

〔特別講演〕

# Corner flow : a classical problem with a new twist

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An informal review is presented of the problem of viscous flow in a corner between two intersecting plane rigid boundaries, with particular attention to the behaviour (i) when the stirring that generates the flow is remote and sinusoidal, and (ii) when the stirring is provoked by torsional oscillation of the fluid domain about the line of intersection of the two boundaries. In each case, weak inertial effects lead to a lag of the response of the fluid behind the forcing mechanism. Conditions determining the existence and evolution of a geometric sequence of eddies in the corner are determined, and the manner in which the associated streamline pattern reverses during each half-period of the flow is described. Full details may be found in the recent paper of Branicki & Moffatt (2006) <sup>1)</sup>.

(KEY WORDS): Stokes flow, corner eddies, singularities, time-periodic, inertial effects, polygonal domain,

## 1 Introduction

It is a privilege to have this opportunity to lecture to the Japanese Society for Fluid Mechanics, and I am most grateful to your President, Professor Yukio Kaneda, and to Professor Yasuhide Fukumoto, for having extended this invitation to me. I am particularly glad to lecture here on the topic of corner flows, because it was here at the University of Kyushu that Professor Taneda in 1979 so brilliantly provided experimental verification of the existence of corner eddies in Stokes flow <sup>2)</sup>. His beautiful photographs were reproduced in Van Dyke's Album of Fluid Mechanics, and are, I'm sure, well known to all of you here, as they are throughout the world. I would like to dedicate this lecture to Professor Taneda in recognition of his remarkable and lasting contributions to experimental fluid mechanics.

## 2 Steady corner flow

I start by reminding you of the universal form of steady two-dimensional low-Reynolds number flow near the sharp corner between two fixed plane rigid boundaries  $\theta = \pm\alpha$  (the motion being driven by some arbitrary stirring mechanism far from the corner), for which it is reasonable to assume that the streamfunction  $\psi(r, \theta)$  takes the asymptotic form  $\psi(r, \theta) \sim r^\lambda f(\theta)$  as  $r \rightarrow 0$ . Substituting this form into the biharmonic equation and applying the no-slip condition on both boundaries leads to a transcendental equation for  $\mu (= \lambda - 1)$  as a function of  $\alpha$ :

$$\sin 2\mu\alpha = \pm\mu \sin 2\alpha, \quad (1)$$

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where the plus or minus is chosen according as the flow is antisymmetric or symmetric about the bisecting plane  $\theta = 0$ . The problem appears to have been first addressed by Lord Rayleigh, who recognised that, for acute angles  $\alpha$  (actually for  $2\alpha < 146^\circ$ ), these equations have no real solutions (other than the irrelevant solution  $\mu = 0$ ), and who concluded (wrongly) that  $\psi$  tends to zero faster than any power of  $r$  as  $r \rightarrow 0$ . The situation was rectified in 1949 by Dean & Montagnon <sup>3)</sup>, who recognised that complex solutions  $\mu = p + iq$  could be relevant; they did not however pursue this insight to the point of understanding the astonishing implication for the structure of the flow. Since  $\text{Re}[r^{p+iq}] = r^p \cos q(\ln r + \epsilon)$ , the velocity components oscillate infinitely (albeit quite strongly damped) as  $r \rightarrow 0$  ( $\ln r \rightarrow -\infty$ ), implying the existence of an infinite geometric sequence of eddies as the corner is approached.

I first encountered this phenomenon in 1962/3 when I was giving a course of lectures on low-Reynolds-number flow to students at Cambridge University. I had to devise suitable problems for a graduate-level examination (Part III of the Mathematical Tripos), and this type of corner flow seemed a good candidate. I soon ran into the above startling phenomenon of an infinite sequence of corner eddies, and it seemed at first so strange that I could hardly believe it to be true! After all, it is well-known that for prescribed boundary velocities, the (unique) Stokes flow in a finite domain has a smaller rate of viscous dissipation than any other kinematically possible flow satisfying the same boundary conditions. How astonishing that this minimally-dissipating flow should have an infinitely reversing sequence of eddies! I discussed this finding with

my research mentor George Batchelor, and was still further astonished when he agreed that the sort of ‘Stokes separation’ that this implied seemed to him altogether plausible! Thus encouraged, I duly submitted my paper on the subject to JFM <sup>4</sup>).

