# Magnetostrophic Turbulence and the Geodynamo

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**Abstract.** The flow generated by a random buoyancy field in a rotating medium permeated by a dynamo-generated magnetic field is considered, under the assumptions that the Rossby number and the magnetic Reynolds number (based on the scale of the buoyancy fluctuations) are both small. This permits linearisation of the governing evolution equations. Provided 'up-down' symmetry is broken, a mean helicity and an associated $\alpha$-effect are generated. These are calculated in terms of the spectrum function of the buoyancy field. Expressions are also obtained for the buoyancy flux and the Reynolds stresses (kinetic and magnetic), and an outline dynamo scenario is proposed. The nature of this type of magnetostrophic turbulence is briefly discussed.

**Keywords:** magnetostrophic, geodynamo, helicity, $\alpha$-effect, mean-field theory

## 1 Introduction

Dynamo theory is concerned with the generation of magnetic fields by fluid motion. When the system considered is sufficiently large, the inductive action of flow across any weak magnetic field induces currents which can, for suitable flow configurations, lead to amplification of the magnetic field. The flow is then unstable to the growth of this field, which grows exponentially until the associated Lorentz force reacts back upon the flow, leading to field saturation.

It is well known that the helicity of the fluid motion (i.e. correlation of velocity and vorticity) is highly conducive to dynamo action. Helicity is a pseudo-scalar quantity, and is generally present only if the fluid is rotating and if the source of energy for the flow breaks the symmetry with respect to the rotation vector, for example if the motion consists of buoyant packets of fluid rising through an otherwise quiescent medium. The precise condition will be clarified in §3 below.

It has been customary in turbulent dynamo theory to start with a kinematic approach in which the statistics of the velocity field are assumed known, and to focus attention on the evolution equation for the magnetic field $\mathbf{B}(\mathbf{x}, t)$



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(i.e. the induction equation). The techniques of mean-field electrodynamics are then available to determine a simplified equation for the slow evolution of the large-scale field (averaged over scales characteristic of the turbulence) in terms of an $\alpha$-effect and a turbulent diffusivity (both tensor in character). It has generally been found difficult to extend this approach to the nonlinear regime in which the back-reaction of the field on the flow is considered; this of course requires parallel analysis of the Navier-Stokes equation, including both Coriolis and Lorentz forces and incorporating explicitly the forces, whether of convective origin or otherwise, that drive the fluid motion.

I propose here an approach, relevant to the particular case of the geodynamo, which overcomes at least some of these difficulties. This takes as its starting point the idealised model of [1], [2], in which the ultimate sources of energy for the geodynamo are assumed to be of thermal and gravitational origin: slow cooling of the Earth leads to slow solidification of the liquid metal core onto the solid inner core. This solidification process takes place in a 'mushy zone', whose depth is of the order of 1km, and within which lighter elements (sulphur, oxygen,...) are 'rejected', the resulting density of the inner core being about 5% greater than the mean density of the liquid core. This results in the continuous creation of a buoyant layer which intermittently erupts from the mushy zone, driving what is primarily a state of compositional convection in the liquid core. The upward flux of buoyancy (equivalently the downward flux of mass) can be estimated on the assumption that the inner core has been growing at a roughly uniform rate over the lifetime of the Earth. This, coupled with the reasonable assumption of geostrophic balance between buoyancy and Coriolis forces, leads to estimates $V \sim 0.2$mm/s and $\theta = \Delta\rho/\rho \sim 3 \times 10^{-9}$ for the typical upward velocity $V$ and buoyancy $\theta$ of upwardly mobile buoyant elements.

It was assumed in [2] for want of better that these buoyant elements remain coherent and roughly spherical throughout their rise towards the core-mantle boundary. Subsequent computations [3] revealed however that this assumption is untenable: when a (dynamo-generated) toroidal magnetic field is present, a spherical blob is apparently subject to a 'slicing' instability as a result of the combined effect of Coriolis and Lorentz forces. Typical localised motions may also be expected to be strongly anisotropic (see for example [4]). It therefore becomes necessary to abandon any assumption concerning the shape of rising elements, this being determined by the full complex dynamics of the convective process.

