Comment on

Natural classification of knots

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Kelvin's (1867) conjecture that the fundamental indivisible particles of matter (atoms in his day) could be represented as knotted vortices imbedded in a background ether has been recently revived in the more esoteric context of quantum chromodynamics (QCD), in which glueballs (metastable, possibly solitonic, solutions to the QCD equations) have been conceived as tightly knotted and/or linked tubes carrying quantised magnetic flux (Buniy & Kephart *Int.J.Mod.Phys.*A**20**, 1252-1259, 2005). Kelvin's original conception lapsed with his inability to find stable knotted vortex equilibrium states. However, it is now recognised that analogous knotted *magnetic flux tubes* in an ideal medium can be minimum-energy (therefore stable) states for prescribed knot topology (Moffatt, *Nature* **347**, 367-369, 1990). Such tubes may have an arbitrary internal twist, related to the helicity of the configuration; for small or zero helicity, the minimum-energy states are those in which the knots are, as it were, *pulled tight*: for a tube of given cross-section (assumed circular), this just means that it has minimal length consistent with the prescribed knot topology. In the terminology of Katritch et al. (*Nature*, **388**, 148-151, 1997) it is, in this minimal energy configuration, an *ideal knot* (perhaps *tight knot* is a more natural description).

Just as a magnetic field in a conducting medium can decay through Joule dissipation with associated reconnection of field lines, so a glueball can decay by various reconnection processes, for example, that in which one strand of a knot is allowed to pass through another. Such a passage frequently (though by no means always) changes the crossing number of the knot by 2. In such a transition, the predecessor (or parent) knot is that with the smaller crossing number, while the successor (or daughter) knot is that with the larger. A parent can obviously have more than one daughter. Less obviously, a daughter can have more than one parent. Two 'family trees' of knots can be constructed on the basis of such reconnection events (one for knots of odd crossing number, and one for even).

It is just such a construction that has now been effected by Flammini & Stasiak, as a result of extensive computational exploration, for all knots of up to nine crossings. The technique is to represent a knot by a random curve consisting of 32 equal straight-line segments which are allowed to evolve through 'random crankshaft motions'. A process is adopted that ensures that each knot starts in a random configuration, and the process terminates after the first subsequent passage of one segment through another. Each knot resulting from such segment transitions is identified by computation of its HOMFLY polynomial. The process for each initial knot type is carried out 20,000 times and the probability (to within about 1%) of any parent-daughter (or daughter-parent) transition in a random evolution is thus obtained.

One curious and as yet unexplained property of tight knots (Katritch et al., *loc.cit.*) is that the writhe (i.e. the sum of the signed crossings averaged over all projections) is apparently quantised, increasing linearly with crossing number within certain welldefined knot families (e.g. torus knots). Flammini & Stasiak have exploited this property in presenting the family trees on a two-dimensional plot, with ordinate the crossing number, and abscissa the writhe of the tight configuration (‘tight writhe’?). This reveals well the discreteness of the transitions in both variables. The relative probabilities of the various transitions are indicated by colour coding. [The writhe of a knot is of course not a topological invariant, but ‘writhe plus twist’, or equivalently helicity, *is* such an invariant. Twist can be converted to writhe by the formation of loops in a tube, like those that can appear spontaneously in a garden hose. The ‘writhe of a knot’ therefore has meaning only if the geometrical configuration is specified, as it is, as least to some extent, for a tight knot.]

Buniy & Kephart (*loc.cit.*) show a remarkable, indeed amazing, linear relationship between glueball masses (as reported by the Particle Data Group) and energies of tight knots/links of up to six crossings, and a few beyond (see also J.P.Ralston, arXiv:hep-ph/0301089 v1 for an interpretation). Just as Kelvin’s conjecture (which may not have been so far off the mark after all!) led to a major surge of interest in the problem of knot classification (sowing the seeds of topology as a new branch of mathematics), so this discovery in QCD may be expected to stimulate intense further activity on the family trees of tight knots, as initiated by Flammini & Stasiak in this pioneering investigation.

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