The interesting thing about the solution is that it is truly universal, providing the asymptotic form of the generic two-dimensional flow near any sharp corner, irrespective of the nature of the remote forcing. Moreover, even if the remote flow conditions are in the high-Reynolds-number regime, the *local* Reynolds number near any perfectly sharp corner with fixed boundaries (based on distance  $r$  from the corner) is always small, so that, provided always that  $2\alpha < 146^\circ$ , the low-Reynolds-number eddies are always present (although possibly on an extremely small scale in the immediate vicinity of the corner!).

It must however be conceded that, although the mathematics implies an infinite geometrical sequence of eddies, not more than two or three of these eddies will ever be observed in practice; this is because the factor  $r^p$  always implies rapid attenuation as  $r \rightarrow 0$ . In fact,  $p$  is always greater than 2.5 for acute angles, and the characteristic circulation time-scale increases by a factor of at least 300 as we pass from one eddy to the next, towards the corner. If this time-scale is one second for the first eddy in the sequence, then it will be 300s in the second,  $300^2$ s ( $\sim 25$  hours) in the third, and so on. Taneda observed the second eddy using a very long photographic exposure (90 minutes), but this was not quite long enough to reveal the third eddy; and to detect the fourth would require an exposure time of the order of one year! We are thus faced with a situation in which mathematics reveals the structure of a singularity that appears to be quite beyond the limits of experimental observability, although experiments do provide incontrovertible evidence for what may be described as the ‘signature’ of the singularity, namely a clearly defined streamline separating a primary eddy from a secondary eddy of opposite circulation. This is a phenomenon that we may indeed describe as ‘Stokes separation’, by analogy with boundary-layer separation at high Reynolds number; but here, separation is induced by viscosity, inertial effects being completely negligible (the more so as the corner is approached!).

### 3 Variants

There are many variants of this corner problem, among which the principal are perhaps the following: (i) the corner may be ‘cusped’ in which case the eddies still survive;

(ii) instead of a corner between two planes, one may consider the analogous axisymmetric flow between two cones with common vertex and axis of symmetry<sup>5</sup>); (iii) the corner may be not quite sharp, in which case the infinite sequence of eddies is simply replaced by a finite sequence, the number depending in an obvious way on the degree of rounding of the corner; (iv) the corner flow may be coupled with a source flow or a flow through a narrow constriction<sup>6</sup>), in which case attached wall eddies appear (which can in some circumstances migrate into the fluid interior; (v) the corner eddies may be in competition with other contributions to the flow that are driven by either local or remote boundary conditions<sup>7</sup>); (vi) the flow in a two-dimensional corner may be three-dimensional<sup>8</sup>); (vii) the corner geometry itself may be three-dimensional; (viii) finally, the flow may be time-periodic or more generally unsteady<sup>1</sup>) (Branicki & Moffatt 2006, hereafter BM’06). [These variants and their numerous possible combinations have been studied in a multitude of papers; only a very small selection can be cited here due to limitations of space.]

The last possibility (viii) is particularly intriguing, because it is generally assumed that Stokes flow is ‘quasi-static’, i.e. the flow is instantaneously determined by motion of the boundaries. This means that if the boundary motion is sinusoidal in time, then at the instant at which the boundaries are at rest, the fluid is at rest also. In practice, for any fluid of finite viscosity, this is not the case: a residual motion obviously remains in the fluid at such instants, as a result of (possibly weak) fluid inertia. For flow in a corner, with its infinite sequence of eddies, the manner in which these reverse in response to reversal of the remote stirring is of particular interest; a primary motivation of the study of BM’06 was to understand this reversal process.

### 4 Oscillating-cylinder problem

There was a second motivation arising from a meeting held at the Isaac Newton Institute in Cambridge in 2004 to commemorate the centenary of the death of George Gabriel Stokes (1821-1904), at which I was asked to give a lecture on Stokes’s contributions to the dynamics of real (i.e. viscous) fluids (<http://www.newton.cam.ac.uk/webseminars/stokes/>). In reading Stokes’s early papers, I found that he had considered the following problem<sup>9</sup>) (Stokes 1843): suppose that an *ideal* fluid is contained inside a rectangular parallelepiped, and that this is subject to an arbitrary rigid body motion (translation plus rotation); the internal flow, assumed

to be irrotational and therefore expressible in terms of a potential  $\mathbf{u} = \nabla\phi$ , is then instantaneously determined by the distribution of normal velocity at the boundary. Stokes found the relevant solution  $\phi(\mathbf{x}, t)$  of Laplace’s equation in terms of Fourier series (quite a novel technique at that epoch).

