, \tag{14}$$

and it is easy to see that a first saddle-node bifurcation occurs at $k_2 x = 3\pi/2$ when $A_2/A_1 \doteqdot 3\pi k_1^2/2k_2^2$. As A_2/A_1 increases further, more saddle-node bifurcations occur until $A_2 k_2 = A_1 k_1$ when the number of critical points of the field (12) has increased to $(k_2 L)^2$.

Similar considerations obviously apply to the field ω. Note that, when A_2/A_1 is in the range

$$\frac{k_1^3}{k_2^3} < \frac{A_2}{A_1} \lesssim \frac{3\pi}{2}\frac{k_1^2}{k_2^2}\,, \tag{15}$$

the number N of 'eddies' (i.e. extrema of ψ) per unit area is $(k_1/\pi)^2$, while the number M of 'vortices' (i.e. extrema of ω) per unit area is $(k_2/\pi)^2$.

Similarly, we may seek to estimate these numbers for a field of 2D turbulence with energy spectrum $E(k)$ having an inertial range $\sim k^{-\lambda}$ between a maximum at k_1 and a viscous cut-off at $k_2 (\gg k_1)$. The corresponding velocity v_k at wavenumber k scales like $(kE(k))^{\frac{1}{2}} \sim k^{\frac{1}{2}(1-\lambda)}$, so the number of eddies per unit area will be of order k_2^2 or k_1^2 according as $\lambda < 1$ or $\lambda > 1$. Similarly the number of vortices per unit area is controlled by $k^2 v_k \sim k^{\frac{1}{2}(5-\lambda)}$, and is of order k_2^2 or k_1^2 according as $\lambda < 5$ or $\lambda > 5$. Note that, for λ in the interesting range $1 < \lambda < 5$, the number of *eddies* per unit area is $O(k_1^2)$ while the number of *vortices* per unit area is $O(k_2^2)$; this is of course consistent with the fact that far more structure is normally seen in the vorticity field than in the ψ-field in DNS of 2D turbulence.

4 Scalar field structure in 3D turbulence

Consider now the generic structure of a scalar field $s(\mathbf{x})$ in 3D. The critical points are again those points where $\nabla s = 0$; and in the neighbourhood of a non-degenerate critical point O, the field admits an expansion of the form

$$s = s_0 + \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + O(|\mathbf{x}|^3)\,, \tag{16}$$

where $\lambda_1\lambda_2\lambda_3 \neq 0$. The index i is again the number of negative eigenvalues, so that $i = 0, 1, 2$ or 3; the index theorem is now

$$\Sigma_{i=0}^{3}(-1)^i n_i = n_0 - n_1 + n_2 - n_3 = 0 \tag{17}$$

for the case of a field periodic in all three directions (topologically, we are then in T^3). If $i = 0$, we have a minimum of s (denoted M_0); if $i = 3$, we have a maximum (denoted M_3); and if $i = 1$ or 2, we have a saddle point of 'type 1' or 'type 2' (denoted S_1, S_2). The separatrix surface $\Sigma : s = s_0$ through a saddle point O is locally a cone with elliptic cross-section,

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_2 z^2 = 0\,, \tag{18}$$

and generically, this cone must 'connect' to itself in homoclinic manner. There are four ways (figure 5) in which it can do this, in which Σ takes the following forms:

1. a 'constricted' sphere Σ_1 (to visualise this, tie a loop of string round the equator of an inflated balloon and pull it tight);

2. an inverted constricted sphere Σ_2 (one 'bulb' of the closure now contains the other);

3. a 'pinched' sphere Σ_3 (push the poles of a balloon inwards till they make contact);

4. a constricted torus Σ_4 (which may be knotted) (tie a string round the small circumference of a toroidal balloon and tighten it).

(We may also pinch a torus, giving a 'pretzel' of two holes; and so on.)