The approach adopted here (somewhat in the spirit of kinematic dynamo theory, but now including essential features of the dynamics) is to suppose that the *statistics* of the buoyancy field $\theta(\mathbf{x}, t)$ are prescribed, and that (following [2]) the scale $L$ of the convective turbulence lies in the range $V/\Omega \ll L \ll \eta/V$ where $\Omega$ is the angular velocity of the Earth, and $\eta$ is the resistivity (or magnetic diffusivity) of the fluid medium, i.e. that the Rossby number $Ro = V/\Omega L$ and the magnetic Reynolds number $R_m = VL/\eta$ based on the 'blob scales' $L$ and $V$ are both small. This restricts $L$ to the range between 10m and 100km, not unreasonable for a buoyancy distribution that is supposed to originate from a layer of thickness ∼1 km. These assumptions allow linearisation of both the induction equation and the equation of motion; the sole remaining nonlinearity is the advective term in the advection-diffusion equation for buoyancy $\theta$; but since we assume that the statistics of $\theta$ are prescribed, this difficulty can be postponed, if not avoided altogether.

## 2 The Magnetostrophic Equations

We suppose that the magnetic field $\mathbf{B}(\mathbf{x}, t)$ in the core of the Earth consists of a mean part $\mathbf{B_0}$, which results from dynamo action and can be considered as locally uniform and steady, and a perturbation field $\mathbf{b}(\mathbf{x}, t)$ induced by the flow $\mathbf{u}(\mathbf{x}, t)$ across $\mathbf{B_0}$. Under the above assumptions that the Rossby number and magnetic Reynolds number are both small, the governing evolution equations may be linearised in the form

$$2\,\mathbf{\Omega} \wedge \mathbf{u} = -\nabla P + (\mu_0 \rho)^{-1} \mathbf{B_0} \cdot \nabla \mathbf{b} - \theta \mathbf{g}\,, \tag{1}$$

$$\partial \mathbf{b}/\partial t = \mathbf{B_0} \cdot \nabla \mathbf{u} + \eta \nabla^{\mathbf{2}} \mathbf{b}\,, \tag{2}$$

$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0\,, \tag{3}$$

where $\mathbf{\Omega}$ is the angular velocity of the Earth, $\mathbf{g}$ is the local gravitational acceleration, $\rho$ is the mean density of the fluid, and $\rho P$ is the sum of fluid and magnetic pressure. These are the ‘magnetostrophic equations’ used in [2], except that here, recognising that magnetic field diffusion in a changing environment is not instantaneous, we retain the local time derivative $\partial \mathbf{b}/\partial t$. With this term included, the equations describe magnetostrophic waves [7] driven by the buoyancy term $-\theta \mathbf{g}$. The approach is thus within the spirit of Braginski's *MAC*-wave scenario [5].

It is perhaps mildly inconsistent to retain the term $\partial \mathbf{b}/\partial t$ in (2) while dropping the nonlinear term $\{\mathbf{u} \cdot \nabla \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{u}\}$ which may be expected to be of similar order of magnitude; this is just a matter of practicality, and we expect that, if the analysis were pursued to higher order in $R_m$, this nonlinear term would have a similar qualitative effect to that of the ‘local’ time-derivative $\partial \mathbf{b}/\partial t$, although involving higher-order statistics of the $\theta$-field.

Neglecting transients, as is appropriate under statistically steady conditions, the equations (1)–(3) evidently establish a linear relationship between $\mathbf{u}$ and $\theta$ (and also between $\mathbf{b}$ and $\theta$); it follows that mean quadratic quantities such as the helicity $\mathcal{H} = <\mathbf{u} \cdot \omega>$ (where $\omega = \nabla \wedge \mathbf{u}$) and the Reynolds stress $\tau_{ij} = < u_i u_j >$ are quadratically related to $\theta$, and should therefore emerge as weighted integrals of the spectrum function of $\theta$. The detailed forms of such integrals are obtained in the following sections.