The case of a cylinder of polygonal cross-section subjected to torsional oscillations about its axis is a special case that can be easily realised experimentally (and it seems possible that Stokes himself attempted such an experiment). Part of his purpose was to demonstrate to what extent the behaviour of real (i.e. viscous) fluids is different from that predicted by potential flow theory. Of course, in this case, the no-slip condition has a profound influence on the development of the flow; indeed for a steady rotation of the container, the flow is ultimately one of uniform vorticity, and certainly therefore not irrotational! For oscillatory rotational motion of the boundary, the interior fluid lags behind the boundary to an extent determined by the fluid viscosity. For high frequency oscillations, this ‘lagging behind’ may be described locally by the ‘Stokes layers’ through which vorticity is diffused into the interior, a mechanism identified somewhat later (Stokes 1850)<sup>9)</sup>. Part of our purpose was to explore the manner in which these Stokes layers interact with the corner eddies which evolve, again in response to the lag between the flow and the boundary in each corner region, and to elucidate the manner in which the flow reversal that must occur during each half-period is actually effected.

**5 Time-periodic corner flow**

It will be sufficient here to outline the technique and to summarise the results obtained for this problem; full details may be found in BM’06<sup>1)</sup> (and in the computer animations linked to the online version of this paper). There are two distinct situations to be considered: first, that in which the corner flow is generated by remote sinusoidal stirring with velocities proportional to  $\cos\omega t$  say, the prototype being the ‘oscillating-lid problem’; and second, that in which the flow is generated by torsional oscillation of the whole fluid container (the ‘oscillating-cylinder problem’ described in §4 above).

**5.1 Oscillating-lid problem**

Retaining only the local contribution  $\partial\mathbf{u}/\partial t$  to acceleration, the streamfunction  $\psi(r, \theta, t)$  satisfies the equation

$$\partial\nabla^2\psi/\partial t = \nu\nabla^4\psi, \tag{2}$$

and, writing  $\psi = \text{Re} [\Psi(r, \theta)e^{i\omega t}]$ , the complex function  $\Psi$  satisfies

$$i\omega\nabla^2\Psi = \nu\nabla^4\Psi. \tag{3}$$

From  $\omega$  and  $\nu$ , we may construct a characteristic length-scale  $\delta = \nu/\omega$ . If  $r \ll \delta$ , then the solution may be expressed in the form

$$\Psi = \Psi_0 + (-i\omega/\nu)\Psi_1 + (-i\omega/\nu)^2\Psi_2 + \dots, \tag{4}$$

where

$$\nabla^4\Psi_0 = 0, \quad \nabla^4\Psi_{n+1} = \nabla^2\Psi_n, \quad (n = 1, 2, 3, \dots). \tag{5}$$

Thus  $\Psi_0 \cos\omega t$  is the quasi-static solution, and  $\Psi_1$  provides the essential inertial correction. This  $\Psi_1$  includes a ‘complementary function’ which has the same structure as  $\Psi_0$  and can be ignored, and a ‘particular integral’ with the property  $\Psi_1 = O(r^2)\Psi_0$ . Taking  $\Psi_0$  to be real,  $\Psi_1$  is also real, and  $\psi$  is then given by

$$\psi = \Psi_0 \cos\omega t + (\omega/\nu)\Psi_1 \sin\omega t. \tag{6}$$

The inertial correction is evidently in quadrature with the quasi-static term and, as  $\Psi_1$  has different spatial structure from  $\Psi_0$ , describes the flow in the neighbourhood of the instants  $\omega t = (n + \frac{1}{2})\pi$  when the quasi-static term changes sign. For example, the flow reversal in the neighbourhood of  $\omega t = \pi/2$  is adequately described by

$$\psi = \Psi_0(r, \theta)(\pi/2 - \omega t) + (\omega/\nu)\Psi_1(r, \theta), \tag{7}$$

and the duration of the reversal (i.e.the period during which the streamline topology changes) is  $\Delta t \sim r^2/\nu$ ; this dependence on  $r$  simply means that, if the process were visible, the change of topology would appear more and more rapid as the corner is approached, i.e. as we move through the sequence of eddies towards the corner.

In the numerical simulations reported in BM’06 (for which a circular ‘lid’  $r = L$  was oscillated in the  $\theta$  direction), the topological evolution of only two successive eddies is visible. This is sufficient however to infer the whole generic evolution of the infinite eddy sequence for this type of oscillating lid flow (for the range of angles  $2\alpha < 146^\circ$  for which this sequence exists): if we designate a ‘primary’ eddy in this sequence at a given instant as  $E_1$  and the succeeding eddies at that instant as  $E_2, E_3, \dots$ , then if the circulation in  $E_1$  is, say, clockwise, then that in  $E_{2n+1}$  is also clockwise, whereas in  $E_{2n}$  it is anti-clockwise (for each  $n > 0$ ). During a flow reversal, computer animations of

$\psi(r, \theta, t)$  (determined analytically) reveal that each eddy in the sequence appears to grow and to move outwards from the corner, so that at the completion of the reversal each  $E_n$  is replaced by  $E_{n+1}$  (and the flow direction at each fixed point in the corner domain is thus reversed). In particular, the primary eddy  $E_1$  is now replaced by the enlarged (and intensified)  $E_2$ . The whole process then repeats itself during each succeeding half-period.