Consider now how we may complexify an s-field starting from a region inside a surface $s =$ cst. of spherical topology within which there is a single extremum, a maximum M_3 say. We may perturb the field in three ways (see figure 6):

1. by introducing another maximum M_3 and a saddle point S_2 of type 2; this topological transition is represented by

$$T_1 : (n_0, n_1, n_2, n_3) \to (n_0, n_1, n_2 + 1, n_3 + 1) \tag{19}$$

   and generates a separatrix surface of type Σ_1;

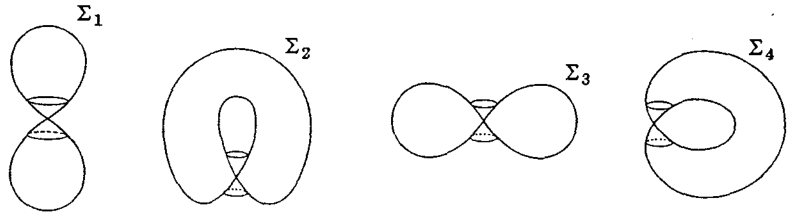


Figure 5: Four possible homoclinic surface connections

2. by introducing a minimum M_0 and a saddle S_1; this corresponds to the transition
$$T_2 : (n_0, n_1, n_2, n_3) \to (n_0 + 1, n_1 + 1, n_2, n_3) \tag{20}$$
and generates a separatrix surface of type Σ_2;

3. by introducing two saddle points S_1 and S_2; here M_3 lies inside a torus Σ_4 constricted at S_2, which in turn lies inside a sphere Σ_3 pinched at S_1; this corresponds to the transition
$$T_3 : (n_0, n_1, n_2, n_3) \to (n_0, n_1 + 1, n_2 + 1, n_3)\,. \tag{21}$$

The transitions T_1, T_2, T_3 are all evidently compatible with (18).

Successive complexifications will lead to a field s whose separatrix surfaces form a complex of nested surfaces of types $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$, with (Σ_3, Σ_4) always occurring as a pair with Σ_4 inside Σ_3. It would appear that this type of construction should provide the generic structure of any scalar field $s(\mathbf{x})$. Note that, because of the possibility of the transition T_3, the number $n_1 + n_2$ of saddles may be much greater than the number $n_0 + n_3$ of extrema.

5 The passive scalar field $\theta(\mathbf{x}, t)$ in turbulent flow

Suppose now that $s = \theta(\mathbf{x}, t)$ is a passive scalar field satisfying the advection-diffusion equation
$$\frac{D\theta}{Dt} \equiv \frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \kappa\nabla^2\theta \tag{22}$$
in a turbulent flow $\mathbf{u}(\mathbf{x}, t)$. Note first that if $\kappa = 0$, then all surfaces (including separatrix surfaces) $\theta =$ cst. are frozen in the fluid; hence their topology is invariant, and the set of numbers $\mathbf{n} = (n_0, n_1, n_2, n_3)$ is constant. If $\mathbf{n} = 0$ at some initial instant (e.g. if $\theta = \mathbf{c}\cdot\mathbf{x}$ at $t = 0$, a field of uniform gradient $\mathbf{c}$), then (when $\kappa = 0$) $\mathbf{n} = 0$ for all $t > 0$: turbulence alone cannot generate maxima or minima or saddle points of θ; it needs the cooperation of diffusion ($\kappa > 0$) to achieve this. Some aspects of this problem have been explored by Gibson (1968).