## 3 Expressions for Mean Helicity and $\alpha$-Effect

We shall suppose that the $\theta$-field is statistically stationary and locally homogeneous (though certainly not isotropic). It then admits Fourier decomposition in the form

$$\theta(\mathbf{x}, t) = \int \hat{\theta}(\mathbf{k}, \omega) e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} \mathrm{d}\mathbf{k}\, \mathrm{d}\omega\,. \tag{4}$$

The spectrum function $\Gamma(\mathbf{k}, \omega)$ is related to the Fourier transform $\hat{\theta}(\mathbf{k}, \omega)$ by

$$< \hat{\theta}^*(\mathbf{k}, \omega)\hat{\theta}(\mathbf{k}', \omega') > = \Gamma(\mathbf{k}, \omega)\delta(\mathbf{k} - \mathbf{k}')\delta(\omega - \omega'). \tag{5}$$

$\Gamma(\mathbf{k}, \omega)$ is real, and, by virtue of the reality of the field $\theta$, satisfies the condition $\Gamma(-\mathbf{k}, -\omega) = \Gamma(\mathbf{k}, \omega)$. Note however that, in general,

$$\Gamma(\mathbf{k}, \omega) \neq \Gamma(\mathbf{k}, -\omega)\,. \tag{6}$$

This inequality corresponds to a breaking of 'up-down' symmetry; thus for example, if the $\theta$-field consists of a distribution of non-overlapping buoyant blobs that rise without change of shape with constant velocity $\mathbf{V}$, with a compensating downward flow of the ambient fluid around the blobs, then

$$\Gamma(\mathbf{k}, \omega) = \Gamma_0(\mathbf{k})\delta(\omega - \mathbf{k}\cdot\mathbf{V})\,, \tag{7}$$

where $\Gamma_0(\mathbf{k})$ is the spectrum function of $\theta$ in the frame of the blobs; this expression is evidently not equal to $\Gamma(\mathbf{k}, -\omega)$. This breaking of up-down symmetry will turn out to be important in what follows.

Equations (1)–(3) can be Fourier transformed, giving

$$2\boldsymbol{\Omega} \wedge \hat{\mathbf{u}} = -\mathrm{i}\mathbf{k}\hat{P} + \mathrm{i}(\mathbf{B_0}\cdot\mathbf{k})\hat{\mathbf{b}} - \hat{\theta}\,\mathbf{g}\,, \tag{8}$$

$$\hat{\mathbf{b}} = \mathrm{i}(\eta k^2 - \mathrm{i}\omega)^{-1}(\mathbf{B_0}\cdot\mathbf{k})\hat{\mathbf{u}}\,, \tag{9}$$

$$\mathbf{k}\cdot\hat{\mathbf{u}} = \mathbf{k}\cdot\hat{\mathbf{b}} = 0\,. \tag{10}$$

(Here, we use Alfven units for $\mathbf{B_0}$ and $\mathbf{b}$, so that the prefactor $(\mu_0\rho)^{-1}$ in (1) disappears.) We may easily solve these equations for $\hat{\mathbf{u}}$ in the form $\hat{\mathbf{u}} = [\mathbf{A}(\mathbf{k}, \omega)/D(\mathbf{k}, \omega)]\hat{\theta}$ where

$$\mathbf{A} = -(\mathbf{k}\cdot\mathbf{B_0})^2(\eta k^2 - \mathrm{i}\omega)^{-1}\mathbf{k} \wedge (\mathbf{k} \wedge \mathbf{g}) - 2(\mathbf{k}\cdot\boldsymbol{\Omega})\mathbf{k} \wedge \mathbf{g}\,, \tag{11}$$

$$D = 4(\mathbf{k}\cdot\boldsymbol{\Omega})^2 + (\mathbf{k}\cdot\mathbf{B_0})^4(\eta k^2 - \mathrm{i}\omega)^{-2}k^2\,. \tag{12}$$