So it would appear that the tiny corner eddies, too weak to be observed under steady conditions, do play a role when the remote (lid) conditions are time-periodic: they emerge successively from the corner, one in each half-period, growing in stature like Kabuki actors on a stage, and ultimately taking the lead role!

In the region much further from the corner, where  $r \gg \delta$ , the situation is very different. Here, the leading-order solution  $\hat{\Psi}_0 \cos \omega t$ , say, satisfies Laplace's equation  $\nabla^2 \hat{\Psi}_0 = 0$ , and the boundary condition  $\hat{\Psi}_0 = 0$  on  $\theta = \pm\alpha$ . Obviously there is no question of a hierarchy of corner eddies in this region; however, Stokes-type oscillatory boundary layers are required in order that the streamfunction  $\psi$  may satisfy the no-slip condition  $\psi = \partial\psi/\partial n = 0$  on  $\theta = \pm\alpha$ . These boundary layers have thickness of order  $(r\delta)^{1/2}$  at distance  $r$  from the corner, and are well separated only for  $r \gg \delta$ . As  $r$  decreases, the two boundary layers overlap when  $r = O(\delta)$ , and in this transitional region, the Stokes layers presumably blend into the corner eddies of the inner region  $r \ll \delta$ .

## 5.2 Oscillating-cylinder problem

The treatment here is very similar, except that at leading order the flow is simply the rigid body motion compatible with the oscillatory motion at the boundaries. In the inner region  $r \ll \delta$ , at first order,  $\Psi_1 \sin \omega t$  gives the first approximation to the flow relative to this rigid motion; this reverses at times  $\omega t = n\pi$ , and it is necessary to go to the next order involving  $\Psi_2 \cos \omega t$  to capture the manner in which the flow reversals are achieved. At each order, the 'eigenfunction' terms, which are oscillatory provided  $2\alpha < 2\alpha^* \approx 146^\circ$ , are in competition with non-oscillatory 'particular integrals' proportional to  $r^4$  at first order and  $r^6$  at second order, and an infinite sequence of eddies exists at a given phase of the cycle if and only if at this phase the oscillatory contributions dominate as  $r \rightarrow 0$ . Transitions in behaviour occur for corner angles  $2\alpha$  at which the smallest positive value of  $\text{Re}\lambda$  equals 4 (when  $2\alpha = 81.9^\circ = 2\alpha_1$ , say), or 6 (when  $2\alpha = 48.7^\circ = 2\alpha_2$ , say).

The conclusions that can be drawn from such considera-

tions are as follows<sup>1)</sup>: (i) for  $2\alpha < 2\alpha_2$  and for  $2\alpha > 2\alpha^*$ , the flow in the corner relative to the moving boundaries is non-oscillatory (i.e. does not display the infinite sequence of eddies) throughout the flow cycle; (ii) if  $2\alpha_2 < 2\alpha < 2\alpha_1$ , then the flow is non-oscillatory except during the short periods of reversal when an infinite sequence of eddies appears; numerical animations indicate that these eddies grow from the walls, forming saddle points on  $\theta = 0$  and separating streamlines that propagate both towards and away from the corner; (iii) if  $2\alpha_1 < 2\alpha < 2\alpha^*$ , then (as for the oscillating-lid situation) an infinite sequence of eddies is present throughout the flow cycle.

The case of a regular polygon of  $n$  sides and internal angle  $2\alpha = (1 - 2/n)\pi$  provides an interesting, if perhaps academic, illustration. If  $n = 3$ , case (ii) above is applicable; if  $4 \leq n \leq 10$ , then case (iii) is applicable; and if  $n \geq 11$ , then  $2\alpha > 2\alpha^*$ , and so case (i) is applicable.

The outer region  $r \gg \delta$  again has a very different character: as assumed by Stokes, the flow is irrotational in a core region, and the no-slip condition requires the existence of non-uniform Stokes layers on both walls  $\theta = \pm\alpha$ , just as described above for the oscillating-lid case.

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