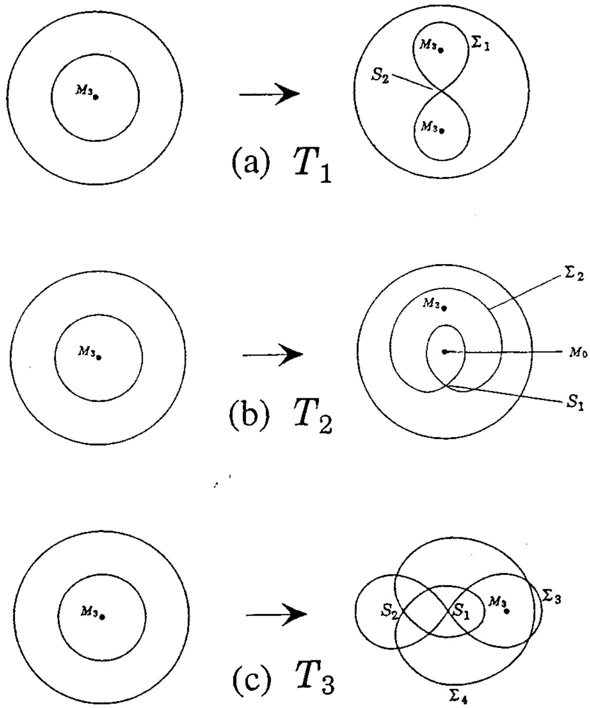


Figure 6: Three topological transitions by which a scalar field may be successively complexified.

If κ is small (equivalently, if the Peclet number is large) it may take a long time for $\mathbf{n}(t)$ to attain statistical equilibrium, starting from a situation in which $\mathbf{n}(0) = 0$. The behaviour of $\mathbf{n}(t)$ as a function of time under the action of a 'prescribed' field of turbulence, is a problem that should be amenable to DNS.

Here, we simply speculate on the (statistical) equilibrium established after a long time, assuming that the turbulence is statistically steady and the Peclet number large. Suppose first that the Prandtl number $Pr = \nu/\kappa$ is $\lesssim O(1)$. Then (Batchelor, Howells & Townsend 1959), the spectrum $\Gamma(\kappa)$ of θ^2 has a $k^{-5/3}$-range ($k_0 \lesssim k \lesssim k_c = (\epsilon/\kappa^3)^{1/4}$) and a relatively steep ($k^{-17/3}$) decrease for $k \gtrsim k_c$. The spectrum of $\nabla\theta$ is $k^2\Gamma(k)$ and

$$(k \cdot k^2\Gamma(k))^{1/2} \sim k^{2/3} \quad (k_0 \lesssim k \lesssim k_c)\,. \tag{23}$$

By analogy with the 2D situation already considered, the increase with k in (23) indicates that the topology of θ is controlled at the upper end of the range, and that the number $N = |\mathbf{n}(t)|$ of critical points of θ is of order $k_c^3 = (\epsilon/\kappa^3)^{3/4}$ per unit volume.

Suppose now that $\nu/\kappa \gg 1$. Then (Batchelor 1959) $\Gamma(k) \sim k^{-5/3}$ for $k_0 \lesssim k \lesssim k_v = (\epsilon/\nu^3)^{1/4}$, and there is a further conduction subrange

$$\Gamma(k) \sim k^{-1} \qquad (k_v \lesssim k \lesssim k_c = (\epsilon/\nu k^2)^{1/4})\,. \tag{24}$$

In this subrange,

$$(k \cdot k^2\Gamma(k))^{1/2} \sim k \tag{25}$$

and this suggests that the topology of θ is again controlled at the conduction cut-off k_c, implying

$$N \sim k_c^3 \sim (\epsilon/\nu k^2)^{3/4} \quad (\kappa \ll \nu) . \tag{26}$$

There is reason however to doubt the validity of this result, which would imply a large number (of order $(\nu/\kappa)^{3/2}$) of critical points of θ within every 'Kolmogorov sphere' of radius $(\nu^3/\epsilon)^{1/4}$; within any such sphere, the velocity gradient is (as assumed by Batchelor) approximately uniform, and it is difficult to see how transitions of types T_1, T_2 and T_3 can be induced by such a flow on such a scale.

For this reason, it seems more likely that in all circumstances, the number of critical points per unit volume is given by

$$N \sim \min\left((\epsilon/\kappa^3)^{3/4}, (\epsilon/\nu^3)^{3/4}\right) . \tag{27}$$

This estimate should again be amenable to testing by DNS.

I am grateful to Boris Khesin and Paul Glendinning for helpful discussions on the topic of this paper.

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