Here, we may note that $D = 0$ is the dispersion relation for magnetostrophic waves damped by magnetic diffusivity; if the forcing by the buoyancy distribution includes contributions from regions of $(\mathbf{k}, \omega)$-space for which $D = 0$ when $\eta = 0$, then a resonant response is to be expected, controlled by magnetic diffusivity. This control is strong in the low-$R_m$ regime. Note further that $\mathbf{A}$ is complex; writing $\mathbf{A} = \mathbf{P} + \mathrm{i}\mathbf{Q}$, we have

$$\mathbf{A}^* \wedge \mathbf{A} = 2\mathrm{i}\mathbf{P} \wedge \mathbf{Q} = 4\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\Omega})\omega(\mathbf{k}\cdot\mathbf{B_0})^2(\eta^2k^4 + \omega^2)^{-1}(\mathbf{k} \wedge \mathbf{g})^2\mathbf{k}\,. \tag{13}$$

We may now construct the mean helicity $\mathcal{H}$ in the form

$$\mathcal{H} =< \mathbf{u} \cdot \omega >= - \iint \frac{\mathrm{i}\mathbf{k} \cdot (\mathbf{A}^* \wedge \mathbf{A})}{|D|^2} \Gamma(\mathbf{k}, \omega) \, \mathrm{d}\mathbf{k} \, \mathrm{d}\omega \,, \tag{14}$$

or, using (13),

$$\mathcal{H} = 4 \iint \frac{(\mathbf{k} \cdot \mathbf{\Omega})\omega}{|D|^2} \frac{(\mathbf{k} \cdot \mathbf{B_0})^2}{\eta^2 k^4 + \omega^2} k^2 (\mathbf{k} \wedge \mathbf{g})^2 \Gamma(\mathbf{k}, \omega) \, \mathrm{d}\mathbf{k} \, \mathrm{d}\omega \,. \tag{15}$$

It is here that the breaking of up-down symmetry is important; for if $\Gamma(\mathbf{k}, -\omega) = \Gamma(\mathbf{k}, \omega)$ for all $(\mathbf{k}, \omega)$, then the integrand in (15) is an odd function of $\omega$, and so the integral (over all $\omega$) vanishes. Thus to get a non-vanishing helicity, we need $\Gamma(\mathbf{k}, -\omega) \neq \mathbf{\Gamma}(\mathbf{k}, \omega)$, i.e. up-down symmetry must be broken, as is the case if the convection is characterised by rising blobs, the compensating downward motion being topologically connected; incidentally, this is just the sort of convection that gives rise to downward topological pumping of horizontal magnetic flux, a process that is recognised and well-understood in stellar contexts [6].

The presence of helicity immediately implies an $\alpha$-effect, i.e. an electromotive force $\mathcal{E}$ linearly related to the large-scale (mean) magnetic field $\mathbf{B_0}$: $\mathcal{E}_i = \alpha_{ij} B_{0j}$, where the pseudo-tensor $\alpha_{ij}$ depends on $\mathbf{g}$, $\mathbf{\Omega}$ and $\mathbf{B_0}$, as well as on $\eta$ and the statistical properties of the $\theta$-field. Thus $\alpha_{ij}$ is certainly highly anisotropic; however, we can still easily evaluate the trace $\alpha = (1/3)\alpha_{ii}$, which is given, for low $R_m$, by

$$\alpha = -\tfrac{1}{3}\eta \iint \frac{k^2 \mathcal{H}(\mathbf{k}, \omega)}{\omega^2 + \eta^2 k^4} \, \mathrm{d}\mathbf{k} \, \mathrm{d}\omega \,, \tag{16}$$

([7], chapter 6) where we now have

$$\mathcal{H}(\mathbf{k}, \omega) = \frac{4(\mathbf{k} \cdot \mathbf{\Omega})\omega}{|D|^2} \frac{(\mathbf{k} \cdot \mathbf{B_0})^2}{\eta^2 k^4 + \omega^2} k^2 (\mathbf{k} \wedge \mathbf{g})^2 \Gamma(\mathbf{k}, \omega) \,. \tag{17}$$

Here we may note the phenomenon of ‘$\alpha$-quenching’, i.e. the reduction of the $\alpha$-effect with increasing $B_0$: since $|D|^2 \sim B_0^8$, (16) gives $\alpha \sim B_0^{-6}$, provided there are no complications associated with resonances (cf [7], pp 254-5). If however resonant magnetostrophic waves are excited, then the quenching effect may be weaker; see [8], where a quenching effect $\alpha \sim B_0^{-3/2}$ was obtained by integrating over the sub-regions of $(\mathbf{k}, \omega)$-space where resonance occurred.

## 4 Buoyancy Flux and Reynolds Stresses

The buoyancy flux $F_z = < \mathbf{u}\, \theta >_z$, where the average is here over horizontal planes, is also a weighted integral of the spectrum function $\Gamma(\mathbf{k}, \omega)$, and is given by

$$F_z = \iint \frac{(\mathbf{k}\cdot\mathbf{B_0})^2 \eta k^2 [\mathbf{k}\wedge(\mathbf{k}\wedge\mathbf{g})]_z \Gamma(\mathbf{k},\omega)}{4(\mathbf{k}\cdot\mathbf{\Omega})^2(\eta^2k^4+\omega^2) + (\mathbf{k}\cdot\mathbf{B_0})^4(\eta^2k^4-\omega^2)k^2}\,\mathrm{d}\mathbf{k}\,\mathrm{d}\omega\,. \tag{18}$$

This flux is in effect provided by a downward mass flux which can be estimated from the slow rate of growth of the inner core (on the assumption that the convection is indeed predominantly compositional in character). This therefore places an implicit constraint on the spectrum function $\Gamma(\mathbf{k},\omega)$ at each horizontal level $z$.

The Reynolds stress tensor $\tau_{ij} = < u_i u_j >$ is likewise a weighted integral of $\Gamma(\mathbf{k},\omega)$,

$$\tau_{ij} = \iint \frac{A_i^* A_j}{|D|^2}\Gamma(\mathbf{k},\omega)\,\mathrm{d}\mathbf{k}\,\mathrm{d}\omega\,. \tag{19}$$

Similarly, the magnetic counterpart of the Reynolds stress tensor is

$$\tau_{ij}^M = < b_i b_j > = \iint \frac{(\mathbf{k}\cdot\mathbf{B_0})^2}{\eta^2k^4+\omega^2}\frac{A_i^* A_j}{|D|^2}\Gamma(\mathbf{k},\omega)\,\mathrm{d}\mathbf{k}\,\mathrm{d}\omega\,. \tag{20}$$

These stresses drive an axisymmetric mean flow $\mathbf{U}$ via an equation of the form

$$2(\mathbf{\Omega}\wedge\mathbf{U})_i = -\frac{\partial}{\partial x_i}(\tau_{ij} - \tau_{ij}^M) + \ldots\,. \tag{21}$$

This mean flow is predominantly differential rotation, associated with the tendency of convecting elements to conserve angular momentum as they rise; it may also however include mean meridional circulation associated with the convection process. This mean flow provides a strain field which is locally uniform on the scale of the turbulence, and whose effect could in principle be included in (1) by an iterative procedure. This mean flow is axisymmetric, and it is important to recognise that, by virtue of Cowling's theorem, it is incapable on its own of maintaining a magnetic field with the same axis of symmetry. The helicity of the turbulence is, in this scenario, essential for dynamo action, the mean velocity field playing a secondary role.

## 5 Summary and Comments on the Nature of Magnetostrophic Turbulence

The picture that we have developed is thus as follows. Buoyancy flux $F_z$ associated with compositional convection establishes a stationary random distribution of buoyancy $\theta$, whose spectrum function $\Gamma(\mathbf{k},\omega)$ exhibits a breaking of up-down symmetry. This $\theta$-field drives an associated velocity field with non-zero helicity, leading to an $\alpha$-effect, and a dynamo process. The mean field $\mathbf{B_0}$ that is generated ultimately leads to $\alpha$-quenching and saturation of the dynamo process. A mean velocity is generated by a combination of dynamic and magnetic Reynolds stresses, and this has a secondary influence on the dynamo process.

This is of course just an outline scenario: the longer term aim must be to obtain a self-consistent set of equations for the axisymmetric mean fields

$\mathbf{B_0}$ and $\mathbf{U}$ (with sole input $F_z$), capable of explaining the evolution of the geomagnetic field over geological time.

The nature of the turbulence considered here is very different from conventional magnetohydrodynamic turbulence, since, as we have seen, all nonlinearities in the equation of motion and the induction equation are here negligible. There is however one remaining nonlinearity which cannot be neglected: this is the advective term in the advection-diffusion equation for $\theta$

$$\frac{\partial \theta}{\partial t} + \mathbf{u}(\theta) \cdot \nabla \theta = S + \kappa \nabla^2 \theta \,. \tag{22}$$

Here, as we have seen, $\mathbf{u}(\theta)$ is linearly related to $\theta$, so we have a quadratic nonlinearity in the equation. This is by no means a 'passive' but rather an 'active' scalar equation, in which all the action is provided by the scalar field itself. The diffusivity $\kappa$ for chemical inhomogeneity in the core is extremely small, and the Péclet number $Pe = VL/\kappa$ correspondingly large (at least $10^8$ in the core). The nonlinearity is therefore dominant. We have included also a source term $S$ which must be interpreted as a source of buoyancy originating in the mushy zone, but in effect continuously regenerating the statistically stationary buoyancy distribution throughout the core.

Now, in the spirit of [9], we may construct an equation for 'thetergy' $<\theta^2>$, which is stationary by assumption:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} < \theta^2 > \, = \, \chi - \kappa < (\nabla \theta)^2 > \, = \, 0 \,, \tag{23}$$

where $\chi \; = < S\theta >$ is the rate of injection of thetergy at the scale $L$. This thetergy cascades to small scales, at which it is ultimately destroyed by molecular diffusion. This cascade is controlled both by the rate of cascade $\chi$ and the velocity scale $V$ that relates $\mathbf{u}(\theta)$ to $\theta$; here, $V \sim g/\Omega$, and $\hat{\mathbf{u}} \sim V\hat{\theta}$ at all $k$. Hence $< \mathbf{u}^2 >$ satisfies an equation like (23) but with $\chi$ replaced by $\epsilon = V^2 \chi$. Dimensional argument then gives the familiar Kolmogorov spectrum and correspondingly

$$\Gamma(k) \sim (\chi/V)^{2/3} k^{-5/3} \tag{24}$$

in an 'inertial' range of wave-numbers $L^{-1} \ll k \ll k_c$, where the conduction cut-off $k_c$ is determined by the diffusive process, i.e.

$$k_c \sim (\kappa^3/V^2\chi)^{1/4} \,. \tag{25}$$

Quite apart from the geomagnetic context considered above, there is an interesting general class of problems here that calls for investigation by direct numerical simulation. Equation (22) with $\mathbf{u}$ an arbitrarily prescribed linear solenoidal functional of $\theta$ is in some respects similar to the Burgers equation, but it has a clearer physical basis, and the velocity field $\mathbf{u}(\theta)$ can here be chosen to be fully three-dimensional. The quadratic nonlinearity is similar to that in the Navier-Stokes equation, but we are dealing here with a simpler problem of scalar field evolution, and complications associated with pressure are absent. These features alone suggest that the problem deserves numerical investigation; perhaps someone at this meeting may be induced to take up this challenge!

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